

## SPECTRAL RADIUS ALGEBRAS OF WCE OPERATORS

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*Abstract.* In this paper, we investigate the spectral radius algebras related to the weighted conditional expectation operators on the Hilbert spaces  $L^2(\mathcal{F})$ . We give a large classes of operators on  $L^2(\mathcal{F})$  that have the same spectral radius algebra. As a consequence we get that the spectral radius algebras of a weighted conditional expectation operator and its Aluthge transformation are equal. Also, we obtain an ideal of the spectral radius algebra related to the rank one operators on the Hilbert space  $\mathcal{H}$ . Finally we get that the operator  $T$  majorizes all closed range elements of the spectral radius algebra of  $T$ , when  $T$  is a weighted conditional expectation operator on  $L^2(\mathcal{F})$  or a rank one operator on the arbitrary Hilbert space  $\mathcal{H}$ .

### 1. Introduction

Let  $(X, \mathcal{F}, \mu)$  be a complete  $\sigma$ -finite measure space. All sets and functions statements are to be interpreted as holding up to sets of measure zero. For a  $\sigma$ -subalgebra  $\mathcal{A}$  of  $\mathcal{F}$ , the conditional expectation operator associated with  $\mathcal{A}$  is the mapping  $f \rightarrow E^{\mathcal{A}}f$ , defined for all non-negative  $f$  as well as for all  $f \in L^2(\mathcal{F}) = L^2(X, \mathcal{F}, \mu)$ , where  $E^{\mathcal{A}}f$  is the unique  $\mathcal{A}$ -measurable function satisfying  $\int_A (E^{\mathcal{A}}f) d\mu = \int_A f d\mu$ , for all  $A \in \mathcal{A}$ . We will often write  $E$  for  $E^{\mathcal{A}}$ . The mapping  $E$  is a linear orthogonal projection from  $L^2(\mathcal{F})$  onto  $L^2(\mathcal{A})$ . For more details on the properties of  $E$  see [20].

We continue our investigation about the class of bounded linear operators on the  $L^p$ -spaces having the form  $M_w E M_u$ , where  $E$  is the conditional expectation operator,  $M_w$  and  $M_u$  are (possibly unbounded) multiplication operators and it is called weighted conditional expectation operator. Our interest in operators of the form  $M_w E M_u$  stems from the fact that such forms tend to appear often in the study of those operators related to conditional expectation. Weighted conditional expectation operators appeared in [4], where it is shown that every contractive projection on certain  $L^1$ -spaces can be decomposed into an operator of the form  $M_w E M_u$  and a nilpotent operator. For stronger results about weighted conditional expectation operators one can see [5, 9, 11, 13]. In these papers one can see that a large classes of operators are of the form of weighted conditional expectation operators.

Let  $\mathcal{H}$  be a Hilbert space and  $\mathcal{B}(\mathcal{H})$  be the algebra of all bounded linear operators on  $\mathcal{H}$ .  $\mathcal{N}(T)$  and  $\mathcal{R}(T)$  denote the null-space and range of an operator  $T$ , respectively. A closed subspace  $\mathcal{M}$  of  $\mathcal{H}$  is said to be invariant for an operator

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$T \in \mathcal{B}(\mathcal{H})$  if  $T\mathcal{M} \subseteq \mathcal{M}$ . The collection of all invariant subspaces of  $T$  is a lattice and it is denoted by  $\text{Lat}(T)$ . If  $\mathcal{M}$  is invariant for all operators commute with  $T$ , then it is called a hyperinvariant subspace for  $T$ . The description  $\text{Lat}(T)$  is an open problem. Some authors describe  $\text{Lat}(T)$  in the special case of  $T$ . In [12], Lambert and Petrovic introduced a modified version of a class of operator algebras that is called spectral radius algebras. Since a spectral radius algebra related to an operator  $T \in \mathcal{B}(\mathcal{H})$  ( $\mathcal{B}_T$ ) contains all operators that commute with  $T$  ( $\{T\}'$ ), then the invariant subspaces of  $\mathcal{B}_T$  are hyperinvariant subspaces of  $T$ . In [12], the authors established several sufficient conditions for  $\mathcal{B}_T$  to have a nontrivial invariant subspace. When  $T$  is compact the results of [12] generalizes the Lomonosov's theorem. In [2], the authors demonstrated that for a subclasses of normal operators  $\mathcal{B}_T$  has a nontrivial invariant subspace. Spectral radius algebras for complex symmetric operators are discussed in [10]. Moreover, spectral radius algebras were studied recently by S. Petrovic in [14, 15, 16, 17, 18, 19].

In this paper we investigate the spectral radius algebras related to the weighted conditional expectation operators on the Hilbert spaces  $L^2(\mathcal{F})$ . We will show that there are lots of operators on  $L^2(\mathcal{F})$  such as  $T$  with  $\mathcal{B}_T \neq \{T\}'$ . In addition, we obtain an ideal of the spectral radius algebra related to the rank one operators on the Hilbert space  $\mathcal{H}$ . Finally we get that the operator  $T$  majorizes all closed range elements of the spectral radius algebra of  $T$ , when  $T$  is a weighted conditional expectation operator on  $L^2(\mathcal{F})$  or a rank one operator on the arbitrary Hilbert space  $\mathcal{H}$ .

### 2. Spectral radius algebras

For notation and basic terminology concerning spectral radius algebras, we refer the reader to [3, 12].

Let  $\mathcal{H}$  be a Hilbert space,  $T \in \mathcal{B}(\mathcal{H})$  and let  $r(T)$  be the spectral radius of  $T$ . For  $m \geq 1$  we define

$$R_m(T) = R_m := \left( \sum_{n=0}^{\infty} d_m^{2n} T^{*n} T^n \right)^{\frac{1}{2}}, \tag{2.1}$$

where  $d_m = \frac{1}{1/m+r(T)}$ . Since  $d_m \uparrow 1/r(T)$ , the sum in (2.1) is norm convergent and for each  $m$ ,  $R_m$  is well defined, positive and invertible. The spectral radius algebra  $\mathcal{B}_T$  of  $T$  consists of all operators  $S \in \mathcal{B}(\mathcal{H})$  such that

$$\sup_{m \in \mathbb{N}} \|R_m S R_m^{-1}\| < \infty.$$

$\mathcal{B}_T$  is an algebra and it contains all operators commute with  $T$ . Throughout this section we assume that  $w, u \in \mathcal{D}(E) := \{f \in L^0(\mathcal{F}) : E(|f|) \in L^0(\mathcal{A})\}$ , where  $L^0(\mathcal{F})$  is the space of almost every where finite valued  $\mathcal{F}$ -measurable functions on  $X$ . Now we recall the definition of weighted conditional expectation operators on  $L^2(\mathcal{F})$ .

DEFINITION 2.1. Let  $(X, \mathcal{F}, \mu)$  be a  $\sigma$ -finite measure space and let  $\mathcal{A}$  be a  $\sigma$ -subalgebra of  $\mathcal{F}$  such that  $(X, \mathcal{A}, \mu_{\mathcal{A}})$  is also  $\sigma$ -finite. Let  $E$  be the conditional

expectation operator relative to  $\mathcal{A}$ . If  $u, w \in L^0(\mathcal{F})$ , such that  $uf$  is conditionable and  $wE(uf) \in L^2(\mathcal{F})$  for all  $f \in \mathcal{D} \subseteq L^2(\mathcal{F})$ , where  $\mathcal{D}$  is a linear subspace, then the corresponding weighted conditional expectation (or briefly WCE) operator is the linear transformation  $M_wEM_u : \mathcal{D} \rightarrow L^2(\mathcal{F})$  defined by  $f \rightarrow wE(uf)$ .

As was proved in [8] we have an equivalent condition for boundedness of the weighted conditional expectation operators  $M_wEM_u$  on  $L^2(\mathcal{F})$  as the next theorem.

**THEOREM 2.2.** *The operator  $T = M_wEM_u : L^2(\mathcal{F}) \rightarrow L^2(\mathcal{F})$  is bounded if and only if  $(E(|w|^2)^{\frac{1}{2}})(E(|u|^2)^{\frac{1}{2}}) \in L^\infty(\mathcal{A})$ , in this case  $\|T\| = \|(E(|w|^2)^{\frac{1}{2}})(E(|u|^2)^{\frac{1}{2}})\|_\infty$ .*

Let  $T = M_wEM_u$  be a bounded operator on  $L^2(\mathcal{F})$ . Direct computations shows that for every  $n \in \mathbb{N}$  (natural numbers) we have

$$\begin{aligned} T^n f &= (E(uw))^{n-1} wE(uf); \\ T^{*n} f &= (\overline{E(uw)})^{n-1} \bar{u}E(\bar{w}f). \end{aligned}$$

Since  $R_m = R_m(M_wEM_u)$  is positive and invertible operator, we obtain

$$R_m = \left( I + M_{(E(|w|^2) \sum_{n=1}^\infty d_m^{2n} |E(uw)|^{2(n-1)})} M_{\bar{u}}EM_u \right)^{\frac{1}{2}}.$$

It is easy to see that the following equality holds almost every where on  $X$ .

$$\sum_{n=1}^\infty d_m^{2n} |E(uw)|^{2(n-1)} = \frac{d_m^2}{1 - d_m^2 |E(uw)|^2}.$$

If we set

$$v_m = \frac{d_m^2 E(|w|^2)}{1 - d_m^2 |E(uw)|^2},$$

then we have

$$R_m = (I + M_{v_m \bar{u}}EM_u)^{\frac{1}{2}}.$$

By an elementary technical method we can compute the inverse of  $R_m$  as follow:

$$R_m^{-1} = \left( I + M_{\frac{v_m \bar{u}}{v_m E(|u|^2) - 1}}EM_u \right)^{\frac{1}{2}}.$$

Here we recall a fundamental lemma in operator theory.

**LEMMA 2.3.** *Let  $T$  be a bounded operator on the Hilbert space  $\mathcal{H}$  and  $\lambda \geq 0$ . Then we have*

$$\|\lambda I + T^*T\| = \lambda + \|T^*T\| = \lambda + \|T\|^2.$$

*Specially, if  $T$  is a positive operator, then  $\|\lambda I + T\| = \lambda + \|T\|$ .*

*Proof.* It is an easy exercise.  $\square$

From now on, we assume that  $E(|u|^2) \in L^\infty(\mathcal{A})$ . Now we characterize the spectral radius algebra corresponding to the WCE operator  $M_wEM_u$  in the next theorem.

**THEOREM 2.4.** *Let  $S \in \mathcal{B}(L^2(\mathcal{F}))$ . Then  $S \in \mathcal{B}_{M_wEM_u}$  if and only if  $\mathcal{N}(EM_u)$  is invariant under  $S$ .*

*Proof.* Since  $R_m$  and  $R_m^{-1}$  are positive operators and  $(EM_u)^* = M_u^{-1}E$ , then by Lemma 2.3 and Theorem 2.2 we have

$$\|R_m\|^2 = \|R_m^2\| = 1 + \|E(|u|^2)v_m\|_\infty$$

and

$$\|R_m^{-1}\|^2 = \|R_m^{-2}\| = 1 + \left\| \frac{E(|u|^2)v_m}{v_mE(|u|^2) - 1} \right\|_\infty.$$

If we decompose  $L^2(\mathcal{F})$  as a direct sum  $\mathcal{H}_1 \oplus \mathcal{H}_2$ , in which

$$\mathcal{H}_2 = \mathcal{N}(EM_u) = \{f \in L^2(\mathcal{F}) : E(uf) = 0\}$$

and

$$\mathcal{H}_1 = \mathcal{H}_2^\perp = \overline{uL^2(\mathcal{A})},$$

then the corresponding block matrix of  $R_m$  is

$$R_m = \begin{pmatrix} M_{(q_m)\frac{1}{2}} & 0 \\ 0 & I \end{pmatrix} \text{ and } R_m^{-1} = \begin{pmatrix} M_{(q_m)\frac{-1}{2}} & 0 \\ 0 & I \end{pmatrix},$$

where  $q_m = 1 + v_mE(|u|^2)$ . Notice that for  $m > m'$  we have  $q_m \geq q_{m'}$  and  $\|q_m\|_\infty \rightarrow \infty$  as  $m \rightarrow \infty$ . If  $S \in \mathcal{B}(L^2(\mathcal{F}))$  say  $S = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ , the block matrix with respect to the decomposition  $\mathcal{H}_1 \oplus \mathcal{H}_2$ , then

$$\begin{aligned} R_mSR_m^{-1} &= \begin{pmatrix} M_{(q_m)\frac{1}{2}} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} M_{(q_m)\frac{-1}{2}} & 0 \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} M_{(q_m)\frac{1}{2}}XM_{(q_m)\frac{-1}{2}} & M_{(q_m)\frac{1}{2}}Y \\ ZM_{(q_m)\frac{-1}{2}} & W \end{pmatrix}. \end{aligned}$$

Since  $\|M_{(q_m)\frac{1}{2}}XM_{(q_m)\frac{-1}{2}}\| \leq \|X\|$ , then we get that  $\sup_m \|R_mSR_m^{-1}\| < \infty$  if and only if  $\sup_m \|M_{(q_m)\frac{1}{2}}Y\| < \infty$ . Direct computations shows that  $\sup_m \|M_{(q_m)\frac{1}{2}}Y\| < \infty$  if and only if  $Y = 0$ . This means that  $\mathcal{H}_2$  is an invariant subspace for  $S$ .  $\square$

Therefore by Theorem 2.4 we get that there are many different operators that have the same spectral radius algebra.

COROLLARY 2.5. *Let  $w, w', u \in \mathcal{D}(E)$ . If  $M_w EM_u$  and  $M_{w'} EM_u$  are bounded operator on the Hilbert space  $L^2(\mathcal{F})$ , then  $\mathcal{B}_{M_{w'} EM_u} = \mathcal{B}_{M_w EM_u}$ .*

Also in the next corollary we have a sufficient condition for  $\mathcal{B}_{M_w EM_u}$  to be equal to  $\mathcal{B}(L^2(\mathcal{F}))$ .

COROLLARY 2.6. *If  $\mathcal{N}(EM_u) = \{0\}$ , then  $\mathcal{B}_{M_w EM_u} = \mathcal{B}(L^2(\mathcal{F}))$ .*

In the next Proposition we find some special elements of  $\mathcal{B}_{M_w EM_u}$ .

PROPOSITION 2.7. *If  $a \in L^0(\mathcal{A})$  such that  $a \geq 0$  and  $M_{a\bar{u}} EM_u \in \mathcal{B}(L^2(\mathcal{F}))$ , then  $M_{a\bar{u}} EM_u \in \mathcal{B}_{M_w EM_u}$ .*

*Proof.* Since  $R_m = (I + M_{v_m \bar{u}} EM_u)^{\frac{1}{2}}$  and  $v_m = \frac{d_m^2 E(|w|^2)}{1 - d_m^2 |E(uw)|^2}$  is an  $\mathcal{A}$ -measurable function, it holds that  $R_m M_{a\bar{u}} EM_u = M_{a\bar{u}} EM_u R_m$ . Therefore we have  $\|R_m M_{a\bar{u}} EM_u R_m^{-1}\| = \|M_{a\bar{u}} EM_u\|$ , and so we get that  $M_{a\bar{u}} EM_u \in \mathcal{B}_{M_w EM_u}$ .  $\square$

Every operator  $T$  on a Hilbert space  $\mathcal{H}$  can be decomposed into  $T = U|T|$  with a partial isometry  $U$ , where  $|T| = (T^*T)^{\frac{1}{2}}$ .  $U$  is determined uniquely by the kernel condition  $\mathcal{N}(U) = \mathcal{N}(|T|)$ . Then this decomposition is called the polar decomposition. The Aluthge transformation  $\tilde{T}$  of the operator  $T$  is defined by  $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ . Here we recall that the Aluthge transformation of  $T = M_w EM_u$  is

$$\tilde{T}(f) = \frac{\chi_{z_1} E(uw)}{E(|u|^2)} \bar{u} E(uf), \quad f \in L^2(\mathcal{F}),$$

in which  $z_1 = z(E(|u|^2))$  (see [8]). Thus  $\tilde{T} = M_{w'} EM_{u'}$  where  $w' = \frac{E(uw)\bar{u}\chi_{z_1}}{E(|u|^2)}$  and  $u' = u$ . We recall that  $r(M_w EM_u) = \|E(uw)\|_\infty$  (see [6]). Direct computations shows that  $E(u'w') = E(uw)$ . Hence  $r(T) = r(\tilde{T})$ . Hence by using Proposition 2.7 we have the next corollary.

COROLLARY 2.8. *If  $w$  and  $u$  are positive measurable functions, then  $\tilde{T} \in \mathcal{B}_T$  where  $T = M_w EM_u$ .*

By the proof of Proposition 2.7 we get that the commutant of  $M_w EM_u$  (in symbol  $\{M_w EM_u\}'$ ) is a proper subset of  $\mathcal{B}_{M_w EM_u}$  when  $w, u$  are positive and  $w \neq u$ . In the next theorem we get that  $\mathcal{B}_T = \mathcal{B}_{\tilde{T}}$  when  $T = M_w EM_u$  and  $w, u \geq 0$ .

COROLLARY 2.9. *If  $T = M_w EM_u$  and  $w, u \geq 0$ , then  $\mathcal{B}_T = \mathcal{B}_{\tilde{T}}$ .*

Recall that for  $f, g \in L^2(\mathcal{F})$  we can define a rank one operator  $f \otimes g$  on  $L^2(\mathcal{F})$  by the action  $(f \otimes g)(h) = \langle h, g \rangle f$  for every  $h \in L^2(\mathcal{F})$ , in which  $\langle \cdot, \cdot \rangle$  is the inner product of the Hilbert space  $L^2(\mathcal{F})$ . In the next proposition we give some conditions under which a rank one operator belongs to the  $\mathcal{B}_{M_w EM_u}$ .

PROPOSITION 2.10. *If  $T = M_wEM_u$  and  $f, g \in L^2(\mathcal{F})$ , then  $f \otimes g \in \mathcal{B}_T$  if and only if*

$$\sup_m \|\alpha_m^{\frac{1}{2}}E(ug)\|^2 \|f\|^2 + \|v_m^{\frac{1}{2}}E(uf)\|^2 (\|g\|^2 + \|\alpha_m^{\frac{1}{2}}E(ug)\|^2) < \infty,$$

where  $\alpha_m = \frac{v_m}{v_mE(|u|^2)-1}$ .

*Proof.* By using the properties of inner product we have

$$\|R_m f\|^2 = \|f\|^2 + \|v_m^{\frac{1}{2}}E(uf)\|^2$$

and

$$\|R_m^{-1} f\|^2 = \|g\|^2 + \|\alpha_m^{\frac{1}{2}}E(ug)\|^2.$$

Now, the desired conclusion follows by [12, Lemma 3.9].  $\square$

By using some results of [12] we get that the conditional expectation corresponding to  $\sigma$ -subalgebra  $\mathcal{A} \subseteq \mathcal{B}$  is in  $\mathcal{B}_{E^{\mathcal{A}}M_u}$  as we mentioned in the next remark.

REMARK 2.11. Let  $T = EM_u \in \mathcal{B}(L^2(\mathcal{F}))$ ,  $u \in L^\infty(\mathcal{A})$  and let  $\mathcal{A}, \mathcal{B}$  be  $\sigma$ -subalgebras of  $\mathcal{F}$  such that  $\mathcal{A} \subseteq \mathcal{B}$ . If  $E = E^{\mathcal{A}}$  and  $S$  is an operator for which  $TS = E^{\mathcal{B}}ST$ , then  $S \in \mathcal{B}_T$ .

*Proof.* It is not hard to see that  $EM_uE^{\mathcal{B}} = E^{\mathcal{B}}EM_u$ . Since  $E^{\mathcal{B}}$  is a projection on  $L^2(\mathcal{F})$ , then it is power bounded. Therefore, by [12, Proposition 2.3] we get that  $S \in \mathcal{B}_T$ .  $\square$

COROLLARY 2.12. *If  $T = M_wEM_u \in \mathcal{B}(L^2(\mathcal{F}))$  and  $a \in L^\infty(\mathcal{A})$ , then  $M_a \in \mathcal{B}_T$ .*

Let  $\mathcal{H}$  be a Hilbert space and  $T \in \mathcal{B}(\mathcal{H})$ . Here we recall the definition of  $Q_T$ , that is defined in [12], as follows:

$$Q_T = \{S \in \mathcal{B}(\mathcal{H}) : \|R_mSR_m^{-1}\| \rightarrow 0\}.$$

In the next theorem we illustrate  $Q_T$  when  $T = M_wEM_u \in \mathcal{B}(L^2(\mathcal{F}))$ .

THEOREM 2.13. *Let  $T = M_wEM_u$  and  $S \in \mathcal{B}(L^2(\mathcal{F}))$ . Then  $S \in Q_T$  if and only if  $\mathcal{N}(EM_u)$  is invariant under  $S$  and  $\mathcal{N}(EM_u) \subseteq \mathcal{N}(S)$ .*

*Proof.* Let  $S = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$ , the block matrix with respect to the decomposition  $\mathcal{H}_1 \oplus \mathcal{H}_2$ , in which  $\mathcal{H}_2 = \mathcal{N}(EM_u)$  and  $\mathcal{H}_1 = \mathcal{H}_2^\perp$ . So similar to the proof of Theorem 2.4 we have

$$R_mSR_m^{-1} = \begin{pmatrix} M_{(q_m)^{\frac{1}{2}}} X M_{(q_m)^{-\frac{1}{2}}} & M_{(q_m)^{\frac{1}{2}}} Y \\ Z M_{(q_m)^{-\frac{1}{2}}} & W \end{pmatrix}.$$

Hence  $\|W\| \leq \|R_mSR_m^{-1}\|$ . Since  $S \in Q_T$ , then  $Y = 0$  and  $\|W\| = 0$ . This means that  $\mathcal{H}_2$  is invariant under  $S$  and  $PSP = 0$  in which  $P = P_{\mathcal{H}_2}$ . Therefore  $SP = PSP = 0$ , and so  $\mathcal{H}_2 \subseteq \mathcal{N}(S)$ . Conversely, if  $\mathcal{N}(EM_u)$  is invariant under  $S$  and  $\mathcal{N}(EM_u) \subseteq \mathcal{N}(S)$ , then we get that  $W = Y = 0$  and

$$R_mSR_m^{-1} = \begin{pmatrix} M_{(q_m)\frac{1}{2}} X M_{(q_m)\frac{-1}{2}} & 0 \\ Z M_{(q_m)\frac{-1}{2}} & 0 \end{pmatrix}.$$

Hence

$$\|R_mSR_m^{-1}\| \leq \|M_{(q_m)\frac{1}{2}} X M_{(q_m)\frac{-1}{2}}\| + \|Z M_{(q_m)\frac{-1}{2}}\|.$$

Since  $\|M_{(q_m)\frac{-1}{2}}\| = \|(q_m)\frac{-1}{2}\|_\infty \rightarrow 0$ , then  $\|R_mSR_m^{-1}\| \rightarrow 0$  when  $m \rightarrow \infty$ . This completes the proof.  $\square$

Now by using [12, Theorem 2.6] and some information about WCE operators we have an equivalent condition for the spectral radius algebra of a WCE operator to be equal to  $\mathcal{B}(L^2(\mathcal{F}))$ .

PROPOSITION 2.14. *If  $T = M_wEM_u$ , then  $\mathcal{B}_T = \mathcal{B}(L^2(\mathcal{F}))$  if and only if*

$$\sup_m (\|E(|u|^2)v_m\|_\infty + \left\| \frac{E(|u|^2)v_m}{v_mE(|u|^2) - 1} \right\|_\infty (1 + \|E(|u|^2)v_m\|_\infty)) < \infty,$$

where  $v_m = \frac{d_m^2 E(|w|^2)}{1 - d_m^2 |E(uw)|^2}$ .

*Proof.* It is a direct consequence of [12, Theorem 2.6] and some information of the proof of Theorem 2.4.  $\square$

By using Proposition 2.14 and some results of [2] we have an equivalent condition for the WCE operator  $M_wEM_u$  to be a constant multiple of an isometry.

THEOREM 2.15. *If  $T = M_wEM_u$  is a bounded operator on the Hilbert space  $L^2(\mathcal{F})$ , then  $T$  is a constant multiple of an isometry if and only if*

$$\sup_m (\|E(|u|^2)v_m\|_\infty + \left\| \frac{E(|u|^2)v_m}{v_mE(|u|^2) - 1} \right\|_\infty (1 + \|E(|u|^2)v_m\|_\infty)) < \infty,$$

where  $v_m = \frac{d_m^2 E(|w|^2)}{1 - d_m^2 |E(uw)|^2}$ .

*Proof.* It is a direct consequence of [2, Theorem 2.7 ] and Proposition 2.14.  $\square$

Now in the next theorem we obtain some sufficient conditions for  $\mathcal{B}_{M_wEM_u}$  to a nontrivial invariant subspace.

THEOREM 2.16. *If the measure space  $(X, \mathcal{A}, \mu)$  is not a non-atomic measure space and  $E(uw) = 0$ , then  $\mathcal{B}_{M_wEM_u}$  has a nontrivial invariant subspace.*

*Proof.* Since  $E(uw) = 0$  then  $M_wEM_u$  is quasinilpotent. Also since the  $\sigma$ -algebra  $\mathcal{A}$  has at least one atom, then we have a compact multiplication operator  $M_a$  for some  $a \in L^\infty(\mathcal{A})$ . Hence by Corollary 2.12 we have  $M_a \in \mathcal{B}_{M_wEM_u}$ . Moreover by using [12, Lemma 3.1] we get that  $M_wEM_u \in Q_{M_wEM_u}$ . Therefore by [12, Theorem 3.4] we get the proof.  $\square$

Here we give a remark on [12, Proposition 2.8] as follows:

REMARK 2.17. For the unit vectors  $u, v, w$  of the Hilbert space  $\mathcal{H}$  we have  $\mathcal{B}_{u \otimes w} = \mathcal{B}_{v \otimes w}$ .

In the next theorem we describe  $Q_{u \otimes v}$  for a rank one operator  $u \otimes v$  in which  $u, v$  are in the Hilbert space  $\mathcal{H}$ .

THEOREM 2.18. Let  $\mathcal{H}$  be a Hilbert space and  $S \in \mathcal{B}(\mathcal{H})$ . If  $u, v \in \mathcal{H}$ , then  $S \in Q_{u \otimes v}$  if and only if  $S = (I - P)SP$ , where  $P = P_{\mathcal{H}_1}$  and  $\mathcal{H}_1$  is the one-dimensional space spanned by  $v$ .

*Proof.* As was computed in [12, Proposition 2.8] we have

$$R_m^2 = I + \frac{d_m^2}{1 - d_m^2 r^2} v \otimes v,$$

in which  $r = r(u \otimes v) = |\langle u, v \rangle|$ . Let  $\lambda_m = \sqrt{1 + \frac{d_m^2}{1 - d_m^2 r^2}}$ . If  $\mathcal{H}_1$  is the one-dimensional space spanned by  $v$  and  $\mathcal{H}_2 = \mathcal{H}_1^\perp$ . For  $S \in \mathcal{B}(\mathcal{H})$ , we have the corresponding block matrix of  $R_m, R_m^{-1}$  and  $S$  with respect to the decomposition  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$  as follows:

$$R_m = \begin{pmatrix} M_{\lambda_m} & 0 \\ 0 & I \end{pmatrix}, \quad R_m^{-1} = \begin{pmatrix} M_{\frac{1}{\lambda_m}} & 0 \\ 0 & I \end{pmatrix}$$

and

$$S = \begin{pmatrix} PSP & PS(I - P) \\ (I - P)SP & (I - P)S(I - P) \end{pmatrix}.$$

Therefore, we have

$$R_mSR_m^{-1} = \begin{pmatrix} PSP & M_{\lambda_m}PS(I - P) \\ M_{\frac{1}{\lambda_m}}(I - P)SP & (I - P)S(I - P) \end{pmatrix}.$$

If  $S \in Q_{u \otimes v}$ , then  $S \in \mathcal{B}_{u \otimes v}$ . Hence by [12, Proposition 2.8] we obtain  $PS(I - P) = 0$ . Since  $S \in Q_{u \otimes v}$ ,  $\|PSP\| \leq \|R_mSR_m^{-1}\|$  and  $\|(I - P)S(I - P)\| \leq \|R_mSR_m^{-1}\|$ , then  $\|PSP\| = 0$  and  $\|(I - P)S(I - P)\| = 0$ . Hence  $PSP = 0$  and  $(I - P)S(I - P) = 0$ . Thus

$$S = \begin{pmatrix} 0 & 0 \\ (I - P)SP & 0 \end{pmatrix} = (I - P)SP.$$



Conversely, If  $S = (I - P)SP$ , then

$$\|R_m SR_m^{-1}\| = \left\| \begin{pmatrix} 0 & 0 \\ M_{\frac{1}{\lambda_m}}(I - P)SP & 0 \end{pmatrix} \right\| = \|M_{\frac{1}{\lambda_m}}(I - P)SP\|.$$

Since  $\|M_{\frac{1}{\lambda_m}}(I - P)SP\| \rightarrow 0$ , then  $\|R_m SR_m^{-1}\| \rightarrow 0$  as  $m \rightarrow \infty$ . So  $S \in Q_{u \otimes v}$ .  $\square$

Let  $X, Y, Z$  be Banach spaces. Assume that  $T \in \mathcal{B}(X, Y)$  and  $S \in \mathcal{B}(X, Z)$ . Then  $T$  majorizes  $S$  if there exists  $M > 0$  such that

$$\|Sx\| \leq M\|Tx\|$$

for all  $x \in X$  (see [1]). Here we recall a result of [1] that gives us an equivalent condition for a closed range operator to majorize another bounded operator.

REMARK 2.19. [1, Proposition 4] Let  $X$  be Banach spaces and  $T, S \in \mathcal{B}(X)$  with  $\mathcal{R}(T)$  closed. Then  $T$  majorizes  $S$  if and only if  $\mathcal{N}(T) \subseteq \mathcal{N}(S)$ .

Now we recall an assertion about closed range weighted conditional expectation operators.

PROPOSITION 2.20. [7, Theorem 2.1] If  $z(E(u)) = z(E(|u|^2))$  and for some  $\delta > 0$ ,  $E(u) \geq \delta$  on  $z(E(|u|^2))$ , then the operator  $EM_u$  has closed range on  $L^2(\mathcal{F})$ .

PROPOSITION 2.21. Let  $T = M_w EM_u$  and  $u \geq 0$ . If  $S \in Q_T$  and  $E(u) \geq \delta$ , then  $EM_u$  majorizes  $S$ .

*Proof.* Since  $u \geq 0$ , then  $z(E(u)) = z(E(|u|^2))$ . Hence by the Remark 2.19, Theorem 2.13 and Proposition 2.20 we get the proof.  $\square$

Finally, since the rank one operator  $x \otimes y$  has closed range, the we can obtain the next proposition.

PROPOSITION 2.22. Let  $x, y \in \mathcal{H}$ . If  $T \in Q_{x \otimes y}$ , then  $x \otimes y$  majorizes  $T$ .

*Proof.* If  $T \in Q_{x \otimes y}$ , then by the proof of Theorem 2.18 we have  $\mathcal{H}_2 = \mathcal{N}(x \otimes y)$  and  $\mathcal{N}(x \otimes y) \subseteq \mathcal{N}(T)$ . Since  $x \otimes y$  has closed range, then by the Remark 2.19 we conclude that  $x \otimes y$  majorizes  $T$ .  $\square$

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