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PSEUDOSPECTRUM OF AN ELEMENT OF A BANACH ALGEBRA

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Abstract. The ε-pseudospectrum $\Lambda_\varepsilon(a)$ of an element a of an arbitrary Banach algebra A is studied. Its relationships with the spectrum and numerical range of a are given. Characterizations of scalar, Hermitian and Hermitian idempotent elements by means of their pseudospectra are given. The stability of the pseudospectrum is discussed. It is shown that the pseudospectrum has no isolated points, and has a finite number of components, each containing an element of the spectrum of a. Suppose for some $\varepsilon > 0$ and $a, b \in A$, $\Lambda_\varepsilon(ax) = \Lambda_\varepsilon(bx)\ \forall x \in A$. It is shown that $a = b$ if:

(i) a is invertible.

(ii) a is Hermitian idempotent.

(iii) a is the product of a Hermitian idempotent and an invertible element.

(iv) A is semisimple and a is the product of an idempotent and an invertible element.

(v) $A = B(X)$ for a Banach space X.

(vi) A is a C^*-algebra.

(vii) A is a commutative semisimple Banach algebra.

1. Introduction

Let A be a complex Banach algebra with unit 1. For $\lambda \in \mathbb{C}$, $\lambda.1$ is identified with λ. Let $\mathrm{Inv}(A) = \{x \in A : x \text{ is invertible in } A\}$ and $\mathrm{Sing}(A) = \{x \in A : x \text{ is not invertible in } A\}$. The spectrum of an element $a \in A$ is defined as:

$$\sigma(a) := \{\lambda \in \mathbb{C} : \lambda - a \in \mathrm{Sing}(A)\}.$$

The spectral radius of an element a is defined as:

$$r(a) := \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

The complement of the spectrum of an element a is called the resolvent set and is denoted by $\rho(a)$. The spectra of elements of Banach algebras can be used to study the properties of the elements. There are generalizations of the spectrum such as Ransford's generalized spectrum [23], ε-pseudospectrum [30] and the ε-condition spectrum [17]. The latter two depend on the norm, and both contain the spectrum as a subset.

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In this note, we attempt to present a systematic study of the pseudospectrum of an element in a Banach algebra. While the ε-pseudospectra of matrices (with the Euclidean norm) and operators on (infinite-dimensional) Hilbert spaces and Banach spaces have been studied, in the literature, the authors have not come across any systematic account about the ε-pseudospectrum of an element of an arbitrary Banach algebra. From this point of view, all the results given in this note are technically new. Some results and also their proofs are very similar to the ones available in the literature for matrices and operators (for example, Theorem 2.3). In some other cases (such as Theorem 3.11), the result is similar, but our proof is different. Lastly, there are some completely new results (Theorem 3.15, Theorem 6.3) in the sense that nothing similar exists in the literature even for matrices and operators.

Some of the basic properties of pseudospectra of matrices and operators on Hilbert spaces also hold for the pseudospectra of elements of an arbitrary Banach algebra, while some exceptions exist. Simple proofs of these properties and examples are given in Section 2.

The pseudospectra of elements of a Banach algebra can also provide some information about the elements. For instance, the pseudospectra of operators on a Hilbert space can be used to characterize self-adjoint operators, projections and so on (see [7] and [8]). This is extended to characterizations of scalar elements, Hermitian elements and Hermitian idempotent elements of a Banach algebra using the ε-pseudospectrum in Section 3. Theorems about the relationships between the spectrum, pseudospectrum and numerical range of an element of an arbitrary Banach algebra are used for these characterizations. In general, for a Banach algebra A, $a \in A$ and $\varepsilon > 0$, the ε-pseudospectrum of a contains the ε-neighbourhood of the spectrum of a. It is shown that if it is equal to the ε-neighbourhood of the spectrum for all $a \in A$, then A is commutative and semisimple (see Theorem 3.15).

Under certain conditions the pseudospectrum has an important property of being stable under perturbations (see Theorem 4.6). This property is not shared by the spectrum (see [14]). In Section 4, it is shown that map $\varepsilon \mapsto \Lambda_\varepsilon(a)$ is always right continuous. Under additional conditions, the continuity of the ε-pseudospectrum of $a \in A$ with respect to a and ε is proved in Section 4.

Some topological properties of the ε-pseudospectrum are discussed in Section 5. It is shown that the pseudospectrum of an element of a Banach algebra always has finitely many components, each of which contains an element of the spectrum. It is also shown that the pseudospectrum has no isolated points.

In [6] the following question is addressed: If A is a semisimple Banach algebra, $a, b \in A$ and

$$\sigma(ax) = \sigma(bx) \quad \forall x \in A, \tag{1}$$

then under what circumstances is $a = b$? An analogous question about pseudospectrum would be: If A is a Banach algebra, $a, b \in A$ and for $\varepsilon > 0$,

$$\Lambda_\varepsilon(ax) = \Lambda_\varepsilon(bx) \quad \forall x \in A, \tag{2}$$

then is $a = b$? This is addressed in Section 6. It is shown that if (2) holds for some $\varepsilon > 0$, then (1) holds. Hence the hypothesis (2) is stronger, and it is shown that this implies $a = b$ in more general cases.

2. Definition and elementary properties

In this section we discuss some elementary properties of the ε-pseudospectrum of an element of a Banach algebra.

NOTATION. Let $B(z_0;r) := \{z \in \mathbb{C} : |z-z_0| < r\}$, $D(z_0;r) := \{z \in \mathbb{C} : |z-z_0| \leqslant r\}$, $\Omega + D(0;r) = \bigcup\limits_{\omega \in \Omega} D(\omega;r)$ for $\Omega \subseteq \mathbb{C}$ and $d(z,K) = \inf\{|z-k| : k \in K\}$ for any closed set $K \subseteq \mathbb{C}$.

Let $\delta\Omega$ be the boundary of a set $\Omega \subseteq \mathbb{C}$.

$\mathbb{C}^{n \times n}$ denotes the space of square matrices of order n and $B(X)$ denotes the set of bounded linear operators on a Banach space X.

DEFINITION 2.1. Let A be a Banach algebra, $a \in A$ and $\varepsilon > 0$.

The ε*-pseudospectrum* $\Lambda_\varepsilon(a)$ of a is defined by

$$\Lambda_\varepsilon(a) := \{\lambda \in \mathbb{C} : \|(\lambda - a)^{-1}\| \geqslant \varepsilon^{-1}\}$$

with the convention that $\|(\lambda - a)^{-1}\| = \infty$ if $\lambda - a$ is not invertible.

The basic reference for pseudospectrum, especially for matrices, is the book [30]. The website [10] (`http://www.comlab.ox.ac.uk/pseudospectra`) (Pseudospectra Gateway) also contains a lot of material about pseudospectra.

REMARK 2.2. Some authors (such as in [1] and [30]) have defined the following set as the ε-pseudospectrum of a:

$$\Lambda_\varepsilon^*(a) := \{\lambda \in \mathbb{C} : \|(\lambda - a)^{-1}\| > \varepsilon^{-1}\}.$$

There are some significant changes in these two definitions.

1. $\Lambda_\varepsilon(a)$ is a compact subset of $\mathbb{C}$ (Theorem 2.3) whereas $\Lambda_\varepsilon^*(a)$ is not.

2. The map $\varepsilon \mapsto \Lambda_\varepsilon(a)$ is right continuous (Theorem 4.1) but the map $\varepsilon \mapsto \Lambda_\varepsilon^*(a)$ is not.

In the case of most of the other results proved in this note about $\Lambda_\varepsilon(a)$, our methods can be easily modified to obtain analogous results for $\Lambda_\varepsilon^*(a)$. In general, $\Lambda_\varepsilon(a)$ is not the closure of $\Lambda_\varepsilon^*(a)$. However, this is true in many cases. Section 4 contains some information about this. One reason given by some authors for accepting $\Lambda_\varepsilon^*(a)$ as the definition of pseudospectrum is that if T is a bounded operator on a Banach space, then

$$\Lambda_\varepsilon^*(T) = \bigcup_{\|S\| < \varepsilon} \sigma(T+S).$$

However, this is not the case for an arbitrary element of a Banach algebra (see Example 2.5).

A more detailed discussion on these two ways of defining pseudospectrum can be found in [26].

The following theorem gives some elementary properties of the pseudospectrum.

THEOREM 2.3. *Let A be a Banach algebra, $a \in A$ and $\varepsilon > 0$. Then*

1. $\sigma(a) = \bigcap_{\varepsilon > 0} \Lambda_\varepsilon(a)$.
2. $\Lambda_{\varepsilon_1}(a) \subseteq \Lambda_{\varepsilon_2}(a)$ $(0 < \varepsilon_1 < \varepsilon_2)$.
3. $\Lambda_\varepsilon(a+\lambda) = \lambda + \Lambda_\varepsilon(a)$ $(\lambda \in \mathbb{C})$.
4. $\Lambda_\varepsilon(\lambda a) = \lambda \Lambda_{\frac{\varepsilon}{|\lambda|}}(a)$ $(\lambda \in \mathbb{C} \setminus \{0\})$.
5. $\Lambda_\varepsilon(a) \subseteq D(0; \|a\| + \varepsilon)$.
6. $\Lambda_\varepsilon(a)$ *is a non-empty compact subset of* $\mathbb{C}$.
7. $\Lambda_\varepsilon(a+b) \subseteq \Lambda_{\varepsilon + \|b\|}(a)$ $(b \in A)$.
8. $\sigma(a+b) \subseteq \Lambda_\varepsilon(a)$ $(b \in A, \|b\| \leqslant \varepsilon)$, *that is,* $\bigcup_{\|b\| \leqslant \varepsilon} \sigma(a+b) \subseteq \Lambda_\varepsilon(a)$.
9. $\Lambda_\varepsilon(a) + D(0;\delta) \subseteq \Lambda_{\varepsilon+\delta}(a)$ $(\delta > 0)$.

Proof. Analogues of (1), (2), (4), (5) and (7) for $\Lambda^*_\varepsilon(a)$ are proved in [1] and (8) is proved in [13]. Our proofs are similar to these proofs. These are included for the sake of completeness.

1. Let $\lambda \in \sigma(a)$. Then $\lambda - a$ is not invertible, hence by the convention in Definition 2.1, $\|(\lambda - a)^{-1}\| = \infty > \frac{1}{\varepsilon}$ $\forall \varepsilon > 0$. On the other hand, if $\lambda \notin \sigma(a)$, then $\lambda - a$ is invertible. Hence $\exists \varepsilon_0 > 0$ such that $\|(\lambda - a)^{-1}\| < \frac{1}{\varepsilon_0}$. Thus $\lambda \notin \Lambda_{\varepsilon_0}(a)$.

2. Let $0 < \varepsilon_1 < \varepsilon_2$, and suppose $\lambda \in \Lambda_{\varepsilon_1}(a)$. If $\lambda - a$ is not invertible, then $\lambda \in \sigma(a) \subseteq \Lambda_{\varepsilon_2}(a)$. Otherwise $\lambda - a$ is invertible and $\|(\lambda - a)^{-1}\| \geqslant \frac{1}{\varepsilon_1} > \frac{1}{\varepsilon_2}$. Hence $\lambda \in \Lambda_{\varepsilon_2}(a)$.

3. Let $z \in \Lambda_\varepsilon(a+\lambda)$. Then $\|((z-\lambda)-a)^{-1}\| = \|(z-(\lambda+a))^{-1}\| \geqslant \frac{1}{\varepsilon}$. Hence $z - \lambda \in \Lambda_\varepsilon(a)$, that is, $z \in \lambda + \Lambda_\varepsilon(a)$. The reverse inclusion follows similarly.

4. Let $z \in \Lambda_\varepsilon(\lambda a)$. Then $\|(\frac{z}{\lambda} - a)^{-1}\| = |\lambda| \|(z - \lambda a)^{-1}\| \geqslant \frac{|\lambda|}{\varepsilon}$. Hence $\frac{z}{\lambda} \in \Lambda_{\frac{\varepsilon}{|\lambda|}}(a)$, that is, $z \in \lambda \Lambda_{\frac{\varepsilon}{|\lambda|}}(a)$. The reverse inclusion follows similarly.

5. Suppose $|z| > \|a\| + \varepsilon > \|a\|$. Then $z - a$ is invertible and
$$\|(z-a)^{-1}\| \leqslant \frac{1}{|z| - \|a\|} < \frac{1}{\varepsilon}.$$
Hence $z \notin \Lambda_\varepsilon(a)$.

6. By (5), $\Lambda_\varepsilon(a)$ is bounded. It is closed by the continuity of the norm and resolvent function. Hence it is compact. It is non-empty because it contains $\sigma(a)$.

7. Let $a,b \in A$ and $\varepsilon > 0$. Suppose $\lambda \notin \Lambda_{\varepsilon+\|b\|}(a)$. Then $\lambda - a$ is invertible and $\|(\lambda-a)^{-1}\| < \frac{1}{\varepsilon+\|b\|}$. Hence

$$\|(\lambda-a-b)-(\lambda-a)\| = \|b\| < \varepsilon+\|b\| < \|(\lambda-a)^{-1}\|^{-1}.$$

Hence $\lambda-a-b$ is invertible, and moreover,

$$\begin{aligned}\|(\lambda-a-b)^{-1}-(\lambda-a)^{-1}\| &= \|(\lambda-a-b)^{-1}((\lambda-a)-(\lambda-a-b))(\lambda-a)^{-1}\| \\ &\leqslant \frac{\|b\|\|(\lambda-a-b)^{-1}\|}{\varepsilon+\|b\|}.\end{aligned}$$

Thus

$$\begin{aligned}\|(\lambda-a-b)^{-1}\| &\leqslant \|(\lambda-a-b)^{-1}-(\lambda-a)^{-1}\| + \|(\lambda-a)^{-1}\| \\ &< \frac{\|b\|\|(\lambda-a-b)^{-1}\|}{\varepsilon+\|b\|} + \frac{1}{\varepsilon+\|b\|}.\end{aligned}$$

Thus we get $\|(\lambda-a-b)^{-1}\| < \frac{1}{\varepsilon}$. Hence $\lambda \notin \Lambda_\varepsilon(a+b)$.

8. Let $a,b \in A$, $\varepsilon > 0$, $\|b\| \leqslant \varepsilon$. Let $\lambda \in \sigma(a+b)$. Then $\lambda-a-b$ is not invertible. If $\lambda \in \sigma(a)$, we are done. Otherwise, $\lambda - a$ is invertible. Now, $\lambda-a-b = (\lambda-a)(1-(\lambda-a)^{-1}b)$. Since $\lambda-a-b$ is not invertible, $(1-(\lambda-a)^{-1}b)$ is not invertible, hence $\|(\lambda-a)^{-1}b\| \geqslant 1$. Thus

$$1 \leqslant \|(\lambda-a)^{-1}b\| \leqslant \|(\lambda-a)^{-1}\|\|b\|,$$

which gives

$$\|(\lambda-a)^{-1}\| \geqslant \frac{1}{\|b\|} \geqslant \frac{1}{\varepsilon}.$$

9. Let $z \in \Lambda_\varepsilon(a)$ and $\lambda \in \mathbb{C}$ with $|\lambda| \leqslant \delta$. If $z \in \sigma(a)$, then $(z+\lambda) \in \sigma(a+\lambda) \subseteq \Lambda_\delta(a)$, by (8). By (2), $\Lambda_\delta(a) \subseteq \Lambda_{\varepsilon+\delta}(a)$.

Suppose $z \in \Lambda_\varepsilon(a) \setminus \sigma(a)$. Let $x = \frac{(z-a)^{-1}}{\|(z-a)^{-1}\|}$. Then

$$\|((z+\lambda)-a)x\| \leqslant \|(z-a)x\| + |\lambda|\|x\| \leqslant \varepsilon+\delta.$$

If $z+\lambda \in \sigma(a)$, then $z+\lambda \in \Lambda_{\varepsilon+\delta}(a)$. If $z+\lambda-a$ is invertible, then

$$1 = \|x\| = \|(z+\lambda-a)^{-1}(z+\lambda-a)x\| \leqslant (\varepsilon+\delta)\|(z+\lambda-a)^{-1}\|.$$

Hence $z+\lambda \in \Lambda_{\varepsilon+\delta}(a)$. $\square$

REMARK 2.4. The inclusion in (8) of Theorem 2.3 can be proper. Several examples have been given in the case of $B(X)$, where X is a Banach space, in [26] and [27]. Also consider the following example:

EXAMPLE 2.5. Let $A = \left\{ a \in \mathbb{C}^{2\times 2} : a = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix} \right\}$ with norm given by $\|a\| = |\alpha| + |\beta|$. Then A is a Banach algebra. Let $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then it can be verified that

$$\bigcup_{\|b\|\leqslant 1} \sigma(a+b) = D(0;1)$$

which is properly contained in

$$\Lambda_1(a) = \{\lambda \in \mathbb{C} : |\lambda|(|\lambda|-1) \leqslant 1\} = D\Big(0; \Big(\frac{1+\sqrt{5}}{2}\Big)\Big).$$

It can be seen that for this example,

$$\bigcup_{\|b\|<1} \sigma(a+b) = B(0;1)$$

which is properly contained in

$$\Lambda_1^*(a) = \{\lambda \in \mathbb{C} : |\lambda|(|\lambda|-1) < 1\} = B\Big(0; \Big(\frac{1+\sqrt{5}}{2}\Big)\Big).$$

Next, we consider the question of reverse inclusion in (8) of Theorem 2.3.

LEMMA 2.6. *Suppose A is a complex Banach algebra with the following property:*

$$\forall a \in Inv(A), \exists b \in Sing(A) \text{ such that } \|a-b\| = \frac{1}{\|a^{-1}\|}. \tag{3}$$

Then $\forall a \in A$ and $\lambda \in \Lambda_\varepsilon(a)$, $\exists b \in A$ such that $\|b\| \leqslant \varepsilon$ and $\lambda \in \sigma(a+b)$.

Proof. If $\lambda \in \sigma(a)$, take $b = 0$. If $\lambda \in \Lambda_\varepsilon(a) \setminus \sigma(a)$, $\exists c \in \mathrm{Sing}(A)$ such that

$$\|\lambda - a - c\| = \frac{1}{\|(\lambda - a)^{-1}\|}.$$

Let $b = \lambda - a - c$. Then

$$\|b\| = \frac{1}{\|(\lambda - a)^{-1}\|} \leqslant \varepsilon.$$

Also $\lambda - a - b = c \in \mathrm{Sing}(A)$, hence $\lambda \in \sigma(a+b)$. $\square$

COROLLARY 2.7. *Let A be a complex Banach algebra satisfying the hypothesis of Lemma 2.6 and $a \in A$. Then*

$$\lambda \in \Lambda_\varepsilon(a) \iff \exists b \in A \text{ with } \|b\| \leqslant \varepsilon \text{ such that } \lambda \in \sigma(a+b).$$

Thus

$$\Lambda_\varepsilon(a) = \bigcup_{\|b\|\leqslant\varepsilon} \sigma(a+b).$$

Proof. Follows from (8) of Theorem 2.3 and Lemma 2.6. □

REMARK 2.8. The above equality has been proved directly in the case that A is a C^* algebra in Theorem 3.27 of [13].

Examples of Banach algebras that satisfy (3) can be found in Examples 2.18, 2.19 and 2.20 in [17]. These include the algebras $C(X)$, for a compact Hausdorff space X, and $\mathbb{C}^{n\times n}\,\forall n \in \mathbb{N}$. In fact, all C^* algebras satisfy the hypothesis as given below. Uniform algebras can also be seen to satisfy (3).

THEOREM 2.9. *If A is a C^* algebra, then* (3) *holds.*

Proof. Suppose $\exists a \in \mathrm{Inv}(A)$ such that $\forall b \in A$, with $\|b-a\| = \frac{1}{\|a^{-1}\|}$, $b \in \mathrm{Inv}(A)$. Let $c = b-a$. Hence

$$\forall c \in A \text{ with } \|c\| \leqslant \frac{1}{\|a^{-1}\|}, \quad a+c \in \mathrm{Inv}(A). \tag{4}$$

Since a is invertible, a^* is also invertible. Let

$$c = \lambda (a^*)^{-1}, \text{ with } 0 < |\lambda| \leqslant \frac{1}{(\|a^{-1}\|)^2}. \tag{5}$$

Since $\|c\| \leqslant \frac{1}{\|a^{-1}\|}$, $a+\lambda(a^*)^{-1}$ is invertible for all λ as in (5). Now

$$a+\lambda(a^*)^{-1} = \lambda a\Big(\frac{1}{\lambda}+a^{-1}(a^*)^{-1}\Big)$$

$$\implies \Big(\frac{1}{\lambda}+a^{-1}(a^*)^{-1}\Big) \text{ is invertible } \forall \lambda \text{ as in (5).}$$

Hence

$$\|(a)^{-1}\|^2 = \|(a^*)^{-1}a^{-1}\| = r((a^*)^{-1}a^{-1}) < \|a^{-1}\|^2$$

$$\implies \|a^{-1}\| < \|a^{-1}\|, \text{ a contradiction.} \quad \square$$

Next, we consider an example of a Banach algebra in which (3) does not hold.

EXAMPLE 2.10. Consider A as in Example 2.5. Let $a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then we claim that

$$b \in A,\ \|a-b\| = \frac{1}{\|a^{-1}\|} \implies b \in \mathrm{Inv}(A).$$

For the given a, $a^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ and $\|a^{-1}\| = 2$. Any $b \in A$ is of the form $\begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix}$ and b is invertible iff $\alpha \neq 0$. Then $\|a-b\| = |1-\alpha| + |1-\beta|$. If $\|a-b\| = \frac{1}{\|a^{-1}\|}$, i.e., $|1-\alpha|+|1-\beta| = \frac{1}{2}$, then $\alpha \neq 0$. Hence b is invertible.

3. Pseudospectrum and numerical range

In this section the relationships between the ε-pseudospectrum, the spectrum and the numerical range of an element of a Banach algebra are given. Scalar, Hermitian and idempotent elements of a Banach algebra are characterized using their ε-pseudospectra.

DEFINITION 3.1. Let A be a Banach algebra and $a \in A$. The *numerical range* (see Definition 1.10.1 in [3]) of a is defined by

$$V(a) := \{f(a) : f \in A', f(1) = 1, \|f\| = 1\},$$

where A' is the dual space of A.

DEFINITION 3.2. Let A be a Banach algebra and $a \in A$. Then a is said to be *Hermitian* if $V(a) \subseteq \mathbb{R}$.

DEFINITION 3.3. Let X be a Banach space and $T \in B(X)$. Let $X^{'}$ be the dual space of X. The *spatial numerical range* of T is defined by

$$W(T) = \{f(Tx) : f \in X^{'}, \|f\| = f(x) = 1, \|x\| = 1\}.$$

For an operator T on a Banach space X, the spatial numerical range $W(T)$ and the numerical range $V(T)$, where T is regarded as an element of the Banach algebra $B(X)$, are related by the following:

$$\overline{\mathrm{Co}}\, W(T) = V(T)$$

where $\overline{\mathrm{Co}}\, E$ denotes the closed convex hull of $E \subseteq \mathbb{C}$. See Theorem 9.4 in [2].

The following theorems establish the relationships between the spectrum, the ε-pseudospectrum and the numerical range of an element of a Banach algebra. The inequalities in the following theorem have been known for operators on a Banach space (see Theorem 1.3.9 of [20] and Problem 6.16 of [16]). Here those are proved for an element of an arbitrary Banach algebra.

THEOREM 3.4. *Let A be a Banach algebra, $a \in A$ and $\varepsilon > 0$. Then*

$$d(\lambda, V(a)) \leqslant \frac{1}{\|(\lambda - a)^{-1}\|} \leqslant d(\lambda, \sigma(a)) \quad \forall \lambda \in \mathbb{C} \setminus \sigma(a). \tag{6}$$

Thus

$$\sigma(a) + D(0;\varepsilon) \subseteq \Lambda_\varepsilon(a) \subseteq V(a) + D(0;\varepsilon). \tag{7}$$

Proof. For the second inequality in (6), we have for $\lambda \notin \sigma(a)$,

$$d(\lambda, \sigma(a)) = \frac{1}{r((\lambda - a)^{-1})} \geqslant \frac{1}{\|(\lambda - a)^{-1}\|}.$$

For the first inequality in (6), for $\lambda \notin \sigma(a)$, by the Hahn-Banach theorem,

$$\exists f \in A' \text{ such that } \|f\| = 1, \text{ and } f((\lambda - a)^{-1}) = \|(\lambda - a)^{-1}\|.$$

Define $g : A \to \mathbb{C}$ by $g(x) = \|(\lambda - a)^{-1}\|^{-1} f(x(\lambda - a)^{-1})$. Then g is linear, $g(1) = 1$ and $\|g\| \leqslant 1 \implies \|g\| = 1$. Hence $g(a) \in V(a)$. Now

$$d(\lambda, V(a)) \leqslant |\lambda - g(a)| = |g(\lambda - a)| = \|(\lambda - a)^{-1}\|^{-1}|f(1)| \leqslant \|(\lambda - a)^{-1}\|^{-1}. \quad \square$$

Next we consider the question of equality in the first inclusion of (7).

DEFINITION 3.5. Let A be a Banach algebra and $a \in A$. We define a to be of G_1*-class* if

$$\|(z - a)^{-1}\| = \frac{1}{\mathrm{d}(z, \sigma(a))} \quad \forall z \in \mathbb{C} \setminus \sigma(a). \tag{8}$$

The following lemma is elementary.

LEMMA 3.6. *Let A be a Banach algebra and $a \in A$. Then*

$$\Lambda_\varepsilon(a) = \sigma(a) + D(0; \varepsilon) \quad \forall \varepsilon > 0 \tag{9}$$

iff a is of G_1-class.

Proof. Suppose (9) holds. Let $z \notin \sigma(a)$. Then $d(z, \sigma(a)) > 0$. For every ε satisyfing $0 < \varepsilon < d(z, \sigma(a))$, by (9), $z \notin \Lambda_\varepsilon(a)$. Thus

$$\|(z - a)^{-1}\| < \frac{1}{\varepsilon} \quad \forall z \notin \sigma(a), \quad 0 < \varepsilon < d(z, \sigma(a)).$$

Hence it follows that $\|(z - a)^{-1}\| \leqslant \frac{1}{d(z, \sigma(a))}$. By (6), we already have

$$\frac{1}{d(z, \sigma(a))} \leqslant \|(z - a)^{-1}\|.$$

The converse implication is trivial. $\square$

REMARK 3.7. The idea of G_1-class is due to Putnam who defined it for operators on Hilbert spaces. See [22] and [21]. It is known that the G_1-class properly contains the class of seminormal operators ($TT^* \leqslant T^*T$ or $T^*T \leqslant TT^*$) and this class properly contains the class of normal operators. Using the Gelfand- Naimark theorem, we can make similar statements about elements in a C^* algebra. In the finite dimensional case, G_1 operators are normal (see [29]). Hence every operator on a finite dimensional Hilbert space satisfying (9) is normal. Also it is easy to see that every element in a uniform algebra is of G_1-class.

REMARK 3.8. The equivalent of Lemma 3.6 for $\Lambda_\varepsilon^*(a)$ is: Let A be a Banach algebra and $a \in A$. Then

$$\Lambda_\varepsilon^*(a) = \sigma(a) + B(0; \varepsilon) \quad \forall \varepsilon > 0 \tag{10}$$

iff a is of G_1-class.

EXAMPLE 3.9. By Theorem 2.2 of [30], if

$$\Lambda_\varepsilon^*(a) = \sigma(a) + B(0;\varepsilon) \tag{11}$$

holds *for every* $\varepsilon > 0$, for $a \in A = \mathbb{C}^{n\times n}(n \in \mathbb{N})$ with $\|\cdot\| = \|\cdot\|_2$, then a is normal. This has been incorrectly interpreted in Example 2.2 in [7], where the authors have given an example of a matrix for which (11) holds for a particular $\varepsilon > 0$. In this example, $\varepsilon = 2$, and the matrix is taken to be $a = a_1 \oplus a_2$, where $a_1 = \operatorname{diag}(1,\omega,\omega^2)$ and $a_2 = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$ where $\omega = e^{\frac{2\pi i}{3}}$ and $\beta > 0$ satisfies

$$\Lambda_\varepsilon^*(a_2) = \{z \in \mathbb{C} : |z| < \sqrt{\varepsilon(\varepsilon+\beta)}\} \subseteq B\Big(0;\frac{\sqrt{13}+1}{2}\Big).$$

a is not normal, or even hyponormal. It can be shown that (11) does not hold for some $\varepsilon > 0$. For example, let $\beta = 0.5$. Note that for every $\varepsilon > 0$,

$$\Lambda_\varepsilon^*(a) = \Lambda_\varepsilon^*(a_1) \cup \Lambda_\varepsilon^*(a_2) = B(1;\varepsilon) \cup B(\omega;\varepsilon) \cup B(\omega^2;\varepsilon) \cup B(0;\sqrt{\varepsilon(\varepsilon+\beta)}),$$

and

$$\sigma(a) + B(0;\varepsilon) = B(1;\varepsilon) \cup B(\omega;\varepsilon) \cup B(\omega^2;\varepsilon) \cup B(0;\varepsilon)$$

When $\varepsilon = 2$, the last of the four discs in $\Lambda_2^*(a)$ is contained in the union of the first three discs, hence (11) is satisfied. On the other hand, when $\varepsilon = 0.3$, all four discs are disjoint and $B(0;\sqrt{\varepsilon(\varepsilon+\beta)})$ properly contains $B(0;\varepsilon)$. Hence (11) is not satisfied. Hence, this example does not show that Theorem 2.2 of [30] is false.

EXAMPLE 3.10. See Remark 5.5 in [18]. Consider the right shift operator R on $l^2(\mathbb{N})$. It is not normal but $\Lambda_\varepsilon(R) = \sigma(R) + D(0;\varepsilon) = D(0;1+\varepsilon)$ $\forall \varepsilon > 0$. R is, however, a hyponormal operator, i.e., $TT^* \leqslant T^*T$.

The following theorem shows that the numerical range $V(a)$ of a is determined by certain closed half-planes related to the pseudospectrum $\Lambda_\varepsilon(a)$. This result has been stated in Theorem 17.5 of [30] for operators on a Banach space. Our proof is different.

THEOREM 3.11. *Let A be a Banach algebra, $a \in A$ and $\varepsilon > 0$. Let H be a closed half-plane in $\mathbb{C}$ such that*

$$\Lambda_\varepsilon(a) \subseteq H + D(0;\varepsilon) \quad \forall \varepsilon > 0. \tag{12}$$

Then $V(a) \subseteq H$.

Proof. The given hypothesis implies that

$$\|(z-a)^{-1}\| \leqslant \frac{1}{d(z,H)} \quad \forall z \notin H.$$

The proof of this assertion is similar to the proof of Lemma 3.6 and hence it is omitted.

It also follows from the first inclusion in (7) that $\sigma(a) \subseteq H$. We can assume without loss of generality that $H = \{z \in \mathbb{C} : \operatorname{Re} z \leqslant 0\}$, for we can then take suitable translations and rotations to prove the theorem in the general case.

We thus have

$$\|(z-a)^{-1}\| \leqslant \frac{1}{\operatorname{Re} z}, \quad \operatorname{Re} z > 0. \tag{13}$$

What follows is a particular case of a classical result of Hille and Yosida. See Theorem 13.37 in [24]. Let $S(\varepsilon) = (1-\varepsilon a)^{-1}$. Then $\varepsilon a S(\varepsilon) = S(\varepsilon) - 1\ \forall \varepsilon > 0$, and $\lim\limits_{\varepsilon \to 0} S(\varepsilon) = 1$. For $t \geqslant 0$, let $T(t,\varepsilon) = \exp(taS(\varepsilon)) = \exp(\frac{t}{\varepsilon}(S(\varepsilon)-1)) = e^{\frac{-t}{\varepsilon}} \exp(\frac{t}{\varepsilon}S(\varepsilon))$.

Hence $\|T(t,\varepsilon)\| \leqslant e^{\frac{-t}{\varepsilon}} e^{\frac{t}{\varepsilon}\|S(\varepsilon)\|} \leqslant 1\ \forall \varepsilon > 0, \forall t \geqslant 0$, for $\|S(\varepsilon)\| = \frac{1}{\varepsilon}\|(\frac{1}{\varepsilon} - a)^{-1}\| \leqslant \frac{1}{\varepsilon}\varepsilon = 1$. Hence $\lim\limits_{\varepsilon \to 0}\|T(t,\varepsilon)\| \leqslant 1\ \forall t \geqslant 0$, i.e. $\|\exp ta\| \leqslant 1\ \forall t \geqslant 0$. By Corollary 1.10.13 in [3], $\operatorname{Re} z \leqslant 0\ \forall z \in V(a)$, that is, $V(a) \subseteq H$. $\square$

The following corollary gives a characterization of Hermitian elements of a Banach algebra in terms of the ε-pseudospectrum.

COROLLARY 3.12. *Let A be a Banach algebra and $a \in A$. Then a is Hermitian iff*

$$\Lambda_\varepsilon(a) \subseteq \{z \in \mathbb{C} : |\operatorname{Im} z| \leqslant \varepsilon\} \quad \forall \varepsilon > 0. \tag{14}$$

Proof. If a is Hermitian, by definition, $V(a) \subseteq \mathbb{R}$. By the second inclusion in (7), $\forall \varepsilon > 0$,

$$\Lambda_\varepsilon(a) \subseteq V(a) + D(0;\varepsilon) \subseteq \mathbb{R} + D(0;\varepsilon) \subseteq \{z \in \mathbb{C} : |\operatorname{Im} z| \leqslant \varepsilon\}.$$

Next, suppose (14) holds. Now

$$\Lambda_\varepsilon(a) \subseteq \{z \in \mathbb{C} : |\operatorname{Im} z| \leqslant \varepsilon\} \quad \forall \varepsilon > 0.$$

$$\Longrightarrow \Lambda_\varepsilon(a) \subseteq \text{ closed upper half plane } + D(0;\varepsilon), \quad \forall \varepsilon > 0,$$

and

$$\Lambda_\varepsilon(a) \subseteq \text{ closed lower half plane } + D(0;\varepsilon), \quad \forall \varepsilon > 0.$$

By Theorem 3.11,

$$V(a) \subseteq \text{closed upper half plane} \cap \text{closed lower half plane} \implies V(a) \subseteq \mathbb{R}.$$

Hence a is Hermitian. $\square$

The numerical range of an element of a Banach algebra is a compact convex subset of $\mathbb{C}$ containing its spectrum, and hence it also contains the convex hull of the spectrum. In some cases, as given below, equality holds.

COROLLARY 3.13. *Let A be a Banach algebra and $a \in A$. Suppose a is of G_1-class. Then $V(a) = Co\ \sigma(a)$, the convex hull of $\sigma(a)$, and $\|a\| \leqslant e\, r(a)$.*

Proof. We first observe that since $\sigma(a)$ is a compact subset of $\mathbb{C}$, so is $\mathrm{Co}\,\sigma(a)$. Hence $\mathrm{Co}\,\sigma(a)$ is closed, and equal to the intersection of all closed half-planes containing $\sigma(a)$. Also, $\mathrm{Co}\,\sigma(a) \subseteq V(a)$, since $V(a)$ itself is convex and $\sigma(a) \subseteq V(a)$.

By the hypothesis, $\Lambda_\varepsilon(a) = \sigma(a) + D(0;\varepsilon)\ \forall \varepsilon > 0$. Let H be any closed half plane such that $\sigma(a) \subseteq H$. Then $\Lambda_\varepsilon(a) \subseteq H + D(0;\varepsilon)\ \forall \varepsilon > 0$. By Theorem 3.11, $V(a) \subseteq H$. Hence $V(a) \subseteq \mathrm{Co}\,\sigma(a)$.

Next, $r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} = \sup\{|\lambda| : \lambda \in V(a)\} \geqslant \frac{1}{e}\|a\|$ by Theorem 1.10.14 of [3]. □

REMARK 3.14. A bounded operator T on a Hilbert space H is said to be convexoid (see [14]) if the closure of its spatial numerical range $\overline{W(T)}$ is equal to the convex hull of the spectrum $\mathrm{Co}\,\sigma(T)$. The numerical range of $T \in B(H)$ is convex, and it has already been observed that for a Banach space operator $\overline{\mathrm{Co}}\,W(T) = V(T)$. Hence in the case of $A = B(H)$, the above theorem shows that a G_1 operator is a convexoid operator (see [19]).

THEOREM 3.15. *Let A be a Banach algebra such that every element of A is of G_1-class. Then A is commutative, semisimple and hence isomorphic and homeomorphic to a function algebra.*

Proof. By Corollary 3.13, $\|a\| \leqslant er(a)\ \forall a \in A$. Hence A is commutative by a theorem of Hirschfeld and Zelazko (see Corollary 2.15.7 of [3]). A is semisimple because $\|a\| \leqslant er(a)\ \forall a \in A$, that is, the spectral radius is a norm in A. □

The following theorem involves the analytical functional calculus for elements of a Banach algebra. See [24].

THEOREM 3.16. *Let A be a Banach algebra and $a \in A$. Let $\Omega \subseteq \mathbb{C}$ be an open neighbourhood of $\Lambda_\varepsilon(a)$ and Γ be a contour that surrounds $\Lambda_\varepsilon(a)$ in Ω. Let f be analytic in Ω. We recall the definition of $\tilde{f}(a)$ in the analytical functional calculus as*

$$\tilde{f}(a) = \frac{1}{2\pi i}\int_\Gamma (z-a)^{-1} f(z)\,dz \tag{15}$$

Then

$$\|\tilde{f}(a)\| \leqslant \frac{Ml}{2\pi\varepsilon} \tag{16}$$

where $l =$ length of Γ and $M = \sup\{|f(z)| : z \in \Gamma\}$.

Proof. The proof is as in (14.10) of [30], but for an arbitrary Banach algebra element.

$$\|\tilde{f}(a)\| \leqslant \frac{1}{2\pi}\int_\Gamma \|(z-a)^{-1}\|\,|f(z)|\,|dz| \leqslant \frac{Ml}{2\pi\varepsilon}$$

since Γ lies outside the interior of $\Lambda_\varepsilon(a)$. □

The following corollary gives an equivalent condition in terms of the ε-pseudospectrum for an element of a Banach algebra to be a scalar (i.e. a scalar multiple of the identity).

COROLLARY 3.17. *Let A be a Banach algebra, $a \in A$ and $\mu \in \mathbb{C}$. Then*

$$a = \mu \iff \Lambda_\varepsilon(a) = D(\mu, \varepsilon) \quad \forall \varepsilon > 0.$$

Proof. If $a = \mu$, it is trivial to see that $\Lambda_\varepsilon(A) = D(\mu, \varepsilon)\ \forall \varepsilon > 0$. For the converse part, by (3) of Theorem 2.3, we may assume that $\mu = 0$. Let $f(z) = z$ and $\Gamma = \{z \in \mathbb{C} : |z| = \varepsilon\}$. Then, with the notations of Theorem 3.16, $M = \varepsilon$ and $l = 2\pi\varepsilon$. Hence by Theorem 3.16, $\|a\| \leqslant \varepsilon$. Since this is true $\forall \varepsilon > 0$, $a = 0 = \mu$.

An alternate proof is as follows: Suppose $\Lambda_\varepsilon(a) = D(0;\varepsilon)\ \forall \varepsilon > 0$. By the first inclusion in (7), $\sigma(a) = \{0\}$. Hence $\Lambda_\varepsilon(a) = \sigma(a) + D(0;\varepsilon)\ \forall \varepsilon > 0$. As in Corollary 3.13, we get $V(a) = \mathrm{Co}\,\sigma(a) = \{0\}$. By Corollary 1.10.14 in [3], $a = 0$. □

The following corollary gives a characterization of Hermitian idempotent elements of a Banach algebra in terms of the ε-pseudospectrum.

COROLLARY 3.18. *Let A be a Banach algebra, $a \in A, a \neq 0$. Then*

$$\Lambda_\varepsilon(a) = D(0;\varepsilon) \cup D(1;\varepsilon) \quad \forall \varepsilon > 0 \tag{17}$$

if and only if a is a non-trivial ($a \neq 0$ and $a \neq 1$) Hermitian idempotent and $\|a\| = 1$.

Proof. Suppose (17) holds. By Corollary 3.12, a is Hermitian. Now, $\sigma(a) \subseteq \{0,1\}$. If $\sigma(a) = \{0\}$, by Theorem 1.10.17 of [3], $\|a\| = r(a) = 0$. Similarly, if $\sigma(a) = \{1\}$, then $\sigma(a-1) = \{0\}$, and by Theorem 1.10.17 of [3], $a = 1$. Hence $\sigma(a) = \{0,1\}$, and by Theorem 1.10.17 of [3], $\|a\| = r(a) = 1$. Let $\varepsilon > 0$ be small enough so that $D(0;\varepsilon) \cap D(1;\varepsilon) = \emptyset$. Let $f(z) = z^2 - z$ and $\Gamma = \{z \in \mathbb{C} : |z| = \varepsilon\} \cup \{z \in \mathbb{C} : |z-1| = \varepsilon\}$. Then, with the notations of Theorem 3.16, $M \leqslant \varepsilon(\varepsilon+1)$ and $l = 4\pi\varepsilon$. Hence by Theorem 3.16, $\|a^2 - a\| = \|\tilde{f}(a)\| \leqslant 2\varepsilon(\varepsilon+1)$ for all sufficiently small $\varepsilon > 0$. Hence $a = a^2$.

Conversely, suppose a is a non-trivial Hermitian idempotent. Then $\sigma(a) = \{0,1\}$ and for $\lambda \neq 0, 1$, $(\lambda - a)^{-1} = \frac{1}{\lambda}(1 + \frac{a}{(\lambda-1)})$. Also $\sigma((\lambda - a)^{-1}) = \{\frac{1}{\lambda}, \frac{1}{\lambda-1}\}$. By Proposition 2 of [28], $\|h + \beta\| = r(h + \beta) \quad \forall \beta \in \mathbb{C}$, for Hermitian $h \in A$. It follows that $\|\alpha h + \beta\| = r(\alpha h + \beta) \forall \alpha, \beta \in \mathbb{C}$. Hence

$$\begin{aligned}
\|(\lambda - a)^{-1}\| &= \left\| \frac{1}{\lambda(\lambda-1)} a + \frac{1}{\lambda} \right\| \\
&= r\Big(\frac{1}{\lambda(\lambda-1)} a + \frac{1}{\lambda}\Big) \\
&= r((\lambda - a)^{-1}) \\
&= \max\left\{ \frac{1}{|\lambda|}, \frac{1}{|\lambda - 1|} \right\},
\end{aligned}$$

and thus (17) holds by Definition 2.1. □

REMARK 3.19. If $a \in A$ is idempotent, but not Hermitian, then by Corollary 3.18, (17) does not hold. The following example illustrates this. Consider the Banach algebra $A = \mathbb{C}^{2\times 2}$ with the induced $\|\cdot\|_\infty$ norm (i.e., the operator norm of a when the underlying space $\mathbb{C}^2$ is equipped with the $\|\cdot\|_\infty$ norm). It is equal to the maximum absolute row sum norm. Let $a = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. Then a is an idempotent but not Hermitian ($r(a) = 1, \|a\|_\infty = 2$). The resolvent norm is given by

$$\|(\lambda - a)^{-1}\|_\infty = \max\left\{\frac{1}{|\lambda|}, \frac{1}{|\lambda||\lambda-1|} + \frac{1}{|\lambda-1|}\right\}, \quad \lambda \in \rho(a).$$

It can be seen that $\lambda = 2.1 \in \Lambda_1(a) \setminus D(0;1) \cup D(1;1)$.

4. Stability

In this section, the stability of the pseudospectrum is discussed. For fixed $a \in A$, the map $\varepsilon \mapsto \Lambda_\varepsilon(a)$ is always right continuous. It is shown that under some conditions, the ε-pseudospectrum $\Lambda_\varepsilon(a)$ varies continuously with respect to a and ε.

THEOREM 4.1. *Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and $K(\mathbb{C})$ denote the set of compact subsets of $\mathbb{C}$ equipped with the Hausdorff metric defined as*

$$d(\Lambda, \Delta) = \max\{\sup_{s\in\Lambda} d(s,\Delta), \sup_{t\in\Delta} d(t,\Lambda)\}.$$

Let A be a Banach algebra and $a \in A$. Define the maps $F_a, K_a, H_a : \mathbb{R}^+ \to K(\mathbb{C})$ by:

$$F_a(\varepsilon) = \Lambda_\varepsilon(a),$$
$$H_a(\varepsilon) = \overline{\bigcup_{0<t<\varepsilon} \Lambda_t(a)}$$

and

$$K_a(\varepsilon) = \bigcap_{s>\varepsilon} \Lambda_s(a).$$

Then

1. *$H_a(\varepsilon) \subseteq F_a(\varepsilon) = K_a(\varepsilon) \quad \forall \varepsilon > 0$, that is, the map F_a is right continuous $\forall \varepsilon > 0$.*
2. *For $\varepsilon_0 > 0$, the following are equivalent:*
   - *(a) F_a is discontinuous at $\varepsilon_0 > 0$.*
   - *(b) F_a is left discontinuous at $\varepsilon_0 > 0$.*
   - *(c) $H_a(\varepsilon_0) \subsetneq F_a(\varepsilon_0)$.*
   - *(d) The level set $\{\lambda \in \mathbb{C} : \|(\lambda - a)^{-1}\| = \frac{1}{\varepsilon_0}\}$ contains a non-empty open set.*

Proof.

1. By (2) of Theorem 2.3, $\Lambda_t(a) \subseteq F_a(\varepsilon)$, $0 < t < \varepsilon$. Since $F_a(\varepsilon) = \Lambda_\varepsilon(a)$ is closed, it follows that $H_a(\varepsilon) \subseteq F_a(\varepsilon)\ \forall \varepsilon > 0$.

Since $F_a(\varepsilon) \subseteq \Lambda_s(a)$ for $\varepsilon < s$, $F_a(\varepsilon) \subseteq K_a(\varepsilon)$ $\forall \varepsilon > 0$. Let $\lambda \in K_a(\varepsilon)$. Then

$$\|(\lambda - a)^{-1}\| \geqslant \frac{1}{s} \quad \forall s > \varepsilon.$$

Hence

$$\|(\lambda - a)^{-1}\| \geqslant \frac{1}{\varepsilon}, \text{ that is, } \lambda \in F_a(\varepsilon).$$

Thus $F_a(\varepsilon) = K_a(\varepsilon)$.

2. (a) $\Longrightarrow$ (b) follows from (1) since F_a is a monotonically increasing and right continuous function.

   Suppose (b) holds, i.e., suppose F_a is left discontinuous at ε_0. Then $\exists r > 0$ such that $\forall \delta > 0$, $\exists \varepsilon > 0$ such that $\varepsilon_0 - \delta < \varepsilon < \varepsilon_0$ and $d(F_a(\varepsilon), F_a(\varepsilon_0)) \geqslant r$. Then $\exists \lambda_0 \in F_a(\varepsilon_0)$ such that $B(\lambda_0; r) \cap F_a(\varepsilon) = \emptyset$. Hence $B(\lambda_0; r) \cap \Lambda_t(a) = \emptyset$, $0 < t < \varepsilon$. Since ε can be arbitrarily close to ε_0, $\lambda_0 \notin H_a(\varepsilon_0)$.

   If (c) holds, $\exists \lambda_0 \in F_a(\varepsilon_0)$ and $r > 0$ such that $B(\lambda_0; r) \cap \bigcup\limits_{0<t<\varepsilon_0} \Lambda_t(a) = \emptyset$. Then $\forall \lambda \in B(\lambda_0; r)$, $\lambda \notin \Lambda_t(a)$, $0 < t < \varepsilon_0$, i.e.,

$$\|(\lambda - a)^{-1}\| < \frac{1}{t}, \quad 0 < t < \varepsilon_0, \forall \lambda \in B(\lambda_0; r).$$

   Hence $\|(\lambda - a)^{-1}\| \leqslant \frac{1}{\varepsilon_0}$ $\forall \lambda \in B(\lambda_0; r)$. Since $\lambda_0 \in F_a(\varepsilon_0)$, $\|(\lambda_0 - a)^{-1}\| \geqslant \frac{1}{\varepsilon_0}$. Hence we have $\|(\lambda_0 - a)^{-1}\| = \frac{1}{\varepsilon_0}$, and $\|(\lambda - a)^{-1}\| \leqslant \frac{1}{\varepsilon_0}$ $\forall \lambda \in B(\lambda_0; r)$.

   We claim that $\|(\lambda - a)^{-1}\| = \frac{1}{\varepsilon_0}$ $\forall \lambda \in B(\lambda_0; r)$. By the Hahn-Banach theorem, $\exists \phi \in A'$ such that $\|\phi\| = 1$ and $\phi((\lambda_0 - a)^{-1}) = \|(\lambda_0 - a)^{-1}\| = \frac{1}{\varepsilon_0}$. Define $f : B(\lambda_0; r) \to \mathbb{C}$ by

$$f(\lambda) = \phi((\lambda - a)^{-1}), \quad \lambda \in B(\lambda_0; r).$$

   Then f is analytic in $B(\lambda_0; r)$, $f(\lambda_0) = \frac{1}{\varepsilon_0}$, and $\forall \lambda \in B(\lambda_0; r)$,

$$|f(\lambda)| = |\phi((\lambda - a)^{-1})| \leqslant \|\phi\| \|(\lambda - a)^{-1}\| \leqslant \frac{1}{\varepsilon_0}.$$

   Hence f is constant in $B(\lambda_0; r)$ by the maximum modulus principle. Hence

$$\frac{1}{\varepsilon_0} = |f(\lambda)| \leqslant \|\phi\| \|(\lambda - a)^{-1}\| = \|(\lambda - a)^{-1}\| \quad \forall \lambda \in B(\lambda_0; r).$$

   If (d) holds, $\exists \lambda_0 \in \rho(a)$ and $r > 0$ such that $\|(z - a)^{-1}\| = \frac{1}{\varepsilon_0}$ $\forall z \in B(\lambda_0; r)$. In particular, $\lambda_0 \in \Lambda_{\varepsilon_0}(a)$. For $\delta > 0$, let $0 < \varepsilon_0 - \delta < \varepsilon_\delta < \varepsilon_0$. Then $\forall z \in B(\lambda_0; r)$, $\|(z - a)^{-1}\| = \frac{1}{\varepsilon_0} < \frac{1}{\varepsilon_\delta}$. Hence $\forall z \in B(\lambda_0; r)$, $z \notin \Lambda_{\varepsilon_\delta}(a)$. Thus, $\exists r > 0$ such that $\forall \delta > 0$, $\exists \varepsilon_\delta > 0$ such that $|\varepsilon_\delta - \varepsilon_0| < \delta$, but $d(\Lambda_{\varepsilon_0}(a), \Lambda_{\varepsilon_\delta}(a)) \geqslant r$. □

REMARK 4.2. The map $\varepsilon \mapsto \Lambda_\varepsilon^*(a)$ is not right- continuous. This will follow from Shargorodsky's example (see [25]) elaborated in Example 4.9 below.

The following theorem shows that continuity of the ε-pseudospectrum of $a \in A$ with respect to ε implies continuity with respect to a as well as joint continuity.

THEOREM 4.3. *Let $a_0 \in A$ and U be an open neighbourhood of a_0. With the notations as in Theorem 4.1, suppose the map $F_a : \mathbb{R}^+ \to K(\mathbb{C})$ is continuous for each $a \in U$. Then the pseudospectrum map $\Lambda_\varepsilon : U \to K(\mathbb{C})$ is continuous with respect to the norm on A, $\forall \varepsilon > 0$. Also the map $\Lambda : \mathbb{R}^+ \times U \to K(\mathbb{C})$ defined by $\Lambda(\varepsilon, a) = \Lambda_\varepsilon(a)$ is continuous with respect to the metric given by $d((\varepsilon_1, a_1), (\varepsilon_2, a_2)) = \|a_1 - a_2\| + |\varepsilon_1 - \varepsilon_2|$.*

Proof. Given $\varepsilon > 0$ and $a \in U$, choose $\delta > 0$ such that $0 < \delta < \frac{\varepsilon}{2}$. Let $b \in U$ and $\varepsilon' > 0$ such that $\|a - b\| + |\varepsilon - \varepsilon'| < \delta$. Let $c = b - a$. Then $\|c\| < \delta - |\varepsilon - \varepsilon'|$.

$$\begin{aligned}
\Lambda_{\varepsilon-\delta}(a) &= \Lambda_{\varepsilon-\delta}(b - c) \\
&\subseteq \Lambda_{\varepsilon-\delta+\|c\|}(b) \\
&\subseteq \Lambda_{\varepsilon'}(b) \qquad (\because \varepsilon - \delta + \|c\| < \varepsilon - |\varepsilon - \varepsilon'| \leqslant \varepsilon') \\
&= \Lambda_{\varepsilon'}(a + c) \subseteq \Lambda_{\varepsilon'+\|c\|}(a) \subseteq \Lambda_{\varepsilon+\delta}(a) \quad (\because \varepsilon' + \|c\| < \varepsilon' + \delta - |\varepsilon - \varepsilon'| \leqslant \varepsilon + \delta).
\end{aligned}$$

Thus we have shown:

$$\Lambda_{\varepsilon-\delta}(a) \subseteq \Lambda_{\varepsilon'}(b) \subseteq \Lambda_{\varepsilon+\delta}(a).$$

Hence

$$d(\Lambda_\varepsilon(a), \Lambda_{\varepsilon'}(b)) \leqslant d(\Lambda_{\varepsilon-\delta}(a), \Lambda_{\varepsilon+\delta}(a))$$

By the hypothesis, $d(\Lambda_{\varepsilon-\delta}(a), \Lambda_{\varepsilon+\delta}(a)) \to 0$ as $\delta \to 0$. Hence the map Λ is jointly continuous in ε and a, and hence also continuous separately in a, i.e. the map $\Lambda_\varepsilon : U \to K(\mathbb{C})$ is continuous. $\square$

We next see a continuity result that uses the right continuity of the map $\varepsilon \mapsto \Lambda_\varepsilon(a)$.

THEOREM 4.4. *Let A be a Banach algebra. Let $\varepsilon > 0$ and $\varepsilon_n \in \mathbb{R}^+$ be such that $\varepsilon_n \geqslant \varepsilon$ $\forall n$, and $\varepsilon_n \to \varepsilon$. Let $a \in A$, and $a_n \in A$ be such that $\Lambda_\varepsilon(a) \subseteq \Lambda_\varepsilon(a_n)$ $\forall n$ and $\|a_n - a\| \to 0$. Then $d(\Lambda_{\varepsilon_n}(a_n), \Lambda_\varepsilon(a)) \to 0$ as $n \to \infty$.*

Proof. Let $b_n = a - a_n$ $\forall n \in \mathbb{N}$. Then

$$\Lambda_\varepsilon(a) \subseteq \Lambda_\varepsilon(a_n) \subseteq \Lambda_{\varepsilon_n}(a_n) = \Lambda_{\varepsilon_n}(a - b_n) \subseteq \Lambda_{\varepsilon_n+\|b_n\|}(a).$$

Hence

$$d(\Lambda_\varepsilon(a), \Lambda_{\varepsilon_n}(a_n)) \leqslant d(\Lambda_\varepsilon(a), \Lambda_{\varepsilon_n+\|b_n\|}(a)) \to 0$$

as $n \to \infty$ by the right continuity of the map $\varepsilon \mapsto \Lambda_\varepsilon(a)$. $\square$

DEFINITION 4.5. A Banach space X is said to be *complex uniformly convex* if for every $\varepsilon > 0$, $\exists \delta > 0$ such that

$$x, y \in X, \|y\| \geqslant \varepsilon \text{ and } \|x + \zeta y\| \leqslant 1 \quad \forall \zeta \in \mathbb{C} \text{ with } |\zeta| \leqslant 1 \implies \|x\| \leqslant 1 - \delta.$$

Note that all uniformly convex spaces are complex uniformly convex. Thus Hilbert spaces and L^p spaces with $1 < p < \infty$ are complex uniformly convex. It is known that L^1 is complex uniformly convex, though not uniformly convex. Also L^∞ is not complex uniformly convex, but $(L^\infty)'$ is (see [25]).

The next theorem shows that under some conditions, the ε-pseudospectrum is stable under perturbations. As mentioned in the Introduction, this is not the case with the spectrum.

THEOREM 4.6. *Let A be a Banach algebra.*

1. *For a fixed $a \in A$, the map F_a, which sends $\varepsilon > 0$ to the ε-pseudospectrum of a in $\mathbb{C}$, is continuous with respect to the usual Euclidean metric in the domain and the Hausdorff metric in the codomain if one of the following holds:*
   (a) *The resolvent set $\rho(a)$ is a connected subset of $\mathbb{C}$.*
   (b) *$A = B(X)$, X a Banach space such that X or X' is complex uniformly convex. In particular, A may be $B(H)$, for a Hilbert space H.*
2. *If $A = B(X)$ where X or X' is complex uniformly convex, then for fixed $\varepsilon > 0$, the map Λ_ε, which sends an element $T \in B(X)$ to its ε-pseudospectrum in $\mathbb{C}$, is continuous with respect to the norm in the domain and the Hausdorff metric in the codomain.*
3. *If $A = B(X)$ is as in (2), then the map Λ, which sends a positive number ε and an element $T \in B(X)$ to the ε-pseudospectrum of T in $\mathbb{C}$, is also continuous with respect to the metric*
   $$d((\varepsilon_1, T_1), (\varepsilon_2, T_2)) = |\varepsilon_1 - \varepsilon_2| + \|T_1 - T_2\|$$
   in the domain and the Hausdorff metric in the co-domain.

Proof. It was proved by Globevnik (see Proposition 1 of [12]) that in any Banach algebra, the resolvent norm cannot be constant on an open subset of the unbounded component of the resolvent set. In particular, if $\rho(a)$ is a connected subset of $\mathbb{C}$, then the resolvent norm cannot be constant on any open subset of $\rho(a)$. This proves (1(a)) by Theorem 4.1.

It is proved in [12] and [25] that the resolvent norm cannot be constant on any open subset of $\rho(a)$, if a is a bounded linear operator on a Banach space X, such that X or X' is complex uniformly convex. Hence (1(b)) is true by Theorem 4.1. (2) and (3) follow from (1(b)) and Theorem 4.3. $\square$

REMARK 4.7. The fact that the resolvent set of an operator on a Hilbert space cannot be constant on an open set was perhaps first used to prove a convergence theorem for pseudospectra in Proposition 6.2 of [4]. Also see Proposition 4.2 in [5]. The following assertion about convergence of pseudospectra in the Haussdorf metric is proved in Theorem 4.4 of [15]. Suppose $\{T_n\} \subseteq B(H)$, for a Hilbert space H, and suppose $\|T_n - T\| \to 0$, for $T \in B(H)$. Let $\varepsilon > 0$. Then $d(\Lambda^*_\varepsilon(T_n), \Lambda^*_\varepsilon(T)) \to 0$ as $n \to \infty$.

REMARK 4.8. Let T be a bounded operator on a Banach space X. Theorem 4.6 shows that if T is a compact operator, then the map $\varepsilon \mapsto \Lambda_\varepsilon(T)$ is continuous because $\rho(T)$ is connected. In particular, if X is a finite dimensional space, then this happens for all $T \in B(X)$, consequently the map $(\varepsilon, T) \mapsto \Lambda_\varepsilon(T)$ is jointly continuous in T and ε. If X is a Hilbert space, or if either X or X' is a complex uniformly convex Banach space, then the map is jointly continuous. In particular, if $X = L^p$, for $1 \leqslant p \leqslant \infty$, then the map is continuous.

In [25], the author has constructed an example of a bounded operator on a Banach space X (where neither X nor X' is complex uniformly convex) such that the norm of its resolvent is constant on an open subset of the resolvent set. Hence in this case the pseudospectrum map has a jump discontinuity. Also, the map $a \mapsto \Lambda_\varepsilon(a)$ has a discontinuity. (See Example 4.9 below.)

In Proposition 2.7 of [8], in the case that T is a bounded operator on a Hilbert space, the authors have used the following inequality without any proof or justification:

$$d(\Lambda_{\varepsilon-\delta}(T), \Lambda_{\varepsilon+\delta}(T)) \leqslant 2\delta.$$

They have remarked that their "proof" also works if X is taken to be a Banach space. This, however, is not true, in view of the example in [25].

In general, for a fixed $\varepsilon > 0$, the map $a \mapsto \Lambda_\varepsilon(a)$ need not be continuous as seen in the example below.

EXAMPLE 4.9. Let X be the Banach space given in Shargorodsky's example (see Theorem 3.1 of [25]). $X = l^\infty(\mathbb{Z})$ with $\|x\|_* = |x(0)| + \sup_{k \neq 0} |x(k)|$. Let $M_i > 2$, for $i = 1, 2$ and $M_1 \neq M_2$. Consider the operators $A_i \in B(X)$ given by the following:

$$A_i(x)(k) = \begin{cases} \frac{1}{M_i} x(k+1), & k = 0, \\ x(k+1), & k \neq 0. \end{cases}$$

It is shown in [25] that the resolvent norm $\|(\lambda - A_i)^{-1}\| = M_i$ for $\lambda \in \mathbb{C}$ with $|\lambda| < \min\left\{\frac{1}{M_i}, \frac{1}{2} - \frac{1}{M_i}\right\}$.

We show that $\sigma(A_i) = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We first show that $\mathbb{T} \subseteq \sigma_p(A_i) \subseteq \sigma(A_i)$, where $\sigma_p(A)$ is the eigen or point spectrum of an operator A. It is easily seen that $1 \in \sigma_p(A)$ with eigenvector $x = (x(k))$, where

$$x(k) = \begin{cases} 1, & k \leqslant 0, \\ M_i, & k > 0. \end{cases}$$

In general, $\mu \in \mathbb{T}$ is an eigenvalue with eigenvector $x = (x(k))$, where

$$x(k) = \begin{cases} \mu^k, & k \leqslant 0, \\ M_i \mu^k, & k > 0. \end{cases}$$

Hence $\mathbb{T} \subseteq \sigma_p(A_i)$.

Next, we observe that $\|A_i\| = 1$ and $\|A_i^{-1}\| = M_i$. It can be shown that $\forall n \in \mathbb{N}$, $\|A_i^n\| = 1$ and $\|A_i^{-n}\| = M_i$. To show the latter, we observe that

$$A_i^{-n}(x)(k) = \begin{cases} M_i x(k-n), & k = 1, \cdots, n, \\ x(k-n), & \text{otherwise}. \end{cases}$$

This shows that $\|A_i^{-n}\| \leqslant M_i \; \forall n$. Choose unit vectors as follows:

$$x_n(k) = \begin{cases} 1, & k = 1-n \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|A_i^{-n} x_n\| = M_i$. Hence $\|A_i^{-n}\| = M_i \; \forall n$. Hence $r(A_i^{-1}) = 1 = r(A_i)$, by the spectral radius formula. This in turn implies that $\sigma(A_i) \subseteq \mathbb{T}$.

Now we compute the pseudospectrum of A_i. We have $\sigma(A_i) + D(0;\varepsilon) \subseteq \Lambda_\varepsilon(A_i)$ and $\Lambda_\varepsilon(A_i) \subseteq D(0;\|A_i\| + \varepsilon) = D(0;1+\varepsilon)$. For $\varepsilon \geqslant \frac{1}{M_i}$ and $|\lambda| < 1$, we get if $z = (A_i - \lambda)^{-1} e_0 = (\sum_{j=0}^{\infty} \lambda^j A_i^{-(j+1)}) e_0$, where e_0 is the vector given by

$$e_0(k) = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0, \end{cases}$$

that

$$z(k) = \begin{cases} 0, & k \leqslant 0, \\ M_i, & k = 1, \\ M_i \lambda^{k-1}, & k \geqslant 2. \end{cases}$$

Hence $\|z\|_* = M_i$, so that $\|(A_i - \lambda)^{-1}\| \geqslant M_i \geqslant \frac{1}{\varepsilon}$, hence $\lambda \in \Lambda_\varepsilon(A_i)$. This implies the following:

For $\varepsilon \geqslant \frac{1}{M_i}$, $\Lambda_\varepsilon(A_i) = D(0;1+\varepsilon)$, but for $\varepsilon < \frac{1}{M_i}$, $B\big(0; \min\{\frac{1}{M_i}, \frac{1}{2} - \frac{1}{M_i}\}\big) \cap \Lambda_\varepsilon(A_i) = \emptyset$.

Now suppose $M_2 > M_1$. Then $\|A_1 - A_2\| = \frac{1}{M_1} - \frac{1}{M_2}$. Let $\varepsilon = \frac{1}{M_2}$.

Then $\Lambda_\varepsilon(A_2) = D(0;1+\varepsilon)$, but $B\big(0; \min\{\frac{1}{M_1}, \frac{1}{2} - \frac{1}{M_1}\}\big) \cap \Lambda_\varepsilon(A_1) = \emptyset$. Hence even if $\|A_1 - A_2\|$ is arbitrarily small, $d(\Lambda_\varepsilon(A_1), \Lambda_\varepsilon(A_2)) \geqslant \min\{\frac{1}{M_1}, \frac{1}{2} - \frac{1}{M_1}\}$. Hence for fixed $\varepsilon > 0$, the map $A \mapsto \Lambda_\varepsilon(A)$ is not continuous in general.

5. Topological properties

In this section we prove some topological properties of the ε-pseudospectrum of an element of a Banach algebra. It is shown that just as in the case of the ε-condition spectrum (see [17]), the ε-pseudospectrum has no isolated points, and that it has a finite number of components.

THEOREM 5.1. *Let A be a Banach algebra, $a \in A$ and $\varepsilon > 0$. Then the ε-pseudospectrum $\Lambda_\varepsilon(a)$ of a has no isolated points.*

Proof. Suppose $\Lambda_\varepsilon(a)$ has an isolated point λ_0. Then $\exists r > 0$ such that

$$\forall \lambda \text{ with } 0 < |\lambda - \lambda_0| < r, \; \|(\lambda - a)^{-1}\| < \frac{1}{\varepsilon}.$$

Case 1: Suppose $\lambda_0 \in \Lambda_\varepsilon(a) \setminus \sigma(a)$. By the Hahn-Banach theorem $\exists \phi \in A'$ such that

$$\phi((\lambda_0 - a)^{-1}) = \|(\lambda_0 - a)^{-1}\| \text{ and } \|\phi\| = 1.$$

Define $f:\mathbb{C}\setminus\sigma(a)\longrightarrow\mathbb{C}$ by $f(z)=\phi((z-a)^{-1})$. Then f is analytic in $B(\lambda_0,r)$. But $\forall\lambda\in B(\lambda_0,r)$ with $\lambda\neq\lambda_0, |f(\lambda)|\leqslant\|(\lambda-a)^{-1}\|<\frac{1}{\varepsilon}$ while

$$f(\lambda_0)=\phi((\lambda_0-a)^{-1})=\|(\lambda_0-a)^{-1}\|\geqslant\frac{1}{\varepsilon}.$$

This is a contradiction to the maximum modulus principle.

Case 2: Suppose $\lambda_0\in\sigma(a)$. Let $\lambda\to\lambda_0$. Then we would have $\|(\lambda-a)^{-1}\|\to\infty$. But for $\lambda\in B(\lambda_0;r), \|(\lambda-a)^{-1}\|<\frac{1}{\varepsilon}$, which gives a contradiction. □

THEOREM 5.2. *Let A be a Banach algebra, $a\in A$ and $\varepsilon>0$. Then the ε-pseudospectrum $\Lambda_\varepsilon(a)$ of a has a finite number of components and each component of $\Lambda_\varepsilon(a)$ contains an element of $\sigma(a)$.*

Proof. For each $\lambda\in\sigma(a)$, $B(\lambda;\varepsilon)\subset\Lambda_\varepsilon(a)$. Also, $\{B(\lambda;\varepsilon):\lambda\in\sigma(a)\}$ is an open cover for $\sigma(a)$. Since $\sigma(a)$ is compact $\exists\lambda_1,\cdots,\lambda_n\in\sigma(a)$ such that $\sigma(a)\subset\bigcup_{i=1}^{n}B(\lambda_i;\varepsilon)$. Since each $B(\lambda_i;\varepsilon)$ is connected and is a subset of $\Lambda_\varepsilon(a)$, it must be contained in some component C_i of $\Lambda_\varepsilon(a)$. Thus we get closed components $C_1,\cdots,C_n$ of $\Lambda_\varepsilon(a)$ such that $\sigma(a)\subset\bigcup_{i=1}^{n}C_i\subset\Lambda_\varepsilon(a)$. We will prove that $\bigcup_{i=1}^{n}C_i=\Lambda_\varepsilon(a)$.

Suppose $z_0\in\Lambda_\varepsilon(a)\setminus\bigcup_{i=1}^{n}C_i$. Then $\|(z_0-a)^{-1}\|\geqslant\frac{1}{\varepsilon}$. Let $r>\|a\|+\varepsilon$. Then $\sigma(a)\subset\Lambda_\varepsilon(a)\subset B(0;r)$. Let $S:=B(0;r)\setminus\bigcup_{i=1}^{n}C_i$. Then S is an open set and $z_0\in S$. Let S_0 be the component of S containing z_0. Then S_0 is open. Also, since $\sigma(A)\subset\bigcup_{i=1}^{n}C_i$, $(\bigcup_{i=1}^{n}C_i)^C\subset\rho(a)$. Hence $S_0\subset\rho(a)$. Define $f:S_0\subset\rho(a)\longrightarrow\mathbb{R}$ as $f(z)=\|(z-a)^{-1}\|$. By the Hahn-Banach theorem,

$$\exists\phi\in A' \text{ such that } \phi((z_0-a)^{-1})=\|(z_0-a)^{-1}\|,\quad\|\phi\|=1.$$

Define $g:S\longrightarrow\mathbb{C}$ by

$$g(z)=\phi((z-a)^{-1})\quad\forall z\in S.$$

Then g is an analytic function on S_0 and

$$|g(z)|=|\phi((z-a)^{-1})|\leqslant\|(z-a)^{-1}\|=f(z)\quad\forall z\in S_0\subset\rho(a).$$

Since $\Lambda_\varepsilon(a)\subset B(0;r)$ which is open, $\delta B(0;r)\subset\Lambda_\varepsilon(a)^C$. Hence

$$f(z)=\|(z-a)^{-1}\|<\frac{1}{\varepsilon}\quad\forall z\in\delta B(0;r).$$

We show that for $z\in\bigcup_{i=1}^{n}\delta C_i, f(z)=\frac{1}{\varepsilon}$. For if $f(z)\gneq\frac{1}{\varepsilon}$, $\exists$ a neighbourhood U of z such that

$$f(\lambda)\gneq\frac{1}{\varepsilon}\quad\forall\lambda\in U.$$

But U intersects points outside the pseudospectrum where $f(z) = \|(z-a)^{-1}\| < \frac{1}{\varepsilon}$. This gives a contradiction. Hence

$$f(z) = \frac{1}{\varepsilon} \quad \forall z \in \bigcup_{i=1}^{n} \delta C_i.$$

Now $\delta S_0 \subset \delta B(0;r) \cup \bigcup_{i=1}^{n} \delta C_i$. Hence if $z \in \delta S_0, |g(z)| \leqslant f(z) \leqslant \frac{1}{\varepsilon}$. But z_0 is in the interior of S_0 and $|g(z_0)| = g(z_0) = \phi((z_0-a)^{-1}) = \|(z_0-a)^{-1}\| \geqslant \frac{1}{\varepsilon}$. By the maximum modulus principle, g must be constant on S_0. Hence

$$\forall z \in S_0, f(z) \geqslant |g(z)| = g(z_0) \geqslant \frac{1}{\varepsilon}.$$

Hence $S_0 \subset \Lambda_\varepsilon(a)$. By the continuity of g, we get $|g(z)| \geqslant \frac{1}{\varepsilon}\ \forall z \in \bar{S_0}$. If $\delta S_0 \cap \delta B(0;r) \neq \emptyset$ we get a contradiction since $|g(z)| \leqslant f(z) < \frac{1}{\varepsilon} \forall z \in \delta B(0;r)$. If $\delta S_0 \cap \delta C_i \neq \emptyset$ for some i, then $S_0 \cup C_i$ is a connected subset of $\Lambda_\varepsilon(a)$. But C_i is a component of $\Lambda_\varepsilon(a) \implies S_0 \subset C_i$, a contradiction since $S_0 \subset B(0;r) \setminus \bigcup_{i=1}^{n} C_i$. Hence $\delta S_0 = \emptyset$, a contradiction. Hence $\Lambda_\varepsilon(a) = \bigcup_{i=1}^{n} C_i$, and each C_i contains a point from the spectrum. □

Theorem 5.2 helps determine certain properties of a matrix when its ε-pseudospectrum is known.

COROLLARY 5.3. *Let $M \in \mathbb{C}^{n\times n}$ and $\varepsilon > 0$.*

1. *If $\Lambda_\varepsilon(M)$ has n components, then M is diagonalizable.*
2. *If each of these components is a disc of radius ε and $\|\cdot\| = \|\cdot\|_2$ then M is normal.*
3. *If $\|\cdot\| = \|\cdot\|_2$, then $\Lambda_\varepsilon(M) = D(\mu;\varepsilon)$ iff $M = \mu I$. (Compare with Corollary 3.17).*
4. *If $\|\cdot\| = \|\cdot\|_2$, then $\Lambda_\varepsilon(M) = D(0;\varepsilon) \cup D(1;\varepsilon)$ iff M is a non-trivial orthogonal projection.*

Proof. By Theorem 5.2, each component of $\Lambda_\varepsilon(M)$ contains an eigenvalue. Hence M has n distinct eigenvalues, and is diagonalizable. This proves (1). A proof is also given in Theorem 2ε of [9]. If, in addition, each component is a disc of radius ε then by the first inclusion in (7) of Theorem 3.4, each component must be equal to $D(\lambda;\varepsilon)$ for λ, an eigenvalue of M. Hence $\Lambda_\varepsilon(M) = \bigcup_{\lambda\in\sigma(M)} D(\lambda;\varepsilon)$. Since each disc is maximally connected in $\Lambda_\varepsilon(M)$, each disc clearly contains a boundary point of $\Lambda_\varepsilon(M)$. By Theorem 2.5 in [7], M is normal. This proves (2). (3) and (4) follow from Corollary 2.6 in [7]. □

6. Determining elements through pseudospectra

In this section, the following question is addressed: If A is a Banach algebra, $\varepsilon > 0$ and $a, b \in A$ satisfy

$$\Lambda_\varepsilon(ax) = \Lambda_\varepsilon(bx) \quad \forall x \in A. \tag{18}$$

Then is $a = b$? This is shown to be true in some cases.

THEOREM 6.1. *Let A be a Banach algebra and $a, b \in A$. Suppose for some $\varepsilon_0 > 0$,*

$$\Lambda_{\varepsilon_0}(ax) = \Lambda_{\varepsilon_0}(bx) \quad \forall x \in A.$$

Then $\forall \varepsilon > 0$,

$$\Lambda_\varepsilon(ax) = \Lambda_\varepsilon(bx) \quad \forall x \in A.$$

Hence

$$\sigma(ax) = \sigma(bx) \quad \forall x \in A.$$

Proof. By the hypothesis,

$$\Lambda_{\varepsilon_0}(tax) = \Lambda_{\varepsilon_0}(tbx) \quad \forall t > 0,\ \forall x \in A.$$

By (4) of Theorem 2.3,

$$t\Lambda_{\frac{\varepsilon_0}{t}}(ax) = t\Lambda_{\frac{\varepsilon_0}{t}}(bx) \quad \forall x \in A,$$

i.e.

$$\Lambda_{\frac{\varepsilon_0}{t}}(ax) = \Lambda_{\frac{\varepsilon_0}{t}}(bx) \quad \forall x \in A.$$

The proof follows by choosing $t = \frac{\varepsilon_0}{\varepsilon}$ and considering (1) of Theorem 2.3. □

COROLLARY 6.2. *Let A be a Banach algebra and $a, b \in A$. Let $\varepsilon_0 > 0$. Suppose*

$$\Lambda_{\varepsilon_0}(ax) = \Lambda_{\varepsilon_0}(bx) \quad \forall x \in A.$$

Then $a = b$ in the following cases:

1. *$A = B(X)$ for a Banach space X.*

2. *A is semisimple and a is a unit regular element (i.e. the product of an idempotent and an invertible element).*

3. *A is a commutative semisimple Banach algebra.*

4. *A is a C^* algebra.*

Proof. (1) follows from Theorem 6.1 and a comment in Section 2 of [6]. (2), (3) and (4) follow from Theorem 6.1 and Theorems 2.4, 2.5 and 2.6 in [6]. □

THEOREM 6.3. *Let A be a Banach algebra, $a,b \in A$ and $\varepsilon_0 > 0$. Suppose*

$$\Lambda_{\varepsilon_0}(ax) = \Lambda_{\varepsilon_0}(bx) \quad \forall x \in A.$$

Then $a = b$ in the following cases:

1. *a is invertible.*
2. *a is Hermitian idempotent.*
3. *$a = hu$, where h is Hermitian idempotent and u is invertible.*

We note here that in all cases, A is not assumed to be semi-simple.

Proof.

1. By Theorem 6.1, $\forall \varepsilon > 0$,

   $$\Lambda_\varepsilon(ax) = \Lambda_\varepsilon(bx) \quad \forall x \in A.$$

   Hence
   $$\Lambda_\varepsilon(ba^{-1}) = \Lambda_\varepsilon(aa^{-1}) = \Lambda_\varepsilon(1) = D(1;\varepsilon) \quad \forall \varepsilon > 0.$$

   By Corollary 3.17, $ba^{-1} = 1$.

2. Suppose a is Hermitian idempotent. Then by Corollary 3.18 it is easy to see that b is also Hermitian idempotent. Now

   $$\Lambda_\varepsilon(b(1-a)) = \Lambda_\varepsilon(a(1-a)) = \Lambda_\varepsilon(0) = D(0;\varepsilon) \quad \forall \varepsilon > 0.$$

   Hence by Corollary 3.17, $b(1-a) = 0$, i.e., $b = ba$. Similarly, considering $\Lambda_\varepsilon(a(1-b))$, we obtain $a(1-b) = 0$, i.e., $a = ab$. Let $h = a-b$. Then h is Hermitian and $h^2 = (a-b)^2 = a^2+b^2-ab-ba = a+b-a-b = 0$. Thus $\sigma(h) = \{0\}$. Hence by Sinclair's result ([28]), $\|h\| = r(h) = 0$. Thus $a = b$.

3. Suppose $a = hu$. Then

   $$\Lambda_\varepsilon(hx) = \Lambda_\varepsilon(au^{-1}x) = \Lambda_\varepsilon(bu^{-1}x) \quad \forall x \in A.$$

   Hence by (2), $h = bu^{-1}$, thus, $b = hu = a$. □

REMARK 6.4. The hypothesis that a is invertible or Hermitian idempotent (or the product of a Hermitian idempotent and an invertible element) in Theorem 6.3 cannot be removed as seen in the following example.

EXAMPLE 6.5. Let A be the Banach algebra defined in Example 2.5. We observe that A is commutative. Let $a = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$. Note that a and b are not invertible. Any $x \in A$ is of the form $x = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix}$, so that $ax = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$,

$bx = \begin{bmatrix} 0 & -\alpha \\ 0 & 0 \end{bmatrix}$. Then for $\lambda \neq 0$, $\|(\lambda - ax)^{-1}\| = \frac{1}{|\lambda|} + |\frac{\alpha}{\lambda^2}| = \|(\lambda - bx)^{-1}\|$, thus for any $\varepsilon > 0$, $\Lambda_\varepsilon(ax) = \Lambda_\varepsilon(bx)\ \forall x \in A$, but $a \neq b$.

Note that A is not semi-simple, for $\sigma(ax) = \sigma(bx) = \{0\}\ \forall x \in A$, but $a \neq 0 \neq b$. Hence, (3) of Corollary 6.2 does not apply. Also, a is neither invertible nor an idempotent, and further, cannot be the product of a Hermitian idempotent and an invertible element because the only idempotent elements of A are the zero and identity matrices. Hence Theorem 6.3 does not apply.

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