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ON GENERALIZED DERIVATION IN BANACH SPACES

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Abstract. In this paper we generalized two important results of B. P. Duggal [4, Theorem 2.1 and 2.6], and other results are also given. If $B(\mathscr{X})$ is the algebra of all bounded linear operators on a complex Banach space $\mathscr{X}$ and $J(\mathscr{X})=\{x\in B(\mathscr{X}): x=x_1+ix_2$, where x_1 and x_2 are hermitian$\}$, two results of orthogonality in the sense of Birkhoff are shown $\|a+b\|\leqslant\|a+b-[x^*,x]\|$ and $\|ab\|\leqslant\|ab-[xx^*,x^*x]\|$ for all $x\in J(\mathscr{X})\cap\delta_{a,b}^{-1}(0)$. As application of our first result the William's theorem "Any hermitian element is finite element" is also established with a shorter and simpler proof.

1. Introduction

Let $\mathscr{X}$ be a separable infinite dimensional complex Banach space, and $B(\mathscr{X})$ denote the algebra of all bounded linear operators on $\mathscr{X}$. In general we define the generalized derivation on $B(\mathscr{X})$ by $\delta_{a,b}x = ax - xb$, the particular case $\delta_a x = \delta_{a,a}x = ax - xa$ is the internal derivation induced by $a \in B(\mathscr{X})$, we define also the elementary operator $\Delta_{a,b}x = axb - x$ for any a,b and x in $B(\mathscr{X})$.

Evidently if a and b are two elements in $B(\mathscr{X})$ such that $a = a_1 + ia_2$, $b = b_1 + ib_2$ then $\delta_{a,b} = \delta_{a_1,b_1} + i\delta_{a_2,b_2}$.

One consider $J(\mathscr{X})$ the algebra of all bounded linear operators x which has the complex representation $x = x_1 + ix_2$, where x_1 and x_2 are hermitian, it's well to recal that $h \in B(\mathscr{X})$ is hermitian if the algebra numerical range

$$V(B(\mathscr{X}),h) = \{f(h) : f \in B(\mathscr{X})^*, f(I) = 1 = \|f\|\}$$

is a subset of the set of reals [Bonsall 3, page 8].

It's easy to prove that each $x \in J(\mathscr{X})$ has a unique complex representation.

We may define also the continuous linear involution on $J(\mathscr{X})$ the mapping

$$x \longrightarrow x^* \text{ by } x^* = x_1 - ix_2, \forall x \in J(\mathscr{X}) \text{ where } x = x_1 + ix_2.$$

Our main results in this paper are two inequalities which give us the notion of orthogonality in sense of Birkhoff

$$\|a+b\| \leqslant \|a+b-[x^*,x]\|$$

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and

$$\|ab\| \leqslant \|ab - [xx^*, x^*x]\|,$$

forall $x \in J(\mathscr{X}) \cap \delta_{a,b}^{-1}(0)$, where $[x,y] = xy - yx, \forall x,y \in B(\mathscr{X})$.

Orthogonality in the sense of Birkhoff is defined as follows x is orthogonal to y in a complex Banach space $\mathscr{X}$ if for all complex λ there holds

$$\|x\| \leqslant \|x + \lambda y\|,$$

this definition has a natural geometric interpretation. Namely, x is orthogonal to y if and only if the complex line $\{x + \lambda y; \lambda$ is a complex number$\}$ is disjoint with the open ball $K(0, \|x\|)$, i.e. if and only if this complex line is a tangent line to $K(0, \|x\|)$. Note that if x is orthogonal to y, then y need not be orthogonal to x. If $\mathscr{X}$ is a Hilbert space, then $\|x\| \leqslant \|x + \lambda y\|$, implies that $\langle x, y\rangle = 0$, i.e. orthogonality in the usual sense.

A simple application of the first result gives us a very nice, simpler and shorter proof of the William's theorem "Any hermitian element in $B(\mathscr{X})$ is finite element", and in the theorem 3.5 we give a new invertibility criterion for the elements of the range of generalized derivation, which gives us a very good applications of our results.

2. Preliminaries

THEOREM 2.1. (12, Corollary 8) *Let $\{T_n\}$ be a sequence of commuting normal operators on a complex Banach space $\mathscr{X}$. Then*

$$\Big(\bigcap_{k=1}^{\infty} N(T_k)\Big) \perp \overline{\sum_{k=1}^{\infty} R(T_k)}.$$

If the space $\mathscr{X}$ is reflexive, then

$$\mathscr{X} = \Big(\bigcap_{k=1}^{\infty} N(T_k)\Big) \oplus^{\perp} \overline{\sum_{k=1}^{\infty} R(T_k)},$$

where $N(T_k)$ and $R(T_k)$ are respectively the kernel and range of T_k.

THEOREM 2.2. (B. P. Duggal 4, Th 2.1) *If $J(\mathscr{X})$ is an algebra and $\delta_a^{-1}(0) \subseteq \delta_{a^*}^{-1}(0)$ for some $a \in J(\mathscr{X})$, then $\|a\| \leqslant \|a - [x^*, x]\|$, for all $x \in J(\mathscr{X}) \cap \delta_a^{-1}(0)$.*

THEOREM 2.3. (B. P. Duggal 4, Th 2.6) *Assume that $\triangle_a^{-1}(0) \subseteq \triangle_{a^*}^{-1}(0)$. If $a \in B(H)$ (resp. $a \in \mathscr{C}_p$ the Schatten p-classes), then $\|a\| \leqslant \|a - [|x|, |x^*|]\|$ for all $x \in B(H) \cap \triangle_a^{-1}(0)$ (resp. $\|a\|_p \leqslant \|a - [|x|, |x^*|]\|_p$ for all $x \in \mathscr{C}_p \cap \triangle_a^{-1}(0)$).*

3. Main results

Let $\mathscr{X}$ be a complex Banach space, $B(\mathscr{X})$ denote the algebra of all bounded linear operators on $\mathscr{X}$, $a, b \in B(\mathscr{X})$, and $\{a\}', \{b\}'$ the commutant of a, and b respectively.

$$\{a\}' = \{x \in B(\mathscr{X}) : ax = xa\}$$

and

$$\{b\}' = \{x \in B(\mathscr{X}) : bx = xb\}.$$

THEOREM 3.1. *If $J(\mathscr{X})$ is a sub algebra of $B(\mathscr{X})$ and if*
(i) $\{a,b\} \subseteq J(\mathscr{X})$,
(ii) $\delta_{a,b}^{-1}(0) \subseteq \delta_{a^*,b^*}^{-1}(0)$,
(iii) $x \in J(\mathscr{X}) \cap \delta_{a,b}^{-1}(0) \cap \delta_a^{-1}(0) \cap \delta_b^{-1}(0)$,
then
(iv) $\|a+b\| \leqslant \|a+b-[x^*,x]\|$.

REMARK 3.2. The result (iv) also holds if the condition (iii) is replaced by
(iii)' $x_1, x_2 \in \delta_{(b-a),(a-b)}^{-1}(0)$, where $x = x_1 + ix_2$.

Proof. Let $x \in J(\mathscr{X}) \cap \delta_{a,b}^{-1}(0)$, then $\delta_{a^*,b^*}(x) = \delta_{a,b}^*(x) = 0$.
We have

$$\begin{aligned} \delta_{a,b}(x) = 0 &\iff (a+b)x - x(a+b) + xa - bx = 0 \\ &\iff \delta_x(a+b) + \delta_{b,a}(x) = 0, \end{aligned}$$

hence

$$\delta_{x_1}(a+b) + i\delta_{x_2}(a+b) + \delta_{b,a}(x) = 0. \tag{3.1}$$

$$\begin{aligned} \delta_{a,b}^*(x) = 0 &\iff a^*x - xb^* = 0 \\ &\iff ax^* - x^*b = 0 \\ &\iff -(bx^* - x^*a) = 0 \\ &\iff \delta_{a,b}(x^*) = 0, \end{aligned}$$

hence by (3.1)

$$\begin{aligned} \delta_{a,b}^*(x) = 0 &\iff \delta_{x^*}(a+b) + \delta_{a,b}(x^*) = 0 \\ &\iff \delta_{x_1}(a+b) - i\delta_{x_2}(a+b) + \delta_{a,b}(x^*) = 0. \end{aligned} \tag{3.2}$$

If $x \in \delta_a^{-1}(0) \cap \delta_b^{-1}(0)$ then $\delta_{b,a}(x) = \delta_{a,b}(x^*) = 0$, hence (3.1) and (3.2) become

$$\begin{cases} \delta_{x_1}(a+b) + i\delta_{x_2}(a+b) = 0 \\ \delta_{x_1}(a+b) - i\delta_{x_2}(a+b) = 0 \end{cases}$$

i.e. $\delta_{x_1}(a+b) = \delta_{x_2}(a+b) = 0$.
It follows by [11, corollary 8] that

$$\|a+b\| \leqslant \min\{\|a+b-\delta_{x_1}(y)\|, \|a+b-\delta_{x_2}(y)\|\}$$

for all $y \in J(\mathscr{X})$. By choosing $y = 2ix_2$ in $\delta_{x_1}(y)$ we have $\delta_{x_1}(y) = [x^*,x]$, then

$$\|a+b\| \leqslant \|a+b-[x^*,x]\|.$$

If the condition (iii) is replaced by (iii)' i.e $x_1, x_2 \in \delta^{-1}_{(b-a),(a-b)}(0)$, then

$$\begin{aligned}\delta_{b,a}(x) + \delta_{a,b}(x^*) &= \delta_{b,a}(x_1) + \delta_{a,b}(x_1) + i\left[\delta_{b,a}(x_2) - \delta_{a,b}(x_2)\right] \\ &= bx_1 - x_1 a + ax_1 - x_1 b + i\left[bx_2 - x_2 a - ax_2 + x_2 b\right] \\ &= (a+b)x_1 - x_1(a+b) + i\left[(b-a)x_2 - x_2(a-b)\right] \\ &= -\delta_{x_1}(a+b) + i\delta_{(b-a),(a-b)}(x_2) \\ &= -\delta_{x_1}(a+b),\end{aligned}$$

and

$$\begin{aligned}\delta_{b,a}(x) - \delta_{a,b}(x^*) &= \delta_{(b-a),(a-b)}(x_1) - i\delta_{x_2}(a+b) \\ &= -i\delta_{x_2}(a+b).\end{aligned}$$

Hence $(1)+(2)$ and $(1)-(2)$ give

$$\begin{cases}\delta_{x_1}(a+b) = 0 \\ \delta_{x_2}(a+b) = 0\end{cases}$$

then

$$\|a+b\| \leqslant \|a+b-[x^*,x]\|. \quad \square$$

COROLLARY 3.3. *For $a = b = \frac{1}{2}e$ where e is the identity we have*

$$\|[x^*,x]-e\| \geqslant 1 \ \text{ for all } \ x \in J(\mathscr{X}),$$

and precisely for all $x, y \in J(\mathscr{X})$ $(x = x_1 + ix_2)$

$$1 \leqslant \min\{\|e - \delta_{x_1}(y)\|, \|e - \delta_{x_2}(y)\|\}.$$

It result that if h is a hermitian element of $J(\mathscr{X})$, then

$$\|[h,g]-e\| \geqslant 1$$

for all $g \in J(\mathscr{X})$.

REMARK 3.4. The last corollary give a shorter and simpler proof of William's result: Any hermitian element is finite element in the William's sense.

LEMMA 3.5. (Bonsall and Duncan [3]) *If E is a complex banach algebra, then $L \in B(E)$ is hermitian if and only if $\|e^{itL}\| \leqslant 1$.*

THEOREM 3.6. *Let $a, b \in J(\mathscr{X})$, where $J(\mathscr{X})$ is a multiplicative sub algebra of $B(\mathscr{X})$, if $\triangle^{-1}_{a,b}(0) \subseteq \triangle^{-1}_{a^*,b^*}(0)$ then, for all $x \in \triangle^{-1}_{a,b}(0)$ such that x commutes with a and b, we have*

$$\|ab\| \leqslant \min\{\|ab - [x^*x, xx^*]\|, \|ab + [x^*x, xx^*]\|\}.$$

Proof. Let $x \in \triangle_{a,b}^{-1}(0)$, then $axb = x$ and $a^*xb^* = x$, i.e. $ax^*b = x^*$, hence

$$\begin{cases} x^*x = a^*xb^*axb = abx^*xab \\ xx^* = axba^*xb^* = abxx^*ab \end{cases}$$

i.e. $x^*x, xx^* \in \triangle_{ab}^{-1}(0)$, by appliying [4, theorem 2.6] we have

$$\|ab\| \leqslant \|ab - [x^*x, xx^*]\|$$

and

$$\|ab\| \leqslant \|ab - [xx^*, x^*x]\|,$$

then

$$\|ab\| \leqslant \min\{\|ab - [x^*x, xx^*]\|, \|ab + [x^*x, xx^*]\|\}.$$

If $\mathscr{X}$ be a separable infinite dimensional complex Hilbert space, $GL(\mathscr{X})$ denote the set of all invertible elements in $B(\mathscr{X})$, we have the nice result. □

THEOREM 3.7. *Let $a, b \in B(\mathscr{X})$, then the following statements are equivalent*
(i) The equation $ax - xb = e$, where e is the identity of $B(\mathscr{X})$, admits a solution (i.e. $e \in R(\delta_{a,b})$).
(ii) There exists an invertible operator w in $R(\delta_{a,b})$ commutes with a or b.
(iii) $R(\delta_{a,b}) \supset GL(\mathscr{X}) \cap [\{a\}' \cup \{b\}']$.

Proof. $(iii) \Longrightarrow (ii)$ is evident since $GL(\mathscr{X}) \cap [\{a\}' \cup \{b\}'] \neq \emptyset$, because $e \in GL(\mathscr{X}) \cap [\{a\}' \cup \{b\}']$.

$(ii) \Longrightarrow (i)$ Let $w \in GL(\mathscr{X}) \cap [\{a\}' \cup \{b\}']$ and $x \in B(\mathscr{X})$ such that $ax - xb = w$. Suppose that $w \in \{a\}'$ and let $y = w^{-1}x$, then

$$ay - yb = aw^{-1}x - w^{-1}xb = w^{-1}(ax - xb) = w^{-1}w = e.$$

$(i) \Longrightarrow (iii)$ Let $x \in B(\mathscr{X})$ such that $ax - xb = e$ and let $w \in GL(\mathscr{X}) \cap [\{A\}' \cup \{B\}']$, suppose that $w \in \{b\}'$.

Let $y = xw$, then

$$ay - yb = axw - xwb = (ax - xb)w = w,$$

hence $w \in R(\delta_{a,b})$. □

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