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INVERTIBILITY ALONG AN OPERATOR

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Abstract. We study the inverse along an element in the case of the algebra of bounded linear operators on a Banach space and characterize it as an outer inverse with prescribed range and nullspace. This inverse generalizes the group, Drazin and Koliha-Drazin inverses.

1. Introduction

Let X be a Banach space. We will denote by $\mathscr{B}(X)$ the algebra of bounded linear operators on X. For an operator $A \in \mathscr{B}(X)$, we will denote by $\mathscr{R}(A)$ the range of A and by $\mathscr{N}(A)$ the nullspace of A. An operator $B \in \mathscr{B}(X)$ is called a (bounded) inverse of $A \in \mathscr{B}(X)$ if $AB = BA = I$. In general, not any $A \in \mathscr{B}(X)$ has a bounded inverse. In this case, sometimes, we are looking for some operator that has certain properties of the inverse. In this way, over the years, there have been appeared several generalizations of invertibility, and many of them are particular cases of outer inverses with prescribed range and nullspace.

Let $A \in \mathscr{B}(X)$. An operator $B \in \mathscr{B}(X)$ is an inner inverse for A if $A = ABA$ holds, in this case we say that A is inner regular and write $B = A^{(1)}$. If $B = BAB$ holds, then A is outer regular, B is an outer inverse for A and we write $B = A^{(2)}$.

Neither inner nor outer inverses are unique. However, if we fix the range and nullspace of the outer inverse, say $\mathscr{R}(B) = M$ and $\mathscr{N}(B) = N$ where M and N are given subspaces of X, then when the outer inverse exists, it is unique, and we denote it by $A^{(2)}_{M,N}$ (see [1, Theorem 1.1.10]).

If V and W are subspaces of X such that $V \cap W = \{0\}$ and $V + W = X$, we say that V is complemented with complement W. Additionally, if V and W are closed, then we write $X = V \oplus W$.

If $A^{(2)}_{M,N}$ exists, then both of M and N are closed and complemented, and there exist two bounded projections P and Q such that $\mathscr{R}(P) = M$ and $\mathscr{R}(Q) = N$. In this way, the outer inverse is linked to two idempotent operators. There is some advantage in considering only one operator, instead of two. In this way, we have interest in an operator $T \in \mathscr{B}(X)$ such that $\mathscr{R}(T) = \mathscr{R}(B)$ and $\mathscr{N}(T) = \mathscr{N}(B)$.

In Theorem 1 we prove that this situation is precisely the notion of inverse along an element, which was introduced by X. Mary ([2]) in the more general context of semigroups. Here we adjust for the case of the Banach algebra $\mathscr{B}(X)$.

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DEFINITION 1. Let $A,T \in \mathscr{B}(X)$. We say that A is invertible along T if there exists $B \in \mathscr{B}(X)$ such that $B = BAB$ and

$$B = LT, \quad T = VB,$$

$$B = TU, \quad T = BW,$$

for some $L,U,V,W \in \mathscr{B}(X)$. In this case, we write $B = A^{\|T}$.

Recall $A \in \mathscr{B}(X)$ is group invertible if there exists $B \in \mathscr{B}(X)$ such that

$$A = ABA, \qquad B = BAB, \qquad AB = BA.$$

We write $B = A^{\sharp}$ for the group inverse of A. We have that A is group invertible if and only if $X = \mathscr{R}(A) \oplus \mathscr{N}(A)$.

In Section 2 we study the inverse of an operator along another operator, present matrix forms for such operators and give relations with group, Drazin and Moore-Penrose inverses.

2. Results

It is well known that any nonzero operator has a nonzero outer generalized inverse (see, for instance, [1, Theorem 1.1.9]). On the other hand, not every nonzero operator has a nonzero inner inverse: an operator $A \in \mathscr{B}(X)$ is inner regular if and only if $\mathscr{N}(A)$ and $\mathscr{R}(A)$ are closed and complemented subspaces of X ([1, Theorem 1.1.3 and Theorem 1.1.4]). In this case, let M and N be subspaces in X such that $X = N \oplus \mathscr{N}(A)$ and $X = \mathscr{R}(A) \oplus M$. Then A has the following matrix form:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} N \\ \mathscr{N}(A) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(A) \\ M \end{bmatrix},$$

where $A_1 \in \mathscr{B}(N, \mathscr{R}(A))$ is invertible. Furthermore, if $B \in \mathscr{B}(X)$ is an inner inverse of A such that $\mathscr{R}(BA) = N$ and $\mathscr{N}(AB) = M$, then B has the following matrix form:

$$B = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & W \end{bmatrix} : \begin{bmatrix} \mathscr{R}(A) \\ M \end{bmatrix} \to \begin{bmatrix} N \\ \mathscr{N}(A) \end{bmatrix},$$

where W is an arbitrary bounded linear operator from M to $\mathscr{N}(A)$. Note that this implies that an inner inverse is not unique even if we fix its range and nullspace.

Now we show that the inverse along an operator is an outer inverse with prescribed range and nullspace.

THEOREM 1. *Let $A,T \in \mathscr{B}(X)$ be nonzero operators. The following statements are equivalent:*

1. *B is the inverse of A along T.*

2. *B is an outer inverse of A such that $\mathscr{R}(B) = \mathscr{R}(T)$ and $\mathscr{N}(B) = \mathscr{N}(T)$.*

Proof. 1 $\Longrightarrow$ 2. Suppose A is invertible along T with inverse $B = A^{\|T}$. Then B is an outer inverse for A and there exist $L,U,V,W \in \mathscr{B}(X)$ such that

$$B = LT, \quad T = VB,$$

$$B = TU, \quad T = BW.$$

Then,

$$\mathscr{R}(T) = \mathscr{R}(BW) \subseteq \mathscr{R}(B) = \mathscr{R}(TU) \subseteq \mathscr{R}(T),$$

hence $\mathscr{R}(T) = \mathscr{R}(B)$. On the other hand,

$$\mathscr{N}(T) \subseteq \mathscr{N}(LT) = \mathscr{N}(B) \subseteq \mathscr{N}(VB) = \mathscr{N}(T),$$

and $\mathscr{N}(T) = \mathscr{N}(B)$.

2 $\Longrightarrow$ 1. Suppose that B is an outer inverse for A and T is such that $\mathscr{R}(B) = \mathscr{R}(T)$ and $\mathscr{N}(B) = \mathscr{N}(T)$. Since B is inner regular, there exist closed subspaces $M,N \subset X$ such that $X = N \oplus \mathscr{N}(B)$ and $X = \mathscr{R}(B) \oplus M$, and we have the following matrix form for B:

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} N \\ \mathscr{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B) \\ M \end{bmatrix}$$

with B_1 invertible. Also, since $\mathscr{R}(T) = \mathscr{R}(B)$ and $\mathscr{N}(B) = \mathscr{N}(T)$, we have that $\mathscr{N}(T)$ and $\mathscr{R}(T)$ are closed and complemented subspaces. Thus, T is inner regular and have the following matrix form with respect to the same decomposition of spaces as above:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} N \\ \mathscr{N}(B) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(B) \\ M \end{bmatrix}$$

with T_1 invertible. Now, let the operators $L,V \in \mathscr{B}(\mathscr{R}(B) \oplus M)$ and $U,W \in \mathscr{B}(N \oplus \mathscr{N}(B))$ be defined by $L = \begin{bmatrix} B_1T_1^{-1} & 0 \\ 0 & L_2 \end{bmatrix}$, $U = \begin{bmatrix} T_1^{-1}B_1 & 0 \\ 0 & U_2 \end{bmatrix}$, $V = \begin{bmatrix} T_1B_1^{-1} & 0 \\ 0 & V_2 \end{bmatrix}$ and $W = \begin{bmatrix} B_1^{-1}T_1 & 0 \\ 0 & W_2 \end{bmatrix}$, where $L_2, V_2 \in \mathscr{B}(M)$ and $U_2, W_2 \in \mathscr{B}(\mathscr{N}(B))$ are arbitrary operators. Then, a simple calculation shows that $LT = B$, $TU = B$, $VB = T$ and $BW = T$. $\square$

Note that the operators L,U,V,W in the proof of previous theorem are not necessarily unique. However, the uniqueness of the outer inverse with prescribed range and nullspace ([1, Theorem 1.1.10]) implies that the operator B is the unique inverse of A along T. On another side, T is not necessarily the only operator such that B is outer inverse of A along it. Indeed, the previous proof shows that any T of the form $\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}$ with T_1 invertible works.

EXAMPLE 1. Consider $X = \ell^2(\mathbb{N})$ the space of square-summable sequences, and let $A,T \in \mathscr{B}(X)$ be defined by

$$Ax := (\tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \tfrac{1}{4}x_4, \ldots),$$

$$Tx := (0, x_2, x_1, 0, 0, \ldots).$$

for $x = (x_1, x_2, x_3, x_4, \dots)$. Then A is invertible along T with $B = A^{\|T}$ defined by

$$Bx = (0, 2x_1, 3x_2, 0, 0, \dots),$$

as can be verified (by Definition 1) taking

$$\begin{aligned} Lx &= (0, 2x_3, 3x_2, 0, \dots), \\ Ux &= (3x_2 2x_1, 0, \dots), \\ Vx &= (0, \tfrac{1}{3}x_3, \tfrac{1}{2}x_2, 0, \dots), \\ Wx &= (x_1, x_2, 0 \dots,). \end{aligned}$$

Now, define an operator $S \in \mathscr{B}(X)$ by

$$Sx := (0, x_1, x_2, 0, 0, \dots),$$

so $\mathscr{R}(B) = \mathscr{R}(S)$ and $\mathscr{N}(B) = \mathscr{N}(S)$. By Theorem 1, A is invertible along S with $B = A^{\|S}$.

Now we give a characterization of the set of operators along which an operator A is invertible.

THEOREM 2. *Let $A, T \in \mathscr{B}(X)$ be nonzero operators. The following statements are equivalent.*

1. *A is invertible along T.*
2. *$\mathscr{R}(T)$ is closed and complemented subspace of X, $A(\mathscr{R}(T)) = \mathscr{R}(AT)$ is closed such that $\mathscr{R}(AT) \oplus \mathscr{N}(T) = X$ and the reduction $A|_{\mathscr{R}(T)} : \mathscr{R}(T) \to \mathscr{R}(AT)$ is invertible.*

Proof. Suppose A is invertible along T with $B = A^{\|T} \in \mathscr{B}(X)$. Then, from Theorem 1, B is an outer inverse for A such that $\mathscr{R}(B) = \mathscr{R}(T)$ and $\mathscr{N}(B) = \mathscr{N}(T)$. Since A is an inner inverse for B, $\mathscr{R}(B)$ and $\mathscr{N}(B)$ (and thus $\mathscr{R}(T)$ and $\mathscr{N}(T)$) are closed and complemented subspaces of X. Furthermore, $I - AB$ is a projection from X on $\mathscr{N}(B) = \mathscr{N}(T)$, thus $X = \mathscr{R}(AB) \oplus \mathscr{N}(T)$, and since $\mathscr{R}(AB) = A(\mathscr{R}(B)) = A(\mathscr{R}(T)) = \mathscr{R}(AT)$ we have that $\mathscr{R}(AT)$ is closed and $X = \mathscr{R}(AT) \oplus \mathscr{N}(T)$. Now, for the invertibility of $A|_{\mathscr{R}(T)} : \mathscr{R}(T) \to \mathscr{R}(AT)$ it is clear that it is onto. To see that $A|_{\mathscr{R}(T)}$ is also $1-1$ on $\mathscr{R}(T)$, suppose that there exists $x \in \mathscr{R}(T)$ such that $Ax = 0$. Since $x \in \mathscr{R}(T) = \mathscr{R}(B)$, there exists $y \in X$ such that $By = x$. Then $0 = Ax$ implies $0 = BAx = BABy = By$ and thus $x = 0$. Therefore $A_{\mathscr{R}(T)}$ is $1-1$ and onto, and hence invertible.

Conversely, suppose that $\mathscr{R}(T)$ and $\mathscr{N}(T)$ are closed and complemented subspaces of X, $X = \mathscr{R}(AT) \oplus \mathscr{N}(T)$, and the reduction $A|_{\mathscr{R}(T)} : \mathscr{R}(T) \to \mathscr{R}(AT)$ is invertible. Let M be a complement of $\mathscr{R}(T)$, so $X = \mathscr{R}(T) \oplus M$. Then A has the following matrix form with respect to these decompositions of spaces:

$$A = \begin{bmatrix} A_1 & A_3 \\ A_4 & A_2 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(T) \\ M \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(AT) \\ \mathscr{N}(T) \end{bmatrix}. \tag{1}$$

Since A maps $\mathscr{R}(T)$ onto $\mathscr{R}(AT)$ (with $A_1 = A|_{\mathscr{R}(T)}$ invertible), it follows that $A_4 = 0$. Now, let B be the operator defined by

$$B = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(AT) \\ \mathscr{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(T) \\ M \end{bmatrix}. \tag{2}$$

A direct verification shows that $BAB = B$, $\mathscr{R}(B) = \mathscr{R}(T)$ and $\mathscr{N}(B) = \mathscr{N}(T)$. Thus, by Theorem 1, B is the inverse of A along T. Therefore, A is invertible along T. □

If A is invertible along T, from the proof of the previous theorem we know that A has the matrix form of (1) with $A_4 = 0$. We claim that also $A_3 = 0$. Indeed, again from the proof of the theorem above, if B is the inverse of A along T, then B has the matrix form of (2). Since BA is the projection from X on $\mathscr{R}(B) = \mathscr{R}(T)$, from the matrix form

$$BA = \begin{bmatrix} I & A_1^{-1}A_3 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(T) \\ M \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(T) \\ M \end{bmatrix},$$

we see that $A_1^{-1}A_3 = 0$, and it follows that $A_3 = 0$. Hence, we have the following:

COROLLARY 1. *Let A be invertible along T. Then A has the following matrix form:*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(T) \\ M \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(AT) \\ \mathscr{N}(T) \end{bmatrix}, \tag{3}$$

with A_1 invertible.

From Theorem 2 we can get that AT is group invertible, as shown in the next theorem (see also [2, Theorem 7]).

THEOREM 3. *Let $A, T \in \mathscr{B}(X)$. The following statements are equivalent:*

1. *The operator A is invertible along T;*
2. *$\mathscr{R}(T) = \mathscr{R}(TA)$ and TA is group invertible;*
3. *$\mathscr{N}(T) = \mathscr{N}(AT)$ and AT is group invertible.*

Moreover, we have $A^{\parallel T} = (TA)^{\sharp}T = T(AT)^{\sharp}$.

Proof. We will prove (1) is equivalent to (2) and (3).

(1) $\Longrightarrow$ (2). Since A is invertible along T, there exist $L, V \in \mathscr{B}(X)$ such that $B = LT$ and $T = VB$. From $TAB = VBAB = VB = T$ we have $\mathscr{R}(T) \subseteq \mathscr{R}(TA)$, and since $\mathscr{R}(TA) \subset \mathscr{R}(T)$ always hold, we get $\mathscr{R}(T) = \mathscr{R}(TA)$. From $T = VB$ and $B = LT$ we have $\mathscr{N}(TA) = \mathscr{N}(VBA) \supseteq N(BA)$ and $\mathscr{N}(BA) = \mathscr{N}(LTA) \supseteq \mathscr{N}(TA)$, so $\mathscr{N}(BA) = \mathscr{N}(TA)$. Finally, since $\mathscr{R}(T) = \mathscr{R}(B) = \mathscr{R}(BA)$ and $X = \mathscr{R}(BA) \oplus \mathscr{N}(BA)$, we have $X = \mathscr{R}(TA) \oplus \mathscr{N}(TA)$, which implies TA is group invertible.

(2) $\Longrightarrow$ (1). Let $B = (TA)^{\sharp}T$. It is straightforward that $B = BAB$. Using Theorem 1, we will prove that $\mathscr{N}(B) = \mathscr{N}(T)$ and $\mathscr{R}(B) = \mathscr{R}(T)$.

We have $\mathscr{R}(B) = \mathscr{R}(BA) = \mathscr{R}((TA)^\sharp TA) = \mathscr{R}(TA) = \mathscr{R}(T)$.

Since $(TA)^\sharp TA$ is a projection on $R(TA) = R(T)$ we have

$$T = (TA)^\sharp(TA).$$

Then, $\mathscr{N}(B) = \mathscr{N}((TA)^\sharp T) \supseteq \mathscr{N}(T)$ and $\mathscr{N}(T) = \mathscr{N}(TA(TA)^\sharp T) \supseteq \mathscr{N}((TA)^\sharp T) = \mathscr{N}(B)$ implies $\mathscr{N}(B) = \mathscr{N}(T)$.

(1) $\implies$ (3). Since A is invertible along T, there exist $U, W \in \mathscr{B}(X)$ such that $B = TU$ and $T = BW$. From $BAT = BABW = BW = T$ we have $\mathscr{N}(T) = \mathscr{N}(BAT) \supseteq \mathscr{N}(AT)$, and since $\mathscr{N}(AT) \supseteq \mathscr{N}(T)$, we get $\mathscr{N}(AT) = \mathscr{N}(T)$. From $B = TU$ and $T = BW$ we have $\mathscr{R}(AT) = \mathscr{R}(ATU) \subseteq \mathscr{R}(AT)$ and $\mathscr{R}(AT) = \mathscr{R}(ABW) \subseteq \mathscr{R}(AB)$, thus $\mathscr{R}(AT) = \mathscr{R}(AB)$. Finally, since $\mathscr{N}(T) = \mathscr{N}(B) = \mathscr{N}(AB)$ and $X = \mathscr{R}(AB) \oplus \mathscr{N}(AB)$, we get $X = \mathscr{R}(AT) \oplus \mathscr{N}(AT)$, which implies AT is group invertible.

(3) $\implies$ (1). Let $B = T(AT)^\sharp$. A simple calculation shows that $B = BAB$. By Theorem 1 we have to show that $\mathscr{N}(B) = \mathscr{N}(T)$ and $\mathscr{R}(B) = \mathscr{R}(T)$.

From $\mathscr{N}(B) = \mathscr{N}(AB) = \mathscr{N}(AT(AT)^\sharp) = N(AT)$ we see $\mathscr{N}(B) = \mathscr{N}(T)$.

Since AT is group invertible and $\mathscr{N}(AT) = \mathscr{N}(T)$, we have $X = \mathscr{R}(AT) \oplus \mathscr{N}(T)$. Now, since $(AT)^\sharp(AT)$ is a projection onto $\mathscr{R}(AT)$ along $\mathscr{N}(AT)$ we have

$$T = T(AT)^\sharp(AT).$$

Then, $\mathscr{R}(B) = \mathscr{R}(T(AT)^\sharp) \subseteq \mathscr{R}(T) = \mathscr{R}(T(AT)^\sharp AT) \subseteq \mathscr{R}(T(AT)\sharp A) = \mathscr{R}(B)$. Therefore, $\mathscr{R}(T) = \mathscr{R}(B)$. $\square$

EXAMPLE 2. Consider X and $A, T \in \mathscr{B}(X)$ as in Example 1. Then AT is group invertible with $(AT)^\sharp x = (3x_2, 2x_1, 0, 0, \dots)$. Also, A is invertible along T and it is easy to check that $B = A^{\|T} = T(AT)^\sharp$.

However, group invertibility of AT is not sufficient for A to be invertible along T, additional conditions on the range or on the nullspace of T in Theorem 3 are needed (see also [2, Theorem 7]).

EXAMPLE 3. Let $X = \ell^2(\mathbb{N})$ and let $A, T \in \mathscr{B}(X)$ be defined by

$$Ax := (x_1, 2x_2, 0, 0, \dots),$$

$$Tx := (x_1, \tfrac{1}{2}x_2, \tfrac{1}{3}x_3, \dots)$$

for $x = (x_1, x_2, x_3, x_4, \dots)$. Then, we have

$$ATx = (x_1, x_2, 0, 0, ..).$$

Thus, AT is group invertible with $(AT)^\sharp = AT$. However, T is compact with infinite range and hence $\mathscr{R}(T)$ is not closed, which by Theorem 2 implies A is not invertible along T.

REMARK 1. In the matrix representation (3) of A given in Corollary 1, based on certain decomposition of the space X, we can replace the subspace M with $\mathscr{N}(TA)$. Indeed, it is easy to see that $M \subset \mathscr{N}(TA)$ (for any $m \in M$, $Am = A_2 m \in \mathscr{N}(T)$ that implies $TAm = 0$) and by group invertibility of TA (see Theorem 3 (ii)) it follows that $X = \mathscr{R}(TA) \oplus \mathscr{N}(TA)$ with $\mathscr{R}(T) = \mathscr{R}(TA)$.

Hence, if A is invertible along T, we have the following matrix representation of A, T and $A^{\|T}$ with respect to decompositions $X = \mathscr{R}(T) \oplus \mathscr{N}(TA) = \mathscr{R}(AT) \oplus \mathscr{N}(T)$:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(T) \\ \mathscr{N}(TA) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(AT) \\ \mathscr{N}(T) \end{bmatrix} \qquad (A_1 \text{ invertible}), \tag{4}$$

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(AT) \\ \mathscr{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(T) \\ \mathscr{N}(TA) \end{bmatrix} \qquad (T_1 \text{ invertible}), \tag{5}$$

and

$$A^{\|T} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathscr{R}(AT) \\ \mathscr{N}(T) \end{bmatrix} \to \begin{bmatrix} \mathscr{R}(T) \\ \mathscr{N}(TA) \end{bmatrix}. \tag{6}$$

It is clear that the inverse along an element generalizes the usual inverse, and it was proved in [3] that this inverse generalizes the group, Moore-Penrose and Drazin inverses.

Recall $A \in \mathscr{B}(X)$ is Drazin invertible if there exists $B \in \mathscr{B}(X)$ and an integer $k \in \mathbb{N}$ such that

$$B = BAB, \quad A^k = A^k BA, \quad AB = BA.$$

The Drazin inverse is unique if it exists, and will be denoted by A^d.

Let H be a Hilbert space and $\mathscr{B}(H)$ the algebra of bounded linear operators on H. Recall the Moore-Penrose inverse of $A \in \mathscr{B}(H)$ is an element $B \in \mathscr{B}(H)$ such that

$$ABA = A, \quad BAB = B, \quad (AB)^* = AB, \quad (BA)^* = BA.$$

The Moore-Penrose inverse is unique if it exists, and will be denoted by $A^\dagger$.

Using Theorem 1, Theorem 2 and Corollary 1 we get at once the following (see also [3, Theorem 11]):

COROLLARY 2. *Let A be invertible along T with $B = A^{\|T}$.*

1. *If T is invertible, $B = A^{-1}$.*
2. *If $\mathscr{R}(T) = \mathscr{R}(A)$, $\mathscr{N}(T) = \mathscr{N}(A)$, then $B = A^\sharp$.*
3. *If $\mathscr{R}(T) = \mathscr{R}(A^k)$, $\mathscr{N}(T) = \mathscr{N}(A^k)$ for some $k \in \mathbb{N}$, then $B = A^d$.*
4. *If $A, T \in \mathscr{B}(H)$, $\mathscr{R}(T) = \mathscr{R}(A^*)$, $\mathscr{N}(T) = \mathscr{N}(A^*)$, then $B = A^\dagger$.*

For instance, for a group invertible operator A, since $\mathscr{R}(A) = \mathscr{R}(A^\sharp) = \mathscr{R}(A^\sharp A) = \mathscr{R}(AA^\sharp)$ and $\mathscr{N}(A) = \mathscr{N}(A^\sharp) = \mathscr{N}(A^\sharp A) = \mathscr{N}(AA^\sharp)$, we have that the inverse of A along A, $A^\sharp$, $A^\sharp A$ or $AA^\sharp$ is its group inverse.

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