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SOME RESULTS ON ABSOLUTE CONTINUITY FOR UNBOUNDED JACOBI MATRICES

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Abstract. This brief paper presents several results which offer a useful approach for studying the absolute continuity of spectral measures associated with some Jacobi matrix operators. The operators are modeled as multiplication operators on a dense domain of a function space with a polynomial basis. Properties of these polynomials and commutator equations are used to obtain results on absolute continuity.

1. Introduction

This paper looks at the spectral properties of special subclasses of tridiagonal matrix operators with subdiagonal sequence $\{a_n\}$ and diagonal sequence $\{b_n\}$ satisfying basic assumptions $a_n > 0$, and b_n real. These operators, known as Jacobi operators, have the following form on the indicated maximal domain:

$$C = \begin{bmatrix} b_1 & a_1 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & \dots \\ 0 & 0 & a_3 & b_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$$D_C = \left\{x\varepsilon\ell^2 : Cx\varepsilon\ell^2\right\}$$

A matrix operator of this type will be self-adjoint if Carleman's condition $\sum \frac{1}{a_n} = \infty$ holds. In addition, Berezanskii [1] showed that C is self-adjoint if either $\{a_{n-1}+b_n+a_n\}$ or $\{a_{n-1} - b_n + a_n\}$ is a bounded sequence. In this paper it will be assumed that the diagonal entries of C are zeroes. The results are generally determined by properties of the difference sequence $\{d_n\}$ defined by $d_n = a_n - a_{n-1}$ with $a_0 = 0$. The conditions imposed guarantee that the resulting operators are self-adjoint, which means that they can be modeled as multiplication operators on appropriate function spaces. If $\{\phi_n\}$ denotes the standard basis for ℓ^2 then, since $a_n > 0$, ϕ_1 is a cyclic vector for the corresponding operator. It follows from the Spectral Theorem that if $C = \int \lambda dE_\lambda$ and if

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$\mu(\beta) = \|E(\beta)\phi_1\|^2$ for any Borel set β, then the matrix operator C is unitarily equivalent to a multiplication operator $M_x : D \to \mathscr{L}^2(\mu)$ defined on a dense subset of D of $\mathscr{L}^2(\mu)$ by $M_x f = xf(x)$. The domain D_C includes all finite linear combinations of the basis vectors $\{\phi_n\}$. The goal is to study the properties of the measure μ. From the tridiagonal structure it follows that the standard basis vectors $\{\phi_n\}$ in ℓ^2 correspond to a sequence of polynomials in $L^2(\mu)$ defined by

$$P_1(x) = 1, \quad P_2(x) = \frac{x - b_1}{a_1}$$

$$P_{n+1}(x) = \frac{(x - b_n)P_n(x) - a_{n-1}P_{n-1}(x)}{a_n}.$$

In the case of a zero diagonal the corresponding polynomials $\{P_n\}$ satisfy the condition $P_n(-x) = (-1)^{n+1}P_n(x)$. It is known that the spectral measure is symmetric about the origin. This symmetry will be used in the results that follow. The proofs of the results on absolute continuity highly depend on properties of the polynomial sequence $\{P_n\}$.

There are a number of papers in the literature, employing a variety of techniques, which study the spectral properties of Jacobi matrices, based on properties of the defining weight sequences in both the bounded and unbounded cases. In this paper commutator equations will play a key role in obtaining results on the absolute continuity of the spectral measure.

2. The commutator equation approach

The following theorem from [2] is the basis for the results in this paper. It extends to unbounded operators a result due to Putnam (See [6]) for bounded operators. The theorem provides an inequality that is sufficient for establishing results on absolute continuity.

THEOREM. *Let C be a cyclic self-adjoint operator with cyclic vector ϕ, and spectral resolution $C = \int \lambda dE_\lambda$. Let I be an interval. Suppose there exists a bounded self-adjoint operator J and positive constants q and Q such that if $CJ - JC = -iK$, and for any bounded subinterval Δ of I,*

$$\langle JE(\Delta)\phi, CE(\Delta)\phi\rangle - \langle CE(\Delta)\phi, JE(\Delta)\phi\rangle = -i\langle KE(\triangle)\phi, E(\Delta)\phi\rangle$$

with

$$(Q|\Delta|)\|E(\Delta)\phi\|^2 \geqq |\langle KE(\Delta)\phi, E(\Delta)\phi_1\rangle| \geqq q\|E(\Delta)\phi\|^4$$

where $|\Delta|$ denotes the Lebesgue measure of Δ. Then the spectral measure of C is absolutely continuous on I.

Proof. If ϕ is a cyclic vector for C, the spectral measure of C is given by $\mu(\beta) = \|E(\beta)\phi\|^2$ for any Borel set β. If Δ is a bounded subinterval of I and $\|E(\Delta)\phi\|^2 \neq 0$, it follows from the given inequalities that $\mu(\Delta) = \|E(\Delta)\phi\|^2 \leqslant \frac{Q}{q}\ |\Delta|$. This inequality can then be extended to Borel subsets of I, from which the result follows. $\square$

In this paper the above theorem will mainly be used to study the spectral measure of an unbounded tridiagonal matrix operator C by choosing the related bounded operator J to be the imaginary part of the unilateral shift operator. If

$$J = \frac{1}{2i}\begin{bmatrix} 0 & -1 & 0 & 0 & \dots \\ 1 & 0 & -1 & 0 & \dots \\ 0 & 1 & 0 & -1 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

the commutator K, defined by $CJ - JC = -iK$ will be five diagonal. Structural properties of five diagonal matrices from [2] will be used to establish the inequalities needed to apply the theorem. Of course the theorem also applies if the matrix operator C is bounded. In this case a suitable choice for J may be:

$$J = \frac{1}{2i}\begin{bmatrix} 0 & -a_1 & 0 & 0 & \cdots \\ a_1 & 0 & -a_2 & 0 & \cdots \\ 0 & a_2 & 0 & -a_3 & \cdots \\ 0 & 0 & a_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and, in this case, the commutator K is diagonal.

The next Lemma shows that if, for a given self-adjoint tridiagonal matrix operator C, the related self-adjoint operator J is chosen to be bounded, then the upper bound for $|\langle KE(\Delta)\phi, E(\triangle)\phi\rangle|$ in the previous theorem can always be found. Results on absolute continuity will then depend on establishing the needed lower bound.

LEMMA. *Let C be a self-adjoint operator with cyclic vector ϕ and spectral resolution $C = \int \lambda dE_\lambda$. Let J be a bounded self-adjoint operator. If $CJ - JC = -iK$, and if for some bounded subinterval Δ,*

$$\langle JE(\Delta)\phi, CE(\Delta)\phi\rangle - \langle CE(\Delta)\phi, JE(\Delta)\phi\rangle = -i\langle KE(\triangle)\phi, E(\Delta)\phi\rangle$$

then $|\langle KE(\Delta)\phi, E(\Delta)\phi\rangle| \leqslant \|J\|(|\Delta|)\|E(\Delta)\phi\|^2$.

Proof. Let λ be the midpoint of Δ. Then

$$\langle JE(\Delta)\phi, (C-\lambda I)E(\Delta)\phi\rangle - \langle (C-\lambda I)E(\Delta)\phi, JE(\Delta)\phi\rangle = -i\langle KE(\triangle)\phi, E(\Delta)\phi\rangle$$

and since $\|(C-\lambda I)E(\Delta)\phi\| = |\int_\Delta (x-\lambda)d\|E_x\phi\|^2| \leqslant \frac{1}{2}|\Delta|\|E(\Delta)\phi\|^2$ it follows that

$$|\langle KE(\Delta)\phi, E(\Delta)\phi\rangle| \leqslant 2\|J\|\left(\frac{1}{2}|\Delta|\right)\|E(\Delta)\phi\|^2. \quad \square$$

The next lemma from [4] plays a key role in the results tht follow. Note that if the self-adjoint matrix operator $C = \int \lambda dE_\lambda$ is viewed as a multiplication operator on $\mathscr{L}^2(\mu)$ with $\mu(\beta) = \|E(\beta)\phi_1\|^2$ for any Borel set β, then the basis vectors $\{\phi_n\}$ correspond to the polynomials $\{P_n(x)\}$ described above, and $\langle E(\beta)\phi_1, \phi_n\rangle = \int_\beta P_n d\mu$.

LEMMA. *Let C be a self adjoint Jacobi matrix defined by the sequence $\{a_n\}$, with $a_n > 0$. Assume $b_n = 0$. If $C = \int \lambda dE_\lambda$, and if $\triangle$ is a subinterval of $(0,\infty)$ then $\sum_{n=1}^{\infty} |\langle E(\triangle)\phi_1, \phi_{2n-1}\rangle|^2 = \sum_{n=1}^{\infty} |\langle E(\triangle)\phi_1, \phi_{2n}\rangle|^2$.*

Proof. Since Δ and $-\Delta$ are disjoint intervals, the corresponding spectral projections $E(\Delta)$ and $E(-\Delta)$ are orthogonal. Thus $\langle E(\Delta)\phi_1, E(-\Delta)\phi_1\rangle = 0$. But $E(\Delta)\phi_1 = \sum_{n=1}^{\infty}\langle E(\Delta)\phi_1, \phi_n\rangle\phi_n = \sum_{n=1}^{\infty}(\int_\Delta P_n d\mu)\phi_n$, and similarly, $E(-\Delta)\varphi_1 = \sum_{n=1}^{\infty}\langle E(-\Delta)\phi_1, \phi_n\rangle = \sum_{n=1}^{\infty}(\int_{-\Delta} P_n d\mu)\varphi_n$. Since $\langle E(\Delta)\phi_1, E(-\Delta)\phi_1\rangle = 0$, and $P_n(-x) = (-1)^{n+1}P_n(x)$ it follows that

$$0 = \langle E(\Delta)\phi_1, E(-\Delta)\phi_1\rangle = \sum_{n=1}^{\infty} |\langle E(\triangle)\varphi_1, \phi_{2n-1}\rangle|^2 - \sum_{n=1}^{\infty} |\langle E(\triangle)\phi_1, \phi_{2n}\rangle|^2.$$

If the matrix operator C is unbounded, as is the case when $lim a_n = \infty$, and if the related operator J is the imaginary part of the unilateral shift, then the commutator K formally defined by $CJ - JC = -iK$ is five diagonal. The next lemma from [2] provides a sufficient condition for K to be self-adjoint. Note that the equation $CJ - JC = -iK$ holds on finite linear combinations of the basis vectors. $\square$

LEMMA. *Assume that the real infinite matrix $K = [k_{ij}]$ has the following structure: $k_{ii} = s_i$, $k_{i,i+2} = k_{i+2,i} = t_i$, with all other entries equal to zero. If the sequence $\{t_i\}$ is bounded or if for all $i \geqslant N$, $|t_i| + |t_{i+1}| > 0$ and $\sum_{i=N}^{\infty} \frac{1}{|t_i|+|t_{i+1}|} = \infty$, then K, defined on the maximal domain $D_K = \left\{x \in \ell^2 : Kx \in \ell^2\right\}$, is self-adjoint.*

The goal of the following sections is to use properties of the commutator K, together with properties of the related sequence of polynomials $\{P_n\}$, to establish the lower bound needed to apply Theorem 2.1 to obtain results on absolute continuity.

3. Main results

THEOREM. *Let C be a self-adjoint Jacobi matrix with $b_n = 0$, $a_n > 0$, $\lim a_n = \infty$. Let $d_n = a_n - a_{n-1}$ with $a_0 = 0$. Suppose there exists a non-negative constant s such that $d_1 - \frac{1}{2}d_2 \geqslant s$, $d_{2n+1} - \frac{1}{2}d_{2n} - \frac{1}{2}d_{2n+2} \geqslant s$, for $n \geqslant 1$, and $d_2 - \frac{1}{2}d_3 + s \geqslant 0$, $d_{2n} - \frac{1}{2}d_{2n-1} - \frac{1}{2}d_{2n+1} + s \geqslant 0$ for $n > 1$. If $d_1 - \frac{1}{2}d_2 - s > 0$ or $d_2 - \frac{1}{2}d_3 + s > 0$ then the spectral measure of C is absolutely continuous on $(-\infty,0)\bigcup(0,\infty)$.*

Proof. Choose J to be the imaginary part of the unilateral shift. Then $CJ - JC = -iK$ where K has the following form:

$$K = \begin{bmatrix} d_1 & 0 & \frac{1}{2}d_2 & 0 & \dots \\ 0 & d_2 & 0 & \frac{1}{2}d_3 & \dots \\ \frac{1}{2}d_2 & 0 & d_3 & 0 & \dots \\ 0 & \frac{1}{2}d_3 & 0 & d_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Let $\triangle$ be a subinterval of $(0,\infty)$. Let $x = E(\triangle)\phi_1$, and $x_n = \langle E(\Delta)\phi_1, \phi_n\rangle = \int_\triangle P_n d\mu$. Then $x = \sum x_n\phi_n$. It follows that $\langle Kx, x\rangle = \sum_{n=1}^{\infty} d_n|x_n|^2 + \sum_{n=2}^{\infty} d_n x_{n-1}x_{n+1}$.

Since

$$d_n|x_{n+1}x_{n-1}| \leqslant \frac{1}{2}d_n\left[\left|\int_\Delta P_{n-1}d\mu\right|^2 + \left|\int_\triangle P_{n+1}d\mu\right|^2\right],$$

it follows that $\langle Kx,x\rangle \geqslant \sum_{n=1}^\infty d_n|x_n|^2 - \sum_{n=2}^\infty \frac{1}{2}d_n|x_{n-1}|^2 - \frac{1}{2}\sum_{n=2}^\infty d_n|x_{n+1}|^2$.

Thus

$$\langle Kx,x\rangle \geqslant \left(d_1 - \frac{1}{2}d_2\right)|x_1|^2 + \left(d_2 - \frac{1}{2}d_3\right)|x_2|^2 + \sum_{n=1}^\infty\left(d_{2n+1} - \frac{1}{2}d_{2n} - \frac{1}{2}d_{2n+2}\right)|x_{2n+1}|^2 + \sum_{n=2}^\infty\left(d_{2n} - \frac{1}{2}d_{2n-1} - \frac{1}{2}d_{2n+1}\right)|x_{2n}|^2$$

Since $-s\sum_{n=1}^\infty |x_{2n+1}|^2 + s\sum_{n=1}^\infty |x_{2n}|^2 = 0$, the assumption $d_1 - \frac{1}{2}d_2 - s > 0$ implies that $\langle Kx,x\rangle \geqslant (d_1 - \frac{1}{2}d_2 - s)|x_1|^2$ for. Thus

$$|\langle Kx,x\rangle| \geqslant \left(d_1 - \frac{1}{2}d_2 - s\right)\|E(\Delta)\phi_1\|^4$$

which establishes the lower bound needed to apply the Theorem above to prove absolute continuity on $(-\infty,0)\bigcup(0,\infty)$. If $d_2 - \frac{1}{2}d_3 + s > 0$ then $\langle Kx,x\rangle \geqslant (d_2 - \frac{1}{2}d_3 + s)|x_2|^2 = (d_2 - \frac{1}{2}d_3 + s)|\int_\triangle \frac{\lambda}{a_1}d\mu|^2$. Thus if $\triangle \subset (\alpha,\infty)$, it follows that $\langle Kx,x\rangle \geqslant (d_2 - \frac{1}{2}d_3 + s)\frac{\alpha^2}{a_1^2}\|E(\triangle)\phi_1\|^4$, which establishes absolute continuity on (α,∞) for every $\alpha > 0$. $\square$

A very similar approach can be used to establish the next result.

THEOREM. *Let C be self-adjoint Jacobi matrix with $b_n = 0$, $a_n > 0$, $\lim a_n = \infty$. Let $d_n = a_n - a_{n-1}$ with $a_0 = 0$. If there exists a non-negative constant s such that $d_1 - \frac{1}{2}d_2 + s > 0$, $d_{2n+1} - \frac{1}{2}d_{2n} - \frac{1}{2}d_{2n+2} + s \geqslant 0$, $d_{2n} - \frac{1}{2}d_{2n-1} - \frac{1}{2}d_{2n+1} \geqslant s$, then the spectral measure of C is absolutely continuous on $(-\infty,0)\bigcup(0,\infty)$.*

Proof. Choose J to be the imaginary part of the unilateral shift. Let K be the commutator defined by $CJ - JC = -iK$. If Δ is a subinterval of $(0,\infty)$ and $x = E(\triangle)\phi_1$ then $x = \sum x_n\phi_n$ and $\langle Kx,x\rangle = \sum_{n=1}^\infty d_n|x_n|^2 + \sum_{n=2}^\infty d_n x_{n-1}x_{n+1}$. Using the same estimates as in the previous proof, it follows that $\langle Kx,x\rangle \geqslant \sum_{n=1}^\infty d_n|x_n|^2 - \sum_{n=2}^\infty \frac{1}{2}d_n|x_{n-1}|^2 - \sum_{n=2}^\infty \frac{1}{2}|x_{n+1}|^2$. Thus

$$\langle Kx,x\rangle \geqslant \left(d_1 - \frac{1}{2}d_2\right)|x_1|^2 + \left(d_2 - \frac{1}{2}d_3\right)|x_2|^2 + \sum_{n=1}^\infty\left(d_{2n+1} - \frac{1}{2}d_{2n} - \frac{1}{2}d_{2n+2}\right)|x_{2n+1}|^2 + \sum_{n=2}^\infty\left(d_{2n} - \frac{1}{2}d_{2n-1} - \frac{1}{2}d_{2n+1}\right)|x_{2n}|^2.$$

Since $s\sum_{n=1}^\infty |x_{2n+1}|^2 - s\sum_{n=1}^\infty |x_{2n}|^2 = 0$, the assumptions of the theorem imply that $\langle Kx,x\rangle \geqslant (d_1 - \frac{1}{2}d_2 + s)|x_1|^2$. Thus it follows that

$$|\langle Kx,x\rangle| \geqslant \left(d_1 - \frac{1}{2}d_2 + s\right)\|E(\triangle)\phi_1\|^4$$

which provides the lower bound needed to prove absolute continuity on $(-\infty,0)\bigcup(0,\infty)$. $\square$

COROLLARY. *Let C be a self-adjoint Jacobi matrix with $b_n = 0$, and $a_n > 0$ defined so that $d_{2n-1} = a_{2n-1} - a_{2n-2} = A$, $d_{2n} = a_{2n} - a_{2n-1} = B$, $n = 1,2,\ldots.$ with $A \geqslant B$. Then C is absolutely continuous on $(-\infty,0)\bigcup(0,\infty)$.*

Proof. Let $s = A - B$ in Theorem 3.1. $\square$

COROLLARY. *Let C be a self-adjoint Jacobi matrix with $b_n = 0$ and a_n defined so that $d_1 - \frac{1}{2}d_2 > 0$, $d_2 - \frac{1}{2}d_3 > 0$, $d_n - \frac{1}{2}d_{n-1} - \frac{1}{2}d_{n+1} \geqslant 0$, for $n > 2$. Then C is absolutely continuous on $(-\infty,0)\bigcup(0,\infty)$.*

Proof. Choose $s = 0$ in Theorem 3.1. $\square$

It is also interesting to apply the technique of the previous theorem to the bounded case.

THEOREM. *Let C be a bounded self-adjoint Jacobi matrix operator with $b_n = 0$, and $a_n > 0$. Suppose there exists a non-negative constant s such that $a_1^2 > s$, $a_{2n+1}^2 - a_{2n}^2 \geqslant s$, $a_{2n}^2 - a_{2n-1}^2 + s \geqslant 0$. Then C is absolutely continuous on $(-\infty,0)\bigcup(0,\infty)$.*

Proof. Choose

$$J = \frac{1}{2i}\begin{bmatrix} 0 & -a_1 & 0 & 0 & \cdots \\ a_1 & 0 & -a_2 & 0 & \cdots \\ 0 & a_2 & 0 & -a_3 & \cdots \\ 0 & 0 & a_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Then $CJ - JC = -iK$ where

$$K = \begin{bmatrix} a_1^2 & 0 & 0 & 0 & \cdots \\ 0 & a_2^2 - a_1^2 & 0 & 0 & \cdots \\ 0 & 0 & a_3^2 - a_2^2 & 0 & \cdots \\ 0 & 0 & 0 & a_4^2 - a_3^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Let Δ be a subinterval of $(0,\infty)$. Let $x = E(\triangle)\phi_1$ and $x_n = \langle E(\Delta)\phi_1, \phi_n\rangle$. Then $x = \sum_{n=1}^{\infty} x_n\phi_n$. It follows that $\langle Kx,x\rangle = a_1^2|x_1|^2 + \sum_{n=1}^{\infty}(a_{n+1}^2 - a_n^2)|x_{n+1}|^2$. Since $-s\sum_{n=1}^{\infty}|x_{2n+1}|^2 + s\sum_{n=1}^{\infty}|x_{2n}|^2 = 0$, the assumptions imply that $\langle Kx,x\rangle \geqslant (a_1^2 - s)|x_1|^2$. It follows that $\langle Kx,x\rangle \geqslant (a_1^2 - s)\|E(\triangle)\phi_1\|^4$ whch establishes the lower bound needed to apply the theorem above to establish the absolute continuity of the spectral measure on $(-\infty,0)\bigcup(0,\infty)$. $\square$

4. Examples

4.1. Bounded case

- Theorem 3.5 can be applied to bounded periodic Jacobi matrices. If $a_{2n-1}=A$, $A>0$, and $a_{2n}=B$, $B>0$, with $A>B$, let $s=A^2-B^2$.

- Choose a_{2n} to be a positive increasing bounded sequence. For some $C>0$ choose $a_{2n-1}=\sqrt{a_{2n}^2+C}$. Then $a_{2n+1}^2-a_{2n}^2=a_{2n+2}^2-a_{2n}^2+C$, and $a_{2n}^2-a_{2n-1}^2+C=0$.

4.2. Unbounded case

- Choose a_n to be a positive sequence such that the difference sequence $d_n=a_n-a_{n-1}$ with $a_0=0$ satisfies the conditions $d_{2n-1}=d$, $d>0$, $d_{2n}=\delta$, $\delta\geqslant 0$, with $d>\delta$. Apply Theorem 3.1 with $s=d-\delta$.

- Choose $0<\omega<1$, $\rho>0$. Let $d_{2n-1}=1+\omega+\rho$, $d_{2n}=1-\omega$. Such operators were considered in [3], which established the existence of the spectral gap $(-\omega-\frac{\rho}{2},\omega+\frac{\rho}{2})$ in the essential spectrum. Theorem 3.1 with $s=2\omega+\rho$ can be used to prove absolute continuity.

- Choose $d_{2n}=1-\frac{1}{n+1}$, $d_{2n-1}=1+\frac{1}{2}(1-\frac{1}{n})+\frac{1}{2}(1-\frac{1}{n+1})$. Then $d_{2n-1}-\frac{1}{2}d_{2n}-\frac{1}{2}d_{2n-2}=1$, and $d_{2n}-\frac{1}{2}d_{2n-1}-\frac{1}{2}d_{2n+1}=-1+\frac{2}{4n(n+1)(n+2)}$. Apply Theorem 3.1 with $s=1$.

- In general, choose the positive difference sequence d_n so that $d_{2n}\to A$, $d_{2n}-\frac{1}{2}d_{2n-2}-\frac{1}{2}d_{2n+2}\geqslant 0$, with $d_0=0$. Choose $C>0$, and let $d_1=\frac{1}{2}d_2+C$, $d_{2n+1}=\frac{1}{2}d_{2n}+\frac{1}{2}d_{2n+2}+C$. Apply Theorem 3.1 with $s=C$. Note that $d_{2n}-\frac{1}{2}d_{2n-1}-\frac{1}{2}d_{2n+1}=d_{2n}-\frac{1}{2}(C+\frac{1}{2}d_{2n-2}+\frac{1}{2}d_{2n})-\frac{1}{2}(C+\frac{1}{2}d_{2n}+\frac{1}{2}d_{2n+2})=-C+\frac{1}{2}(d_{2n}-\frac{1}{2}d_{2n-2}-\frac{1}{2}d_{2n+2})$

- Choose the positive difference sequence d_n so that $d_{2n}-\frac{1}{2}d_{2n-2}-\frac{1}{2}d_{2n+2}\geqslant 0$ with $d_2>0$, $d_0=0$. Choose $C<0$, $|C|<\frac{1}{2}d_2$. Let $d_1=\frac{1}{2}d_2+C$, $d_{2n+1}=\frac{1}{2}d_{2n}+\frac{1}{2}d_{2n+2}+C$. Apply Theorem 3.2 with $s=-C$. Note that $d_{2n}-\frac{1}{2}d_{2n-1}-\frac{1}{2}d_{2n+1}=d_{2n}-\frac{1}{2}(C+\frac{1}{2}d_{2n-2}+\frac{1}{2}d_{2n})-\frac{1}{2}(C+d_{2n}+d_{2n+2})=-C+\frac{1}{2}(d_{2n}-\frac{1}{2}d_{2n-2}-\frac{1}{2}d_{2n})$.

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