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PERTURBATION OF (m,p)–ISOMETRIES BY NILPOTENT OPERATORS AND THEIR SUPERCYCLICITY

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Abstract. A bounded linear operator T on a Hilbert space H is an (m,p)-isometry if

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}||T^kx||^p=0$$

for all $x\in H$, in which $p\in[1,\infty)$ and $m\geqslant 1$. In this paper, two significant results will be proved. First, we introduce some perturbations of (m,p)-isometries which are (n,p)-isometries for some suitable n. Indeed, we show that the sum of an (m,p)-isometry and a commuting nilpotent operator of degree r is a $(pr-p+m,p)$-isometry for every even number p. As an application, the second result is to prove that such operators are not N-supercyclic for any positive integer N, even if p is a rational number. These results generalize the previous works on m-isometries.

1. Introduction

Let H denote a Hilbert space and $\mathscr{B}(H)$ be the algebra of all bounded linear operators on H. For a positive integer m and $p\in[1,+\infty)$ the operator T in $\mathscr{B}(H)$ is called an (m,p)-isometry if

$$\sum_{k=0}^{m}(-1)^k\binom{m}{k}||T^kx||^p=0,$$

for all $x\in H$. When $p=2$ these operators are called m-isometric operators and have been studied in [1, 2, 3]. The dynamics of such operators is discussed in [14] and [8]. In 2011, Bayart introduced (m,p)-isometric operators on Banach spaces [6]; see also [22].

On the other hand, an operator Q in $\mathscr{B}(H)$ is a nilpotent of degree $r\geqslant 1$, if $Q^r=0$. The dynamical properties of an isometry A plus a nilpotent operator Q commuting with A are studied in [23]. After that Bermúdez, et al. [11] proved that the operator $A+Q$ is a $(2r-1)$-isometry. Recently, this result is generalized by proving that the sum of an m-isometry and an r-nilpotent operator, commuting with each other, is a $(2r+m-2)$-isometry [10, 17, 20]. We will generalize this result to the case that A

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be an (m,p)-isometry, where p is any even number. We prove that $A+Q$ is a $(pr-p+m,p)$-isometry. As an application, we will see that if p is a rational number, the operator $A+Q$ is not N-supercyclic; this improves the results obtained in [23], [11] and [10]. Throughout this paper, unless stated otherwise, we assume that $Q\in\mathscr{B}(H)$ is a nilpotent operator of degree r, p is an even number and $A\in\mathscr{B}(H)$ is an (m,p)-isometric operator such that $AQ=QA$.

A few comments are in order. For nonnegative integer numbers n and k, we denote

$$n^{(k)}=\begin{cases}1, & (n=0 \quad \text{or} \quad k=0)\\ n(n-1)\cdots(n-k+1), & (n\neq 0 \quad \text{and} \quad k\neq 0).\end{cases}$$

Moreover, if $a_k\in\mathbb{C}$, $(k=0,1,\ldots,n)$ then $a_0a_1\ldots a_{i-1}a_{i+1}\ldots a_n$ (obtained by removing a_i from the product $\prod\limits_{k=0}^{n}a_k$) is denoted by $D_i(\prod\limits_{k=0}^{n}a_k)$ for $0\leqslant i\leqslant n$. By convention, if $i=n=0$, then $D_i(\prod\limits_{k=0}^{n}a_k)=1$.

Let $x\in H$. By Proposition 2.1 of [6]

$$||A^nx||^p=\sum_{k=0}^{m-1}\frac{n^{(k)}}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}||A^ix||^p \tag{1}$$

for all nonnegative integers n. Moreover, Proposition 2.1 of [19] states that

$$||A^nx||^p=\sum_{k=0}^{m-1}\frac{(-1)^{m-1-k}}{k!(m-1-k)!}D_k(\prod_{i=0}^{m-1}(n-i))||A^kx||^p \tag{2}$$

for all $n\geqslant m$. Note that (2) is obvious for $n<m$. Also, every (m,p)-isometry is an $(m+1,p)$-isometry ([6]) but not vice versa (Proposition 8 of [5]). If T is an (m,p)-isometry but not $(m-1,p)$-isometry then T is called a strict (m,p)-isometry.

We borrow two lemmas from [13] and [9].

LEMMA 1. (Lemma 1 of [13]) *If n is any positive integer then*

$$\sum_{k=0}^{n}(-1)^{n-k}k^i\binom{n}{k}=0$$

for $i=0,1,\ldots,n-1$, where $0^0=1$, by convention.

LEMMA 2. (Lemma 3.6 of [9]) *Let h be a real number and m be a positive integer. If $a_k=h-k$ then*

$$\sum_{i=0}^{m-1}(-1)^{m-i-1}\frac{D_i(\prod_{k=0}^{m-1}a_k)}{i!(m-i-1)!}=1.$$

2. The sum of an (m,p)-isometry and a nilpotent

For $T \in \mathscr{B}(H)$, m a positive integer, and $x \in H$, let φ_x be a mapping from the set $\{1,2,\dots,m\}$ to $[0,\infty)$ defined by $\varphi_x(k) = ||T^k x||^p$. Using Lemma 1, it can be easily seen that if for every $x \in H$, φ_x is a polynomial in k of degree at most $m-1$, then T is an (m,p)-isometry. The main result of this paper, runs as follows:

THEOREM 1. *Suppose that $Q \in \mathscr{B}(H)$ is a nilpotent operator of degree r, p is an even number and $A \in \mathscr{B}(H)$ is an (m,p)-isometry commuting with Q. Then the operator $T = A+Q$ is a $(pr-p+m,p)$-isometry.*

Proof. Take $x \in H$ and put

$$y = \sum_{j=0}^{r-1} \binom{n}{j} Q^j A^{r-1-j} x.$$

Then for every $n \geqslant r-1$, (2) implies that

$$\begin{aligned} ||T^n x||^p &= ||A^{n-(r-1)} y||^p \\ &= \sum_{k=0}^{m-1} \frac{(-1)^{m-1-k}}{k!(m-1-k)!} D_k\Big(\prod_{i=0}^{m-1}(n-r+1-i)\Big)||A^k y||^p. \qquad (3) \end{aligned}$$

Now, suppose that $n < r-1$. For the simplicity of notation, let $c_k = \frac{(-1)^{m-1-k}}{k!(m-1-k)!}$ and $d_k = D_k(\prod_{i=0}^{m-1}(n-r+1-i))$, $0 \leqslant k \leqslant m-1$. Then, applying Lemma 2, we see that $\sum_{k=0}^{m-1} c_k d_k = 1$. Therefore,

$$\begin{aligned} &\sum_{k=0}^{m-1} c_k d_k ||A^k y||^p = \sum_{k=0}^{m-1} c_k d_k ||A^{r-1-n+k}(T^n x)||^p \\ &= \sum_{k=0}^{m-1} c_k d_k \Big(\sum_{j=0}^{m-1}\Big(\sum_{t=j}^{m-1} \frac{(-1)^{t-j}}{t!}(r-1-n+k)^{(t)}\binom{t}{j}\Big)||A^j(T^n x)||^p\Big) \quad \text{(by (1))} \\ &= \sum_{j=0}^{m-1}\sum_{t=j}^{m-1}\sum_{k=0}^{m-1} c_k d_k \frac{(-1)^{t-j}}{t!}(r-1-n+k)^{(t)}\binom{t}{j}||A^j(T^n x)||^p \\ &= \sum_{t=0}^{m-1}\sum_{k=0}^{m-1} c_k d_k \frac{(-1)^t}{t!}(r-1-n+k)^{(t)}||T^n x||^p \\ &\quad -\Big[\prod_{i=0}^{m-1}(n-r+1-i)\Big]\Big[\sum_{j=1}^{m-1}\sum_{t=j}^{m-1}\frac{(-1)^{t-j}}{t!}\binom{t}{j}\sum_{k=0}^{m-1} c_k (r-n+k-2)^{(t)}||A^j(T^n x)||^p\Big] \\ &= \sum_{k=0}^{m-1} c_k d_k ||T^n x||^p + \sum_{t=1}^{m-1}\frac{(-1)^t}{t!}\sum_{k=0}^{m-1} c_k d_k (r-1-n+k)^{(t)}||T^n x||^p \quad \text{(by Lemma 1)} \\ &= ||T^n x||^p - \Big[\prod_{i=0}^{m-1}(n-r+1-i)\Big]\sum_{t=1}^{m-1}\frac{(-1)^t}{t!}\sum_{k=0}^{m-1} c_k (r-n+k-2)^{(t)}||T^n x||^p \\ &= ||T^n x||^p \quad \text{(by Lemma 1)}. \end{aligned}$$

Thus, (3) holds for every nonnegative integer number n and every $x \in H$. On the other hand, since $\binom{n}{r-1}$ is a polynomial in n of degree $r-1$, we conclude that $||A^k y||^p$ is a polynomial in n of degree $pr-p$ (here we use the facts that H is a Hilbert space and p is an even number). Furthermore, the coefficient of $||A^k y||^p$ in (3) is a polynomial in n of degree $m-1$; therefore, the mapping $n \longmapsto ||T^n x||^p$ is of degree at most $pr-p+m-1$. Hence T is a $(pr-p+m,p)$-isometry. □

If A is an isometry we can say more. Recall that the operator Q is of order $r \geqslant 1$ if $Q^r = 0$ and $Q^{r-1} \neq 0$.

COROLLARY 1. *Suppose that A is an isometry and Q is a nilpotent operator of order r. Then $T = A+Q$ is a strict $(pr-p+1,p)$-isometry.*

Proof. Assume, on the contrary, that T is a $(pr-p,p)$-isometry. Proposition 2.1 of [6] state that the mapping $n \mapsto ||T^n x||^p$ is a polynomial of degree at most $pr-p-1$, for every $x \in H$. But by (3)

$$||T^n x||^p = ||y||^p = ||\sum_{j=0}^{r-1} \binom{n}{j} Q^j A^{r-1-j} x||^p,$$

which, in turn, implies that the coefficient of n^{pr-p} in $||T^n x||^p$ is $\frac{1}{((r-1)!)^p} Q^{r-1} x$. Hence we get $Q^{r-1} = 0$, which is absurd. □

Although nilpotent operators are not (m,p)-isometry, as we see in the next result the perturbation of these operators by a unimodular scalar of the identity is (m,p)-isometry for some suitable m and p.

COROLLARY 2. *Suppose that Q is a nilpotent operator of order r and λ is a complex number with $|\lambda| = 1$. Then $\lambda I + Q$ is a strict $(pr-p+1,p)$-isometry.*

COROLLARY 3. *Suppose that A is an isometry. Then $A+Q$ is a $(pr-p+1)$-isometry for every integer number $p \geqslant 2$.*

Proof. Note that $pr-p+1 \geqslant 2r-1$ and $A+Q$ is a $(2r-1)$-isometry. □

The following examples show that in Theorem 1, it is essential that p be an even number and the underlying space be a Hilbert space.

EXAMPLE 1. Let $(e_n)_{n\in\mathbb{Z}}$ be the ordinary orthonormal basis for $\ell^2(\mathbb{Z})$. Define the weighted shift operator Q by $Qe_n = w_n e_{n+1}$, where $w_{2n} = 0$ for all integers n, $w_{2n-1} = \frac{1}{(1-2n)^2}$ for all $n \geqslant 1$ and $w_{2n-1} = \frac{1}{1-2n}$ for all $n \leqslant 0$. Then, $Q^2 = 0$. A simple computation shows that

$$\sum_{k=0}^{4} (-1)^k \binom{4}{k} ||(I+Q)^k e_1||^3 \neq 0;$$

thus, the operator $I+Q$ is not a (4,3)-isometry.

EXAMPLE 2. Let $(e_n)_{n\in\mathbb{Z}}$ be the ordinary basis for $\ell^3(\mathbb{Z})$, and consider the nilpotent operator Q as in the preceding example. Since

$$\sum_{k=0}^{3}(-1)^k\binom{3}{k}||(I+Q)^k e_1||^2 \neq 0,$$

the operator $I+Q$ is not a (3,2)-isometry.

3. N-supercyclicity

Let T be a bounded linear operator on a Banach space X and $E\subseteq X$. The orbit of E under T is defined by

$$\mathrm{orb}(T,E)=\{T^k x : x\in E, k\geqslant 0\}.$$

If there exists an N-dimensional subspace E of X such that $\mathrm{orb}(T,E)$ is dense in X, then T is called an N-supercyclic operator. Every 1-supercyclic operator is called a supercyclic operator. As good sources on the dynamics of linear operators, one can see [7] and [16]. Supercyclicity of operators was introduced by Hilden and Wallen in [18]. Moreover, Feldman initiated the study of N-supercyclicity of operators [15] (see also [12]). In 1974, Hilden and Wallen proved that isometries on Hilbert spaces of dimension greater than 1 are not supercyclic [18]. After that, Ansari and Bourdon in [4] and Miller in [21] generalized their result to isometries on Banach spaces. In 2012, Faghih-Ahmadi and Hedayatian extended this result to the class of m-isometries on Hilbert spaces [14]. Moreover, Bermúdez et al. [8] have given sufficient conditions under which m-isometric operators on Hilbert spaces are not N-supercyclic. Later, Bayart [6] generalized this result on Banach spaces and showed that m-isometric operators on infinite dimensional Banach spaces are not N-supercyclic. On the other hand, in 2011, Yarmahmoodi et al. [23] proved that if A is an isometry and Q is a nilpotent operator on a normed space that commutes with A, then the operator $A+Q$ is not supercyclic. Later, in 2013, Bermúdez et al. [11] showed that the operator $A+Q$ is not N-supercyclic on a Hilbert space. Recently, the result is proved when A is an m-isometry [10]. By applying Theorem 1, we improve this result for each (m,p)-isometric operator A.

THEOREM 2. *Suppose that H is an infinite dimensional Hilbert space, m is a positive integer, and $p\geqslant 1$ is a rational number. If A is an (m,p)-isometry, then the operator $T=A+Q$ is never N-supercyclic for any N.*

Proof. Let $p=t/s$ where the greatest common divisor of t and s is 1. Put $k=2s$. If A is a strict (m,p)-isometry then by Corollary 4.6 of [19] it is also a $(2s(m-1)+1,2t)$-isometry. Hence Theorem 1 states that T is a $(2(tr-t+sm-s)+1,2t)$-isometry. So the result follows using Theorem 3.3 of [6]. Otherwise, A is an $(m-1,p)$-isometry, and again an argument similar to the above, shows that T is not N-supercyclic. Now, continuing this process, and noting that isometries are not N-supercyclic [6], we conclude that the operator T is not N-supercyclic. □

Note that the operator $T = I + Q$ in Examples 1 and 2 are not N-supercyclic. Indeed, if X is any infinite dimensional Banach space and Q is a nilpotent operator of degree r, then $T = I + Q$ is not N-supercyclic for every $N \geqslant 1$. Because if $x \in X$ and $k \geqslant 0$ then $T^k x = \sum_{j=0}^{r-1} \binom{k}{j} Q^j x$. Suppose that $x_1, x_2, \ldots, x_N$ are N linearly independent vectors in X such that $E = \text{span}\{x_1, x_2, \ldots, x_N\}$. Thus $T^k x_i$ is in $\text{span}(\bigcup_{j=0}^{r-1} \bigcup_{t=1}^{N} \{Q^j x_t\})$ for $1 \leqslant i \leqslant N$ and $k \geqslant 0$. Therefore, $\text{orb}(T, E)$ is not dense in X. However, on a Banach space the question that whether an (m, p)-isometry plus a nonzero nilpotent, that commute with each other, is N-supercyclic or not is still an open question.

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