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INEQUALITIES FOR EIGENVALUES OF COMPACTLY PERTURBED UNITARY OPERATORS

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Abstract. We consider the operator $A = U + K$, where U is a unitary operator and K is a compact one. An eigenvalue λ of A is said to be a non-unitary one, if $|\lambda| \neq 1$. We derive inequalities for sums of absolute values of the non-unitary eigenvalues. Applications of these inequalities to operator functions, spectrum perturbations and operator equations are also discussed.

1. Introduction

Denote by $\mathscr{H}$ a separable Hilbert space with a scalar product $(.,.)$, the norm $\|.\| = \sqrt{(.,.)}$, and the unit operator I. By $[\mathscr{H}]$ we denote the set of all bounded linear operators in $\mathscr{H}$. For a linear operator A, A^* is the adjoint operator; $\sigma(A)$ is the spectrum, $r_s(A)$ is the spectral radius. $SN_p = SN_p(\mathscr{H})$ $(1 \leqslant p < \infty)$ is the Schatten-von Neumann ideal of operators K_0 in $\mathscr{H}$ with the finite norm

$$N_p(K_0) := [\text{trace}\left((K_0^* K_0)^{p/2}\right)]^{1/p}.$$

Let $B \in [\mathscr{H}]$ and its Hermitian component $\Im B = (B - B^*)/2i$ be compact. Recall the Weyl inequality [11, Lemma II.6.1]:

$$\sum_{k=1}^{j} |\Im \mu_k| \leqslant \sum_{k=1}^{j} s_k(\Im B) \ \ (j = 1, 2, \ldots),$$

where μ_k are the nonreal eigenvalues of B with their (algebraic) multiplicities enumerated in nonincreasing order, $s_k(\Im B)$ are the singular values of $\Im B$ with their multiplicities enumerated in nonincreasing order. Note that B is a sum of a selfadjoint operator and a compact one. The aim of this paper is to derive a similar result for the operator $A = U + K$, where U is a unitary operator and K is a compact one. Note that although the literature on the inequalities eigenvalues of linear operators is very rich, but mainly compact operators have been investigated, cf. [1, 2, 15, 16, 18, 19, 9, 10] and references given therein.

A few words about the contents. The paper consists of 6 sections.

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Section 2 contains the preliminary results.

The main result and some of its corollaries are presented in Section 3.

In Section 4 we suggest norm estimates for operator functions under the condition $K \in SN_1$. Besides, we considerably refine the corresponding result from [7].

Section 5 is devoted to spectrum perturbations of A under the condition $K \in SN_1$. Although the theory of operator perturbations is very well developed [14, 4, 5, 6, 12, 13] etc, the norm estimates for resolvents from Section 4 enable us to obtain a new result.

In Section 6, applications of our results to the Sylvester operator equations are discussed. These equations play an essential role in the theory of differential and difference equations, control theory and other applications, cf. [3, 17].

2. Preliminary results

Recall that we consider the operator $A = U + K$, where U is a unitary operator and K is compact. So the operator

$$A^*A - I = (U+K)^*(U+K) - I = U^*K + K^*U + K^*K \text{ is compact .}$$

An isolated eigenvalue of A having a finite algebraic multiplicity and satisfying $|\lambda| \neq 1$ will be called *a non-unitary eigenvalue*. In the sequel it is assumed that the set of all non-unitary eigenvalues is non-empty.

For example, due to Theorem I.5.3 [11], under condition (2.1), if A has a regular point in the disc $\{z \in \mathbb{C} : |z| < 1\}$, then any point λ, such that $|\lambda| \neq 1$ is either regular or a non-unitary eigenvalue of A. About other similar results see [14, p. 244].

Everywhere below the eigenvalues and singular values are enumerated with the algebraic multiplicities taken into account.

Denote by $s_k(A^*A - I)$ $(k = 1, 2, \ldots)$ the singular values of $A^*A - I$ enumerated in the non-increasing order.

2.1. Eigenvalues outside the unit circle

In this subsection it is supposed that all the non-unitary eigenvalues of A lie outside the unit circle. Denote by $\lambda_k(A)$ these eigenvalues enumerated in the non-increasing order of their modulus: $|\lambda_k(A)| \geqslant |\lambda_{k+1}(A)|$ $(k = 1, 2, \ldots)$. So

$$|\lambda_k(A)| \geqslant 1 \;\; (k = 1, 2, \ldots). \tag{2.1}$$

Here and below without loss of the generality it is assumed that the total multiplicity of all non-unitary eigenvalues is infinity. If the total multiplicity equals $m_0 < \infty$, we put $|\lambda_k(A)|^2 - 1 = 0$ for $k > m_0$. Condition (2.1) is provided by the inequality

$$A^*A \geqslant I. \tag{2.2}$$

LEMMA 2.1. *Let condition (2.1) hold. Then*

$$\sum_{k=1}^{j} (|\lambda_k(A)|^2 - 1) \leqslant \sum_{k=1}^{j} s_k(A^*A - I) \;\; (j = 1, 2, \ldots). \tag{2.3}$$

Proof. Denote by $\mathscr{E}$ the linear closed convex hull of all the root vectors of A corresponding to non-unitary eigenvalues. That is, $\phi \in \mathscr{E}$ means that $(A-\lambda_j(A)I)^n\phi = 0$ for some finite positive integer n. The set $\mathscr{L}_j$ $(j=1,2,\dots)$ of all root vectors corresponding to some and the same eigenvalue is called the root subspace. Choose in each root subspace a Jordan basis. Then we obtain vectors ϕ_k for each of which either $A\phi_k = \lambda_k(A)\phi_k$, or $A\phi_k = \lambda_k(A)\phi_k + \phi_{k+1}$. Orthogonalizing the system $\{\phi_k\}$, we obtain an (orthonormal) Schur basis $\{e_k\}$. That is,

$$Ae_k = a_{1k}e_1 + a_{2k}e_2 + \ldots + a_{kk}e_k. \tag{2.4}$$

Besides, $a_{kk} = \lambda_k(A)$ $(k=1,2,\dots)$. We have

$$((A^*A-I)e_k,e_k) = (A^*Ae_k,e_k) - (e_k,e_k) = (Ae_k,Ae_k) - 1.$$

Due to (2.4) this gives

$$((A^*A-I)e_k,e_k) = |a_{1k}|^2 + |a_{2k}|^2 + \ldots + |a_{kk}|^2 - 1 \geqslant |\lambda_k(A)|^2 - 1. \tag{2.5}$$

Applying the well-known Lemma II.4.1 from [11] we can write

$$\sum_{k=1}^{j} |((A^*A-I)e_k,e_k)| \leqslant \sum_{k=1}^{j} s_k(A^*A-I). \tag{2.6}$$

This and (2.5) prove the result. □

From (2.3) it follows

$$\sum_{k=1}^{j} h(|\lambda_k(A)|^2 - 1) \leqslant \sum_{k=1}^{j} h(s_k(A^*A-I)) \quad (j=1,2,\dots) \tag{2.7}$$

for any convex continuous function $h(x)$ $(x \geqslant 0)$ satisfying $h(0)=0$, cf. [11, Lemma II.3.4]. In particular,

$$\sum_{k=1}^{j} (|\lambda_k(A)|^2 - 1)^p \leqslant \sum_{k=1}^{j} s_k^p(A^*A-I) \quad (j=1,2,\dots) \tag{2.8}$$

for any $p \geqslant 1$. If, in addition, $A^*A - I \in SN_p$ $(p \geqslant 1)$, then

$$\sum_{k=1}^{\infty} (|\lambda_k(A)|^2 - 1)^p \leqslant N_p^p(A^*A-I). \tag{2.9}$$

2.2. Eigenvalues inside the unit circle

Throughout this subsection it is supposed that A is invertible and all the non-unitary eigenvalues of A lie inside the unit circle $|z| < 1$. By $\hat{\lambda}_k(A)$ we denote these eigenvalues enumerated in the non-decreasing order: $|\hat{\lambda}_k(A)| \leqslant |\hat{\lambda}_{k+1}(A)|$ $(k=1,2,\dots)$.

Note that

$$A^{-1}(A^*)^{-1} - I = A^{-1}(A^*)^{-1}(I - A^*A)$$

is compact. Then $\lambda_k(A^{-1}) = 1/\hat{\lambda}_k(A)$ and therefore $|\lambda_k(A^{-1})| = 1/|\hat{\lambda}_k(A)| > 1$. Here $\lambda_k(A^{-1})$ are the non-unitary eigenvalues of A^{-1} enumerated in the non-increasing order. Due to Lemma 2.1,

$$\sum_{k=1}^{j}(|\lambda_k(A^{-1})|^2 - 1) \leqslant \sum_{k=1}^{j} s_k(A^{-1}(A^{-1})^* - I).$$

Here $s_k(A^{-1}(A^{-1})^* - I)$ are the singular values of $A^{-1}(A^{-1})^* - I$, respectively, enumerated in the non-increasing order. Since

$$s_k(A^{-1}(A^*)^{-1} - I) = s_k(A^{-1}(A^*)^{-1}(I - A^*A) \leqslant \|A^{-1}\|^2 s_k(AA^* - I),$$

we obtain

$$\sum_{k=1}^{j}\left(\frac{1}{|\hat{\lambda}_k(A)|^2} - 1\right) = \sum_{k=1}^{j}\frac{1 - |\hat{\lambda}_k(A)|^2}{|\hat{\lambda}_k(A)|^2} \leqslant \|A^{-1}\|^2 \sum_{k=1}^{j} s_k(A^*A - I).$$

Hence for all integer $j \geqslant 1$ we obtain

$$\sum_{k=1}^{j}(1 - |\hat{\lambda}_k(A)|^2) \leqslant |\hat{\lambda}_j(A)|^2\|A^{-1}\|^2 \sum_{k=1}^{j} s_k(AA^* - I) \leqslant r_s^2(A)\|A^{-1}\|^2 \sum_{k=1}^{j} s_k(A^*A - I). \tag{2.10}$$

We thus arrive at

LEMMA 2.2. *Let all the non-unitary eigenvalues be inside the unit circle and A be invertible. Then inequality (2.10) is valid. If, in addition, $A^*A - I \in SN_p$ $(p \geqslant 1)$, then*

$$\sum_{k=1}^{\infty}(1 - |\hat{\lambda}_k(A)|^2)^p \leqslant r_s^{2p}(A)\|A^{-1}\|^{2p} N_p^p(AA^* - I).$$

3. The main result

In the sequel operator A can have the eigenvalues inside and outside the unit circle. Besides, $\tilde{\lambda}_k(A)$ $(k = 1, 2, \ldots)$ are *the non-unitary eigenvalues of A are enumerated as*

$$||\tilde{\lambda}_k(A)|^2 - 1| \geqslant ||\tilde{\lambda}_{k+1}(A)|^2 - 1| \quad (k = 1, 2, \ldots).$$

So

$$||\tilde{\lambda}_k(A)|^2 - 1| = 1 - |\hat{\lambda}_k(A)|^2 \text{ if } |\tilde{\lambda}_k(A)| < 1$$

and

$$||\tilde{\lambda}_k(A)|^2 - 1| = |\lambda_k(A)|^2 - 1 \text{ if } |\tilde{\lambda}_k(A)| > 1.$$

Here $\lambda_k(A)$ $(\hat{\lambda}_k(A))$ are the non-unitary eigenvalues of A outside (inside) the unit circle enumerated in the non-increasing (non-decreasing) order of their absolute values.

Now we are in a position to formulate and prove our main result.

THEOREM 3.1. *Let A be invertible and $A^*A-I \in SN_p$ for an integer $p \geqslant 1$. Then*

$$\sum_{k=1}^{\infty} |1-|\tilde{\lambda}_k(A)|^2|^p \leqslant r_s^{2p}(A)\|A^{-1}\|^{2p}N_p^p(AA^*-I). \tag{3.1}$$

If, in addition, condition (2.2) holds, then inequalities (2.1) and (2.9) are valid (due to Lemma 2.1).

Proof. Let P and $\overline{P}$ be the invariant projections of A, corresponding to the eigenvalues inside and outside the unit circle, such that $|\tilde{\lambda}_k(A\overline{P})| < 1$ and $|\tilde{\lambda}_k(AP)| > 1$ $(k=1,2,\ldots)$, and $P \leqslant I-\overline{P}$. So the eigenvalues of PA are ordered in the decreasing way and the eigenvalues of $A\overline{P}$ are ordered in the increasing way of their absolute values. In addition,

$$\sum_{k=1}^{\infty} |1-|\tilde{\lambda}_k(A)|^2|^p = \sum_{j=1}^{\infty}(|\lambda_j(AP)|^2-1)^p + \sum_{j=1}^{\infty}(1-|\lambda_j(A\overline{P})|^2)^p. \tag{3.2}$$

Operators A^*A and AA^* are unitarily equivalent. Hence, A^*A-I and AA^*-I are unitarily equivalent, and therefore, $N_p(A^*A-I) = N_p(AA^*-I)$. Similarly, for the operator $PA = PAP$ considered in $P\mathscr{H}$ we have $N_p(A^*PA-P) = N_p(PAA^*P-P)$. Operator AP satisfies the conditions of Lemma 2.1, which implies

$$\sum_{k=1}^{j}(|\lambda_k(AP)|^2-1) \leqslant \sum_{k=1}^{j} s_k(A^*PA-P) \quad (j=1,2,\ldots).$$

Hence, according to the classical Lemma II.3.4 [11] we can write

$$\sum_{j=1}^{\infty}(|\tilde{\lambda}_j(AP)|^2-1)^p \leqslant N_p^p(A^*PA-P) = N_p^p(PAA^*P-P).$$

We thus have

$$\sum_{j=1}^{\infty}(|\tilde{\lambda}_j(AP)|^2-1)^p \leqslant N_p^p(P(AA^*-I)P) \leqslant N_p(A^*A-I). \tag{3.3}$$

Moreover, Lemma 2.2 implies

$$\sum_{k=1}^{\infty}(1-|\tilde{\lambda}_k(A\overline{P})|^2)^p \leqslant r_s^{2p}(A\overline{P})\|(A\overline{P})^{-1}\|^{2p}N_p^p(\overline{P}A^*A\overline{P}-\overline{P}). \tag{3.4}$$

Here $(A\overline{P})^{-1}$ is understood as the inverse of the restriction of $A\overline{P}$ in $\overline{P}\mathscr{H}$. Since A is Fredholm, we can write

$$\|(A\overline{P})^{-1}\| = \frac{1}{\|A\overline{P}\|_{low}}$$

where

$$\|A\overline{P}\|_{low} = \inf_{x\in\overline{P}\mathscr{H}} \|A\overline{P}x|/\|x\| \geqslant \inf_{y\in\mathscr{H}} \|Ay|/\|y\|.$$

So

$$\|(A\overline{P})^{-1}\| \leqslant \frac{\|y\|}{\inf_{y\in\mathscr{H}}\|Ay\|} = \|A^{-1}\|.$$

Take into account that $r_s(A\overline{P}) \leqslant r_s(A)$ and

$$N_p(\overline{P}A^*A\overline{P}-\overline{P}) = N_p(\overline{P}(A^*A-I)\overline{P}) \leqslant N_p(A^*A-I).$$

Then from (3.4) it follows

$$\sum_{j=1}^{\infty}(1-|\lambda_j(A\overline{P})|^2)^p \leqslant r_s^{2p}(A)\|A^{-1}\|^{2p}N_p^p(\overline{P}(A^*A-I)\overline{P}).$$

Now (3.2) and (3.3) imply the required result since $P+\overline{P}\leqslant I$. □

4. Functions of compactly perturbed unitary operators

In this subsection we consider operator functions under the condition

$$AA^*-I \in SN_1. \tag{4.1}$$

Put

$$\vartheta(A) := [\text{trace}\,(A^*A-I) - \sum_{k=1}^{\infty}(|\lambda_k(A)|^2-1)]^{1/2}.$$

It can be checked that $\vartheta(A)\geqslant 0$; if A is a normal operator, then $\vartheta(A)=0$, cf. [7, Section 7.15].

THEOREM 4.1. *Under condition (4.1), let A have a regular point on the unit circle. Then*

$$\|R_\lambda(A)\| \leqslant \sum_{k=0}^{\infty}\frac{\vartheta^k(A)}{\sqrt{k!}\rho^{k+1}(A,\lambda)} \tag{4.2}$$

and

$$\|R_\lambda(A)\| \leqslant \frac{e^{1/2}}{\rho(A,\lambda)}\exp\left[\frac{\vartheta^2(A)}{2\rho^2(A,\lambda)}\right] \quad (\lambda\notin\sigma(A)). \tag{4.3}$$

For the proof see [7, Theorem 7.15.2]. Inequality (4.2) is sharper than (4.3), but (4.3) is more compact. This theorem is sharp: (4.2) becomes the equality if A is normal.

The calculations of $\vartheta(A)$ is a not easy task, in general. We are going to apply our above results to estimate $\vartheta(A)$.

Let A have a purely unitary spectrum. That is, $\sigma(A)$ lies on the unit circle. Then

$$\vartheta(A) = [Tr\,(A^*A-I)]^{1/2}. \tag{4.4}$$

Note that

$$N_1(A^*A-I) \leqslant 2N_1(K)+N_2^2(K) \leqslant 2N_1(K)+N_1^2(K)$$

and $r_s(A)\|A^{-1}\| \geqslant r_s(A)r_s(A^{-1}) \geqslant 1$.

LEMMA 4.2. *Under condition (4.1), let A be invertible. Then $\vartheta(A) \leqslant \zeta(A)$, where*

$$\zeta(A) := \sqrt{N_1(A^*A-I)} \begin{cases} 1 & \text{if } A^*A \geqslant I, \\ (r_s^2(A)\|A^{-1}\|^2-1)^{1/2} & \text{if } A^*A < I, \\ (r_s^2(A)\|A^{-1}\|^2+1)^{1/2} & \text{otherwise.} \end{cases} \tag{4.5}$$

Proof. If condition (3.1) hold, then $|\lambda_k(A)| \geqslant 1$. and

$$\sum_{k=1}^{\infty}(|\lambda_k(A)|^2-1) \geqslant 0$$

and therefore,

$$\vartheta(A) \leqslant [\text{trace } (A^*A-I)]^{1/2} = N_1^{1/2}(A^*A-I).$$

So in the case (3.1) the lemma is proved.

If $A^*A < I$, then $|\lambda_k(A)| < 1$ and trace $(A^*A-I) = -N_1(A^*A-I)$. Due to Lemma 2.2

$$\vartheta^2(A) = -N_1(A^*A-I) + \sum_{k=1}^{\infty}(1-|\hat{\lambda}_k(A)|^2) \leqslant (r_s^2(A)\|A^{-1}\|^2-1)N_1(AA^*-I).$$

So in the case $A^*A < I$ the lemma is also proved.

In the general, case due to Theorem 3.1,

$$\vartheta^2(A) \leqslant N_1(A^*A-I) + \sum_{k=1}^{\infty}|1-|\tilde{\lambda}_k(A)|^2| \leqslant (1+r_s^2(A)\|A^{-1}\|^2)N_1(AA^*-I).$$

This finishes the proof. □

Note that in the case $A^*A \geqslant I$, A is automatically invertible, since it is a Fredholm operator.

Theorem 4.1 and Lemma 4.2 imply

COROLLARY 4.3. *Under the hypothesis of Theorem 4.1 let A be invertible. Then*

$$\|R_\lambda(A)\| \leqslant \sum_{k=0}^{\infty}\frac{\zeta^k(A)}{\sqrt{k!}\rho^{k+1}(A,\lambda)} \tag{4.6}$$

and

$$\|R_\lambda(A)\| \leqslant \frac{e^{1/2}}{\rho(A,\lambda)} \exp\left[\frac{\zeta^2(A)}{2\rho^2(A,\lambda)}\right] \quad (\lambda \notin \sigma(A)). \tag{4.7}$$

The following theorem is proved in [7, Theorem 17.5.4].

THEOREM 4.4. *Let A satisfy condition (4.1) and have a regular point on the unit circle. If, in addition, f is a holomorphic function on a neighborhood of closed convex hull $co(A)$ of $\sigma(A)$, then*

$$\|f(A)\| \leqslant \sum_{k=0}^{\infty} \sup_{\lambda \in co(A)} |f^{(k)}(\lambda)| \frac{\vartheta^k(A)}{(k!)^{3/2}}. \tag{4.8}$$

This theorem is sharp: inequality (4.8) becomes equality if A is a unitary operator and

$$\sup_{\lambda \in co(A)} |f(\lambda)| = \sup_{\lambda \in \sigma(A)} |f(\lambda)|,$$

because $\vartheta(A) = 0$ in this case. The latter theorem and Lemma 4.2 imply

COROLLARY 4.5. *Under the hypothesis of Theorem 4.4 let A be invertible. Then*

$$\|f(A)\| \leqslant \sum_{k=0}^{\infty} \sup_{\lambda \in co(A)} |f^{(k)}(\lambda)| \frac{\zeta^k(A)}{(k!)^{3/2}}.$$

EXAMPLE 4.6. Let A satisfy the hypothesis of Theorem 4.4 and be invertible. Then

$$\|A^m\| \leqslant \sum_{k=0}^{m} \frac{m! r_s^{m-k}(A) \zeta^k(A)}{(m-k)!(k!)^{3/2}} \tag{4.10}$$

for any integer $m \geqslant 1$. In addition,

$$\|e^{At}\| \leqslant e^{\alpha(A)t} \sum_{k=0}^{\infty} \frac{t^k \zeta^k(A)}{(k!)^{3/2}} \quad \text{for all} \quad t \geqslant 0, \tag{4.11}$$

where $\alpha(A) = \sup Re\, \sigma(A)$.

5. Spectral variations

DEFINITION 5.1. Let A and B be linear operators in $\mathscr{H}$. Then the quantity

$$sv_A(B) := \sup_{\mu \in \sigma(B)} \inf_{\lambda \in \sigma(A)} |\mu - \lambda|$$

is called the spectral variation of a B with respect to A.

We need the following technical lemma

LEMMA 5.2. *Let $A, B \in \mathscr{H}$ and $q := \|A - B\|$. In addition, let*

$$\|R_\lambda(A)\| \leqslant F\left(\frac{1}{\rho(A,\lambda)}\right) \ \ (\lambda \notin \sigma(A)),$$

where $F(x)$ is a monotonically increasing non-negative function of a non-negative variable x, such that $F(0) = 0$ and $F(\infty) = \infty$. Then $sv_A(B) \leqslant z(A,q)$, where $z(A,q)$ is the unique positive root of the equation $qF(1/z) = 1$.

For the proof see [7, Lemma 8.4.2, p. 129]. The previous lemma and (4.7) imply

COROLLARY 5.3. *Let A satisfy the hypothesis of Theorem 4.1. Then for any $B \in \mathscr{H}$ one has $sv_A(B) \leqslant z_2(q,A)$, where $z_2(q,A)$ is the unique positive root of the equation*

$$\frac{q}{z}\ \exp\left[\frac{1}{2}+\frac{\zeta^2(A)}{2z^2}\right]=1. \tag{5.1}$$

We need also the following

LEMMA 5.4. *The unique positive root z_a of the equation*

$$\sum_{j=0}^{p-1}\frac{1}{y^{j+1}}\ \exp\left[\frac{1}{2}(1+\frac{1}{y^{2p}})\right]=a\ \ (p=1,2,\dots\ ;\ a\equiv const>0) \tag{5.2}$$

satisfies the inequality $z_a \leqslant \delta_p(a)$, where

$$\delta_p(a):=\begin{cases} pe/a & if\ a\leqslant pe,\\ [ln\ (a/p)]^{-1/2p} & if\ a>pe \end{cases}.$$

Furthermore, substitute into (5.1) the equality $z=x\zeta(A)$ and apply the previous lemma. Then we can assert that $z_2(q,A)\leqslant \delta_1(A,q)$, where

$$\delta_1(A,q)=\begin{cases} eq & \text{if } \zeta(A)\leqslant eq,\\ \zeta(A)\ [ln\ (\zeta(A)/q)]^{-1/2} & \text{if } \zeta(A)>eq \end{cases}.$$

Hence, Corollary 5.3 yield the inequality $sv_A(B)\leqslant \delta_1(q,A)$.

6. The Sylvester operator equation

Consider the Sylvester equation

$$AX-XB=C, \tag{6.1}$$

where A,B and C are given bounded linear operators in $\mathscr{H}$ and X should be found. The Lyapunov equation

$$AX+XA^*=C. \tag{6.2}$$

is an example of (6.1). It plays an important role in the theory of differential equations.

LEMMA 6.1. *Let*

$$\int_0^\infty \|e^{-At}\|\|e^{Bt}\|dt<\infty.$$

Then (6.1) has a unique solution X, which is representable as

$$X=\int_0^\infty e^{-At}Ce^{Bt}dt. \tag{6.3}$$

Proof. It is well-known that $\|A\| \geqslant r_s(A)$. Hence due to the Spectral Mapping theorem $\|f(A)\| \geqslant \sup_{s\in\sigma(A)} |f(s)|$ for any function f regular on the spectrum of A. In particular, $\|e^{Bt}\| \geqslant e^{\alpha(B)t}$ and $\|e^{-At}\| \geqslant e^{-\beta(A)t}$ $(t \geqslant 0)$, where

$$\beta(A) := \inf \Re\, \sigma(A).$$

Thus,

$$\int_0^\infty e^{(-\beta(A)+\alpha(B))t} dt \leqslant \int_0^\infty \|e^{-At}\| \|e^{Bt}\| dt < \infty.$$

Hence $\alpha(B) < \beta(A)$. Now the existence and uniqueness of solutions to (6.1) is due Theorem I.3.2 [3]. From (6.3) it follows

$$AX - XB = \int_0^\infty (Ae^{-At}Ce^{Bt} - e^{-At}CBe^{Bt})dt = \int_0^\infty \Big(-\frac{de^{-At}}{dt} Ce^{Bt} - e^{-At}C\frac{de^{Bt}}{dt}\Big)dt$$
$$= -\int_0^\infty \frac{d}{dt}(e^{-At}Ce^{Bt})dt = -e^{-At}Ce^{Bt}|_{t=0}^{t=\infty} = C.$$

This proves the result. $\square$

THEOREM 6.2. *Let A and B satisfy the hypothesis of Theorem 4.3. In addition, let the condition $\beta(A) > \alpha(B)$ hold. Then the unique solution X to equation (6.1) is subject to the inequality*

$$\|X\| \leqslant \|C\| \sum_{j,k=0}^\infty \frac{(k+j)!\zeta^k(A)\zeta^j(B)}{(\beta(A)-\alpha(B))^{k+j+1}(k!j!)^{3/2}}.$$

Proof. Due to (4.11) we have

$$\|e^{-As}\| \leqslant e^{-\beta(A)s} \sum_{k=0}^\infty \frac{s^k\zeta^k(A)}{(k!)^{3/2}} \text{ and } \|e^{Bs}\| \leqslant e^{\alpha(B)s} \sum_{j=0}^\infty \frac{s^j\zeta^j(B)}{(j!)^{3/2}}$$

for all $s \geqslant 0$. So

$$\|e^{-As}\| \|e^{Bs}\| \leqslant e^{-(\beta(A)-\alpha(B))s} \sum_{j=0}^\infty \sum_{k=0}^\infty \frac{s^{k+j}\zeta^k(A)\zeta^j(B)}{(k!j!)^{3/2}}.$$

But

$$\int_0^\infty s^{k+j} e^{-(\beta(A)-\alpha(B))s} ds = \frac{(k+j)!}{(\beta(A)-\alpha(B))^{k+j+1}}.$$

This and the previous lemma proves the theorem. $\square$

Similarly one can consider equation (6.1) with $A = a_1 + a_2\tilde{A}$, $B = b_1 + b_2\tilde{B}$, where a_1, a_2, b_1, b_2 are constants, $\tilde{A}, \tilde{B}$ satisfy the hypothesis of Corollary 4.4.

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