Operators and Matrices
Volume 11, Number 1 (2017), 289–299 doi:10.7153/oam-11-19

MOORE–PENROSE INVERSE OF CONDITIONAL TYPE OPERATORS

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(*Communicated by R. Curto*)

Abstract. We prove some basic results on some Moore-Penrose inverse of conditional type operators on $L^2(\Sigma)$. For instance, we show, among other results, that a weighted conditional operator $T = M_w E M_u$ is centered if and only if $T^\dagger$, the Moore-Penrose inverse of T, is centered. In addition, we establish lower and upper bounds for the numerical range of T and $T^\dagger$.

1. Introduction and preliminaries

Let (X,Σ,μ) be a complete σ-finite measure space. For any σ-finite subalgebra $\mathscr{A} \subseteq \Sigma$ the Hilbert space $L^2(X,\mathscr{A},\mu_{|\mathscr{A}})$ is abbreviated to $L^2(\mathscr{A})$ where $\mu_{|\mathscr{A}}$ is the restriction of μ to $\mathscr{A}$. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$ and $L^0_+(\Sigma) = \{f \in L^0(\Sigma) : f \geqslant 0\}$. The support of a measurable function f is defined by $\sigma(f) = \{x \in X : f(x) \neq 0\}$. All sets and functions statements are to be interpreted as being valid almost everywhere with respect to μ. For each non-negative $f \in L^0(\Sigma)$ or $f \in L^2(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique $\mathscr{A}$-measurable function $E^{\mathscr{A}}(f)$ such that

$$\int_A f d\mu = \int_A E^{\mathscr{A}}(f) d\mu,$$

where A is any $\mathscr{A}$-measurable set for which $\int_A f d\mu$ exists. Now associated with every complete σ-finite subalgebra $\mathscr{A} \subseteq \Sigma$, the mapping $E^{\mathscr{A}} : L^2(\Sigma) \to L^2(\mathscr{A})$ uniquely defined by the assignment $f \mapsto E^{\mathscr{A}}(f)$, is called the conditional expectation operator with respect to $\mathscr{A}$. Put $E = E^{\mathscr{A}}$. The mapping E is a linear orthogonal projection. Note that $\mathscr{D}(E)$, the domain of E, contains $L^2(\Sigma) \cup \{f \in L^0(\Sigma) : f \geqslant 0\}$. For more details on the properties of E see [10, 14, 16].

Given a complex separable Hilbert space H, let $B(H)$ denotes the linear space of all bounded linear operators on H. $\mathscr{N}(T)$ and $\mathscr{R}(T)$ denote the null-space and range of an operator T, respectively. Recall that for $T \in B(H)$ there is a unique factorization $T = U|T|$, where $\mathscr{N}(T) = \mathscr{N}(U) = \mathscr{N}(|T|)$, U is a partial isometry; i.e. $UU^*U = U$ and $|T| = (T^*T)^{1/2}$ is a positive operator. This factorization is called the polar decomposition of T. It is a classical fact that the polar decomposition of T^* is $U^*|T^*|$.

Mathematics subject classification (2010): 47B20, 47B38.
Keywords and phrases: Moore-Penrose inverse, Aluthge transformation, conditional expectation, polar decomposition.

Paper OaM-11-19

Associated with $T \in B(H)$ there is a useful related operator $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$, called the Aluthge transform of T. For important properties of Aluthge transform see [8, 12].

Let $CR(H)$ be the set of all bounded linear operators on H with closed range. For $T \in CR(H)$, the Moore-Penrose inverse of T, denoted by $T^\dagger$, is the unique operator $T^\dagger \in CR(H)$ that satisfies following:

$$TT^\dagger T = T,\ \ T^\dagger TT^\dagger = T^\dagger,\ \ (TT^\dagger)^* = TT^\dagger,\ \ (T^\dagger T)^* = T^\dagger T. \tag{1.1}$$

We recall that $T^\dagger$ exists if and only if $T \in CR(H)$. The Moore-Penrose inverse is designed as a measure for the invertibility of an operator. If $T = U|T|$ is invertible, then $T^{-1} = T^\dagger$, U is unitary and so $|T|$ is invertible. For other important properties of $T^\dagger$ see [1, 3].

A combination of conditional expectation, multiplication and composition operators appears more often in the service of the study of other operators, such as Frobenius-Perron operators [2], integral operators and operators generated by random measures [9] and probabilistic conditional operators [15].

In this paper, we consider the weighted conditional operator M_wEM_u and the weighted conditional composition operator $M_wEM_uC_\varphi$ on $L^2(\Sigma)$. We prove some basic results on some Moore-Penrose inverse of these type operators. For instance, we obtain a lower and upper bound for the numerical range of T and $T^\dagger$, respectively.

2. Weighted conditional operators

LEMMA 2.1. *Let $\omega \in L^0(\Sigma)$, $0 \leqslant v \in L^0(\mathscr{A})$ and let $A := M_{v\bar{\omega}}EM_\omega \in B(L^2(\Sigma))$. Then for each $p \in (0,\infty)$ and $f \in L^2(\Sigma)$, $A^p(f) = v^p\bar{\omega}E(|\omega|^2)^{p-1}E(\omega f)$.*

Proof. First note that, because v is $\mathscr{A}$-measurable then the positive multiplication operator M_v commutes with the positive operator $M_{\bar{\omega}}EM_\omega$, and so A is positive. Suppose $f \in L^2(\Sigma)$, then by induction we obtain

$$A^{\frac{1}{n}}(f) = v^{\frac{1}{n}}\bar{\omega}E(|\omega|^2)^{\frac{1}{n}-1}E(\omega f), \quad n \in \mathbb{N}.$$

Now the reiteration of powers of operator $A^{\frac{1}{n}}$, yields

$$A^{\frac{m}{n}}(f) = v^{\frac{m}{n}}\bar{\omega}E(|\omega|^2)^{\frac{m}{n}-1}E(\omega f), \quad m,n \in \mathbb{N}.$$

Finally, by using of the functional calculus the desired formula is proved. □

For $f \in L^2(\Sigma)$, it is easy to see that $\|M_wEM_uf\|_2 = \|EM_\upsilon f\|_2$ where $\upsilon := u(E(|w|^2))^{\frac{1}{2}}$. But we know that a multiplication operator has closed range if and only if the inducing function is bounded away from zero on its support. As a result it can easily be checked that for some $\delta > 0$ such that $E(\upsilon) \geqslant \delta$ on $\sigma(\upsilon)$, T has closed range (see also [11, Theorem 2.8(ii)]). Some basic results concerning the conditional type operators are given by Herron [10], Estaremi et al. [4] and the first author in [11]. Here we recall some results of [4] that state our results is valid for M_wEM_u.

LEMMA 2.2. *Let $T = M_wEM_u$ be a weighted conditional operator on $L^2(\Sigma)$. Then the following assertions hold.*

(a) $T \in B(L^2(\Sigma))$ if and only if $E(|w|^2)E(|u|^2) \in L^\infty(\mathscr{A})$, and in this case $\|T\| = \|E(|w|^2)E(|u|^2)\|_\infty^{1/2}$.

(b) Let $T \in B(L^2(\Sigma))$, $0 \leqslant u \in L^0(\Sigma)$ and $\upsilon = u(E(|w|^2))^{\frac{1}{2}}$. If $E(\upsilon) \geqslant \delta$ on $\sigma(\upsilon)$, then T has closed range.

(c) Let $U|T|$ be the polar decomposition of T. Then

$$|T|(f) = \left(\frac{E(|w|^2)}{E(|u|^2)}\right)^{\frac{1}{2}} \chi_S \bar{u} E(uf);$$

$$U(f) = \left(\frac{\chi_{S\cap G}}{E(|w|^2)E(|u|^2)}\right)^{\frac{1}{2}} wE(uf),$$

where where $S = \sigma(E(u))$, $G = \sigma(E(w))$ and $f \in L^2(\Sigma)$.

(d) The Aluthge transformation of T is

$$\widetilde{T}(f) = \frac{\chi_S E(uw)}{E(|u|^2)} \bar{u} E(uf), \quad f \in L^2(\Sigma).$$

From now on, we assume that $u, w \in L^0_+(\Sigma)$, $T = M_wEM_u \in B(L^2(\Sigma))$ and $K := S \cap G$, where $G = \sigma(E(w))$ and $S = \sigma(E(u))$.

PROPOSITION 2.3. *$T \in CR(L^2(\Sigma))$. Then $T^\dagger = M_{\frac{\chi_K}{E(u^2)E(w^2)}} T^*$.*

Proof. It is easy to check that T satisfy all equations in (1.1). □

PROPOSITION 2.4. *Let $T \in CR(L^2(\Sigma))$ and let $U_\dagger|T^\dagger|$ be the polar decomposition of $T^\dagger$. Then*

$$|T^\dagger|(f) = \left(\frac{\chi_K}{E(u^2)(E(w^2))^3}\right)^{\frac{1}{2}} wE(wf);$$

$$U_\dagger(f) = \left(\frac{\chi_K}{E(u^2)E(w^2)}\right)^{\frac{1}{2}} uE(wf).$$

Proof. Let $f \in L^2(\Sigma)$. Then $(T^\dagger)^*(T^\dagger)(f) = (E(u^2)(E(w^2))^2)^{-1}\chi_K wE(wf)$. Now $|T^\dagger|$ follows from Lemma 2.1. Moreover, it is easy to check that $U_\dagger|T^\dagger| = T^\dagger$, $U_\dagger U_\dagger^* U_\dagger = U_\dagger$ and $\mathscr{N}(U_\dagger) = \mathscr{N}(T^*) = \mathscr{N}(T^\dagger)$. This completes the proof. □

We now turn to the computation of $(\widetilde{T})^\dagger$ and $\widetilde{T^\dagger}$. By combining the previous results we obtain the following proposition.

PROPOSITION 2.5. *Let $T, \widetilde{T} \in CR(L^2(\Sigma))$. Then*

(i) $(\widetilde{T})^\dagger = M_{\frac{u\chi_{\sigma(E(uw))\cap S}}{E(u^2)E(uw)}} EM_u$.

(ii) $\widetilde{T^\dagger} = M_{\frac{\chi_K wE(uw)}{E(u^2)(E(w^2))^2}} EM_w$

REMARK 2.6. If $w \neq u$, then $(\widetilde{T})^{\dagger} \neq \widetilde{T^{\dagger}}$. Moreover, by Lemma 2.2(b), $\widetilde{T} \in CR(L^2(\Sigma))$ whenever $E(u)\frac{E(uw)}{\sqrt{E(u^2)}} \geqslant \delta$ for some $\delta > 0$ on S.

Now, we determine a lower and upper estimates for the numerical range of $T^{\dagger}$. Let B be largest $\mathscr{A}$-measurable set contained in K with $\mu(B) < \infty$. Then by Proposition 2.3 and definition of $\omega(T^{\dagger})$ we have

$$\omega(T^{\dagger}) \geqslant \left\langle T^{\dagger}\frac{\chi_B}{\sqrt{\mu(B)}}, \frac{\chi_B}{\sqrt{\mu(B)}}\right\rangle = \frac{1}{\mu(B)}\int_B \frac{\chi_{S\cap G}}{E(u^2)E(w^2)} uE(w)d\mu$$
$$\geqslant \frac{1}{\mu(B)}\int_B \frac{E(u)E(w)}{E(u^2)E(w^2)} d\mu.$$

On the other hand, by the conditional Hölder inequality we have

$$|E(u\bar{f}E(wf))| \leqslant (E(u^2))^{\frac{1}{2}}(E(w^2))^{\frac{1}{2}}E(|f|^2).$$

Put $A = \{f \in L^2(\Sigma) \cap L^{\infty}(\Sigma) : \|f\|_2 \leqslant 1\}$. Then

$$\omega(T^{\dagger}) = \sup_{\|f\|_2 \leqslant 1} |\langle T^{\dagger}f, f\rangle| = \sup_{f\in A} |\langle T^{\dagger}f, f\rangle| \leqslant \int_K \frac{d\mu}{\sqrt{E(u^2)E(w^2)}}.$$

By a similar argument we obtain $\omega(T) \leqslant \|T\|$ and $\int_B E(u)E(w)d\mu \leqslant \mu(B)\omega(T)$, for each $B \in \mathscr{A}$ with $0 < \mu(B) < \infty$. So

$$\|E(u)E(w)\|_{\infty} = \sup_{0<\mu(B)<\infty} \frac{1}{\mu(B)}\int_B E(u)E(w)d\mu \leqslant \omega(T).$$

Consequently, we have the following proposition.

PROPOSITION 2.7. *Let* $T, \widetilde{T} \in CR(L^2(\Sigma))$. *Then*

$$\|E(u)E(w)\|_{\infty} \leqslant \omega(T) \leqslant \|\sqrt{E(u^2)E(w^2)}\|_{\infty};$$

$$\frac{1}{\mu(B)}\int_B \frac{E(u)E(w)}{E(u^2)E(w^2)} d\mu \leqslant \omega(T^{\dagger}) \leqslant \int_K \frac{d\mu}{\sqrt{E(u^2)E(w^2)}},$$

where B *is the largest* $\mathscr{A}$*-measurable set contained in* K *with* $\mu(B) < \infty$.

EXAMPLE 2.8. Let $X = [-\frac{1}{2}, \frac{1}{2}]$, $d\mu = dx$, Σ be the Lebesgue sets, and let $\mathscr{A} \subseteq \Sigma$ be the σ-algebra generated by the symmetric sets about the origin. Then for each $f \in \mathscr{D}(E)$, $E(f)(x) = \frac{f(x)+f(-x)}{2}$. Put $u(x) = 2x+5$, $w(x) = \cos x$ and $T = M_wEM_u$. Then $K = B = X$, $E(u) = 5$, $E(w) = \cos x$, $E(u^2) = 4x^2+25$ and $E(w^2) = \cos^2(x)$. Note that

$$u\sqrt{E(w^2)} = (2x+5)(\cos x) \geqslant 3.9;$$
$$E(u)\frac{E(uw)}{\sqrt{E(u^2)}} = \frac{125\cos x}{\sqrt{4x^2+25}} \geqslant \frac{125\cos\frac{1}{2}}{\sqrt{26}} \geqslant 24.5\,.$$

So by Lemma 2.2, $T, \widetilde{T} \in CR(L^2(\Sigma))$. Also, it is easy to check that

$$\int_{[-\frac{1}{2},\frac{1}{2}]} \frac{E(u)E(w)}{E(u^2)E(w^2)} d\mu = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{5\cos x dx}{(4x^2+25)(\cos^2(x))} = 0.2060;$$

$$\int_{[-\frac{1}{2},\frac{1}{2}]} \frac{d\mu}{\sqrt{E(u^2)E(w^2)}} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{(x^2+4)(x^2+9)}} = 0.2074;$$

$$\|T\| = \|\sqrt{(4x^2+25)(\cos^2(x))}\|_\infty = 5;$$

$$\|T^\dagger\| = \|\frac{1}{\sqrt{E(u^2)E(w^2)}}\|_\infty = 0.2235;$$

$$\|\widetilde{T}\| = \|E(uw)\|_\infty = 5.$$

Thus, $\|\widetilde{T}\| = \|T\| = \omega(T)$ and by Proposition 2.7 we have

$$0.2060 \leqslant \omega(T^\dagger) \leqslant 0.2074 \leqslant \|T^\dagger\| \leqslant \frac{1}{2}\omega(T).$$

PROPOSITION 2.9. *Let $T \in CR(L^2(\Sigma))$. If $T^\dagger$ is p-hyponormal, then $E(u^2)(E(w))^2 \geqslant (E(u))^2 E(w^2)$ on K.*

Proof. Let $f \in L^2(\Sigma)$. Then by Lemma 2.1, we have

$$((T^\dagger)^* T^\dagger)^p = \frac{\chi_K}{(E(u^2))^p (E(w^2))^{2p}} w(E(w^2))^{p-1} E(wf);$$

$$(T^\dagger (T^\dagger)^*)^p = \frac{\chi_K}{(E(u^2))^{2p} (E(w^2))^p} u(E(u^2))^{p-1} E(uf).$$

Thus $T^\dagger$ is p-hyponormal if and only if

$$M_{\frac{\chi_K}{(E(u^2))^p(E(w^2))^p}} (M_{\frac{\chi_K}{E(w^2)}} wEM_w - M_{\frac{\chi_K}{E(u^2)}} uEM_u) \geqslant 0.$$

Put $P := M_{\frac{\chi_K}{E(w^2)}} wEM_w - M_{\frac{\chi_K}{E(u^2)}} uEM_u$. Since $M_{\frac{\chi_K}{(E(u^2))^p(E(w^2))^p}}$ is positive and commute with P, it follows that $T^\dagger$ is p-hyponormal if and only if $P \geqslant 0$. But this implies that

$$\langle Pf, f\rangle = \int_K \left\{ \frac{wE(wf)}{E(w^2)} - \frac{uE(uf)}{E(u^2)} \right\} \bar{f} d\mu \geqslant 0.$$

Choose $0 < f_0 \in L^2(\mathscr{A})$. By replacing f to f_0, we obtain

$$\int_K \left\{ \frac{(E(w))^2}{E(w^2)} - \frac{(E(u))^2}{E(u^2)} \right\} f_0^2 d\mu \geqslant 0,$$

and so $E(u^2)(E(w))^2 \geqslant (E(u))^2 E(w^2)$ on K. □

In [6], Estaremi determined when weighted conditional operators were A-class, $*$-A-class and quasi-$*$-A-classes. Now, we discuss measure theoretic characterizations for $T^\dagger$ in some A-classes of operators on $L^2(\Sigma)$. An operator $T \in B(H)$ is an A-class operator if $|T^2| \geqslant |T|^2$, quasi-A-class if $T^*|T^2|T \geqslant T^*|T|^2T$ and quasi-$*$-A-class if $T^*|T^2|T \geqslant T^*|T^*|^2T$.

PROPOSITION 2.10. *Let $T = M_wEM_u \in CR(L^2(\Sigma))$. Then the followings are equivalent.*

(i) $T^\dagger$ is A-class.

(ii) $T^\dagger$ is quasi-A-class.

(iii) $T^\dagger$ is quasi-$*$-A-class.

(iv) $(E(uw))^2 \geqslant (E(u^2))(E(w^2))$ on K.

Proof. (i) $\iff$ (iv) Let $f \in L^2(\Sigma)$. Then we obtain

$$\begin{aligned}\langle(|(T^\dagger)^2| - |T^\dagger|^2)f,\, f\rangle &= \int_X \left\{ \frac{\chi_K E(uw) w\bar{f} E(wf)}{(E(u^2))^{\frac{3}{2}}(E(w^2))^{\frac{5}{2}}} - \frac{\chi_K w\bar{f}E(wf)}{E(u^2)(E(w^2))^2} \right\}) d\mu \\ &= \int_K \left\{ \frac{E(uw)}{(E(u^2))^{\frac{3}{2}}(E(w^2))^{\frac{5}{2}}} - \frac{1}{E(u^2)(E(w^2))^2} \right\} |E(wf)|^2 d\mu.\end{aligned}$$

This implies that if $(E(uw))^2 \geqslant (E(u^2))(E(w^2))$ on K, then $|(T^\dagger)^2| - |T^\dagger|^2 \geqslant 0$.

Conversely, if $T^\dagger$ is an A-class operator, then $\langle(|(T^\dagger)^2| - |T^\dagger|^2)f,\, f\rangle \geqslant 0$ for all $f \in L^2(\Sigma)$. Let $B \in \mathscr{A}$, with $B \subseteq K$ and $0 < \mu(B) < \infty$. By replacing f to χ_B, we get that

$$\int_B \left\{ \frac{E(uw)}{(E(u^2))^{\frac{3}{2}}(E(w^2))^{\frac{5}{2}}} - \frac{1}{E(u^2)(E(w^2))^2} \right\} (E(w))^2 d\mu \geqslant 0.$$

Since $B \in \mathscr{A}$ is arbitrary, then $(E(uw))^2 \geqslant (E(u^2))(E(w^2))$ on K. The proofs of the other implications are similar. ☐

In [13] Morrel and Muhly introduced the concept of a centered operator. An operator $T = U|T|$ on a Hilbert space H is said to be centered if the doubly infinite sequence $\{T^nT^{*n}, T^{*m}T^m : n,m \geqslant 0\}$ consists of mutually commuting operators. For $T \in B(H)$ and $n \in \mathbb{N}$, let $U_n|T^n|$ be the polar decomposition of T^n. It is shown in [13, Theorem I] that T is centered if and only if $U_n = U^n$. In the following theorem we give a necessary and sufficient condition for the Moore-Penrose of M_wEM_u to be centered.

PROPOSITION 2.11. *Let $T \in CR(L^2(\Sigma))$. Then the followings are equivalent.*

(i) T is centered.

(ii) $T^\dagger$ is centered.

(iii) $(E(uw))^2 = E(u^2)E(w^2)$ on $\sigma(E(uw))$.

Proof. Put $Q=\sigma(E(uw))$ and let $n\in\mathbb{N}$, $f\in L^2(\Sigma)$. Then by induction we obtain

$$(T^\dagger)^n(f)=\frac{\chi_K(E(uw))^{n-1}}{(E(u^2))^n(E(w^2))^n}uE(wf);$$

$$U_n(f)=\frac{\chi_Q E(uw)^{n-1}uE(wf)}{(E(u^2))^{\frac{1}{2}}(E(w^2))^{\frac{1}{2}}(E(uw))^{n-1}};$$

$$U^n(f)=\frac{\chi_K E(uw)^{n-1}uE(wf)}{(E(u^2))^{\frac{n}{2}}(E(w^2))^{\frac{n}{2}}}.$$

If $(E(uw))^2=E(u^2)E(w^2)$, then a calculation shows that $U_n=U^n$, and so $T^\dagger$ is centered. Conversely, suppose that $U_n=U^n$. Then

$$\left\{\frac{E(uw)^{n-1}}{(E(u^2))^{\frac{1}{2}}(E(w^2))^{\frac{1}{2}}(E(uw))^{n-1}}-\frac{E(uw)^{n-1}}{(E(u^2))^{\frac{n}{2}}(E(w^2))^{\frac{n}{2}}}\right\}\chi_Q uE(wf)=0.$$

In particular, it is holds for any strictly positive $f\in L^2(\mathscr{A})$. Therefore, $(E(uw))^2=E(u^2)E(w^2)$ on Q. The equivalence $(i)\iff(iii)$ follows from [7]. □

3. Weighted conditional composition operators

Let φ be a measurable transformation from X into X such that $\mu\circ\varphi^{-1}$ is absolutely continuous with respect to μ, that is μ is non-singular. Let h be the Radon-Nikodym derivative $d\mu\circ\varphi^{-1}/d\mu$ and we always assume that h is almost everywhere finite valued or, equivalently $\varphi^{-1}(\Sigma)$ is a sub-sigma finite algebra. In this section we investigated some classic properties of weighted conditional composition operators $T_\varphi:=M_wEM_uC_\varphi$ on $L^2(\Sigma)$, where $u,w\in L^0_+(\Sigma)$. Let $\varphi^{-1}(\Sigma)\subseteq\mathscr{A}$. Since for each $f\in L^0_+(\Sigma)$, $E(f\circ\varphi)=f\circ\varphi$, so $T_\varphi=M_{wEM_u}C_\varphi$ is a weighted composition operator. Put $E_\varphi=E^{\varphi^{-1}(\Sigma)}$. It is easy to check that $\|T_\varphi f\|_2=\|M_{\sqrt{J}}f\|_2$, where $J=hE_\varphi(w^2(E(u))^2)\circ\varphi^{-1}$. Thus, $T_\varphi\in B(L^2(\Sigma))$ if and only if $J\in L^\infty(\Sigma)$ and in this case $\|T_\varphi\|=\|\sqrt{J}\|_\infty$ (see [5]). Moreover, $T_\varphi\in CR(L^2(\Sigma))$ if and only if J is bounded away from zero on $\sigma(J)$. Set again $K=S\cap G$, where $G=\sigma(E(w))$ and $S=\sigma(E(u))$.

Let $U_\varphi|T_\varphi|$ be the polar decomposition of T_φ. Since $T^*_\varphi(f)=hE_\varphi(wE(u)f)\circ\varphi^{-1}$, we obtain $|T_\varphi|(f)=\sqrt{J}f$ and $U_\varphi(f)=\chi_{\sigma(wE(u))}(J\circ\varphi)^{-1/2}T_\varphi(f)$. It follows that

$$\widetilde{T_\varphi}f=|T_\varphi|^{\frac{1}{2}}U_\varphi|T_\varphi|^{\frac{1}{2}}f=\chi_{\sigma(wE(u))}\{\frac{J}{J\circ\varphi}\}^{\frac{1}{4}}wE(u)f\circ\varphi.$$

Now, let $T_\varphi\in CR(L^2(\Sigma))$. Put

$$P(f)=\frac{\chi_{\sigma(J)}}{E_\varphi(w^2(E(u))^2)\circ\varphi^{-1}}E_\varphi(wE(u)f)\circ\varphi^{-1}.$$

Then P satisfy all equations in (1.1). Thus $P=T^\dagger_\varphi$. In fact we can write $T^\dagger_\varphi=M_{\frac{\sigma(J)}{J}}T^*_\varphi$. Hence

$$(T^\dagger_\varphi)^*T^\dagger_\varphi(f)=\frac{\chi_{\sigma(wE(u))}}{h\circ\varphi\{E_\varphi(w^2(E(u))^2)\}^2}wE(u)E_\varphi(wE(u)f).$$

In Lemma 2.1, set $v = \frac{\chi_{\sigma(wE(u))}}{h\circ\varphi\{E_\varphi(w^2(E(u))^2)\}^2}$ and $\omega = wE(u)$. Then we obtain

$$|T_\varphi^\dagger|(f) = \frac{wE(u)\chi_{\sigma(wE(u))}}{(h\circ\varphi)^{\frac{1}{2}}\{E_\varphi(w^2(E(u))^2)\}^{\frac{3}{2}}}E_\varphi(wE(u)f);$$

$$|T_\varphi^\dagger|^{\frac{1}{2}}(f) = \frac{wE(u)\chi_{\sigma(wE(u))}}{(h\circ\varphi)^{\frac{1}{4}}\{E_\varphi(w^2(E(u))^2)\}^{\frac{5}{4}}}E_\varphi(wE(u)f).$$

Define

$$U_{\varphi^\dagger}(f) = \left\{\frac{h\chi_{\sigma(J)}}{E_\varphi(w^2(E(u))^2)\circ\varphi^{-1}}\right\}^{\frac{1}{2}}E_\varphi(wE(u)f)\circ\varphi^{-1}.$$

Then $T_\varphi^\dagger = U_{\varphi^\dagger}|T_\varphi^\dagger|$, $U_{\varphi^\dagger}U_{\varphi^\dagger}^*U_{\varphi^\dagger} = U_{\varphi^\dagger}$ and $\mathcal{N}(U_{\varphi^\dagger}) = \mathcal{N}(T_\varphi^\dagger)$. Note that $U_{\varphi^\dagger} = U_\varphi^*$ and $|T_\varphi^\dagger| = |T_\varphi^*|^\dagger$. So we have the following proposition.

PROPOSITION 3.1. *Let $T_\varphi \in CR(L^2(\Sigma))$ and let $U_{\varphi^\dagger}|T_\varphi^\dagger|$ be the polar decomposition of $T_\varphi^\dagger$. Then*

$$|T_\varphi^\dagger|(f) = \frac{wE(u)\chi_{\sigma(wE(u))}}{(h\circ\varphi)^{\frac{1}{2}}\{E_\varphi(w^2(E(u))^2)\}^{\frac{3}{2}}}E_\varphi(wE(u)f);$$

$$U_{\varphi^\dagger}(f) = \left\{\frac{h\chi_{\sigma(J)}}{E_\varphi(w^2(E(u))^2)\circ\varphi^{-1}}\right\}^{\frac{1}{2}}E_\varphi(wE(u)f)\circ\varphi^{-1}.$$

Let $\widetilde{T_\varphi} \in CR(L^2(\Sigma))$ and put $B(f) = \chi_{\sigma(J)}hJ^{-\frac{3}{4}}E_\varphi(\chi_{\sigma(J)}J^{-\frac{1}{4}}wE(u)f)\circ\varphi^{-1}$. Then it is easy to check that B satisfy all equations in (1.1). Thus $B = (\widetilde{T_\varphi})^\dagger$. Now, let $T_\varphi \in CR(L^2(\Sigma))$. Set $W = U_{\varphi^\dagger}|T_\varphi^\dagger|^{\frac{1}{2}}$. A calculation show that $W(f) = \chi_{\sigma(J)}hJ^{-\frac{3}{4}}E_\varphi(wE(u)f)\circ\varphi^{-1}$, and so we obtain

$$\begin{aligned}\widetilde{T_\varphi^\dagger}(f) &= |T_\varphi^\dagger|^{\frac{1}{2}}W(f) = |T_\varphi^\dagger|^{\frac{1}{2}}(\chi_{\sigma(J)}hJ^{-\frac{3}{4}}E_\varphi(wE(u)f)\circ\varphi^{-1})\\ &= \frac{\chi_{\sigma(wE(u))\cap\sigma(J)}wE(u)}{(h\circ\varphi)^{\frac{1}{4}}\{E_\varphi(w^2E(u)^2)\}^{\frac{5}{4}}}E_\varphi(wE(u)hJ^{-\frac{3}{4}}E_\varphi(wE(u)f)\circ\varphi^{-1}).\end{aligned}$$

These observations establish the following proposition.

PROPOSITION 3.2. *Let $k = wE(u)$ and $T \in CR(L^2(\Sigma))$. Then the following assertions hold.*

(i) $T_\varphi^\dagger(f) = \frac{\chi_{\sigma(J)}}{E_\varphi(k^2)\circ\varphi^{-1}}E_\varphi(kf)\circ\varphi^{-1}$.

(ii) Let $U_{\varphi^\dagger}|T_\varphi^\dagger|$ be the polar decomposition of $T^\dagger$. Then

$$|T_\varphi^\dagger|\,(f) = \frac{k\chi_{\sigma(k)}}{(h\circ\varphi)^{\frac{1}{2}}\{E_\varphi(k^2)\}^{\frac{3}{2}}}E_\varphi(kf);$$

$$U_{\varphi^\dagger}(f) = \left\{\frac{h\chi_{\sigma(J)}}{E_\varphi(k^2)\circ\varphi^{-1}}\right\}^{\frac{1}{2}}E_\varphi(kf)\circ\varphi^{-1}.$$

(iii) If $\widetilde{T_\varphi} \in CR(L^2(\Sigma))$*, then* $(\widetilde{T_\varphi})^\dagger(f) = \chi_{\sigma(J)} h J^{-\frac{3}{4}} E_\varphi(\chi_{\sigma(J)} J^{-\frac{1}{4}} kf) \circ \varphi^{-1}$.

(iv) $\widetilde{T_\varphi^\dagger}(f) = \frac{\chi_{\sigma(k)\cap\sigma(J)} k}{(h\circ\varphi)^{\frac{1}{4}} \{E_\varphi(k^2)\}^{\frac{5}{4}}} E_\varphi(\chi_{\sigma(J)} khJ^{-\frac{3}{4}} E_\varphi(kf) \circ \varphi^{-1})$.

EXAMPLE 3.3. Let $X = [0,1]$ equipped with the Lebesgue measure $d\mu = dx$ on the Lebesgue measurable subsets of X and let $\psi, \varphi : X \to X$ be a non-singular measurable transformations defined by $\psi(x) = x^3$ and

$$\varphi(x) = \begin{cases} 2x & 0 \leqslant x \leqslant \frac{1}{2}, \\ 2-2x & \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

Then $\psi^{-1}(\Sigma) = \Sigma$, and hence $E^{\psi^{-1}(\Sigma)} = I$. Moreover, for each $f \in L^2(\Sigma)$ and $x \in X$ we have

$$h(x) = \left|\frac{d}{dx}\left(\frac{x}{2}\right)\right| + \left|\frac{d}{dx}\left(\frac{2-x}{2}\right)\right| = 1;$$
$$E_\varphi(f)(x) = \frac{f(x)+f(1-x)}{2};$$
$$(E_\varphi(f)\circ\varphi^{-1})(x) = \frac{1}{2}\left(f\left(\frac{x}{2}\right) + f\left(1-\frac{x}{2}\right)\right).$$

Put $u(x) = x$ and $w(x) = 2$. Then $k(x) = (wE(u))(x) = 2x$ and

$$E_\varphi(k)\circ\varphi^{-1} = 1;$$
$$E_\varphi(k^2)\circ\varphi^{-1} = x^2 - 2x + 2;$$
$$J = x^2 - 2x + 2;$$
$$J\circ\varphi = 4x^2 - 2x + 2.$$

Hence we get that

$$T_\varphi^\dagger f(x) = \left(\frac{1}{2x^2-4x+4}\right)\left\{xf\left(\frac{x}{2}\right) + (2-x)f\left(1-\frac{x}{2}\right)\right\};$$

$$U_{\varphi^\dagger}(x)f = \left(\frac{1}{4(x^2-2x+2)}\right)^{\frac{1}{2}}\left\{xf\left(\frac{x}{2}\right) + (2-x)f\left(1-\frac{x}{2}\right)\right\};$$

$$T_\varphi f(x) = \begin{cases} 2xf(2x) & 0 \leqslant x \leqslant \frac{1}{2}, \\ 2xf(2-2x) & \frac{1}{2} \leqslant x \leqslant 1; \end{cases}$$

$$U_\varphi f(x) = \begin{cases} (4x^2-2x+2)^{\frac{-1}{2}} 2xf(2x) & 0 \leqslant x \leqslant \frac{1}{2}, \\ (4x^2-2x+2)^{\frac{-1}{2}} 2xf(2-2x) & \frac{1}{2} \leqslant x \leqslant 1; \end{cases}$$

$$|T_\varphi| f(x) = \sqrt{J} f(x) = \sqrt{x^2-2x+2}\, f(x);$$

$$|T_\varphi^\dagger|f(x) = \frac{2x}{(4x^2-2x+2)^{\frac{3}{2}}}\{xf(x)+(1-x)f(1-x)\};$$

$$(\widetilde{T_\varphi})^\dagger f(x) = \frac{1}{2(x^2-2x+2)^{\frac{3}{4}}}\left\{\frac{xf\left(\frac{x}{2}\right)}{\left(\frac{x^2}{4}-x+2\right)^{\frac{1}{4}}}+\frac{(2-x)f\left(1-\frac{x}{2}\right)}{\left(\left(1-\frac{x}{2}\right)^2+x\right)^{\frac{1}{4}}}\right\}.$$

EXAMPLE 3.4. (i) Let $X=[0,1]\times[0,1]$, $d\mu = dxdy$, Σ be the Lebesgue subsets of X, $\mathscr{A} = \{[0,1]\times A\colon\ A$ is a Lebesgue set in $[0,1]\}$. Then for each $f\in L^2(\Sigma)$, $(Ef)(x,y)=\int_0^1 f(t,y)dt$, which is independent of the first coordinate. Now, if we take $u(x,y)=x^2e^y$, $w(x,y)=x^2\sin(y)$. Then $E(u^2)(x,y)=\frac{e^{2y}}{5}$, $E(w^2)(x,y)=\frac{\sin^2(y)}{5}$. It follows that

$$(E(uw))^2(x,y)=\frac{e^{2y}\sin^2(y)}{25}=E(u^2)(x,y)E(w^2)(x,y).$$

Thus, by Theorem 2.10, $T^\dagger$ belongs to A-classes of operator and quasi-A-class, quasi-$*$-A-class and by Theorem 2.11 the operator $T^\dagger$ is centered.

(ii) Let $X=[-1,1]$, $d\mu=\frac{1}{2}dx$. With the same assumptions of Example 2.8 let $\mathscr{A}=\langle\{(-a,a):0\leqslant a\leqslant 1\}\rangle$. Then for each $f\in L^2(\Sigma)$, $E^{\mathscr{A}}(f)$ is the even part of f. Let $u(x)=e^x$, $w(x)=1$. Then $E(u)(x)=\cosh(x)$, $S(E(u))=X$ and $E(u^2)(x)=\cosh(2x)$. Since $\cosh^2(x)\neq\cosh(2x)$ then by Theorem 2.11, T and $T^\dagger$ are not centered. Now, if $u(x)=x^2$ and $w(x)=\cos(x)$ then $E(u^2)(x)=x^4$, $E(w^2)(x)=\cos^2(x)$ and $E(uw)(x)=x^2\cos(x)$, and thus $T^\dagger$ is centered.

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(Received March 8, 2016)

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