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THE SPECTRAL EQUALITY FOR UPPER TRIANGULAR OPERATOR MATRICES WITH UNBOUNDED ENTRIES

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Abstract. Let

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : D(M_C) \subset X \times X \to X \times X$$

be a 2×2 unbounded upper triangular operator matrix on the complex Hilbert space $X\times X$. We investigate the conditions under which $\sigma(M_C) = \sigma(A)\cup\sigma(B)$ holds in the diagonally dominant ($D(M_C) = D(A)\times D(B)$) and upper dominant case ($D(M_C) = D(A)\times D(C)$). Some necessary and sufficient conditions are obtained. The results generalize some results of Han, Du, and Barraa in the bounded case.

1. Introduction

Because of the important applications of block operator matrices in mathematics and physics, the spectral properties of block operator matrices are studied by many researchers (see [1, 2], [4]–[8], [10]–[14]). Motivated by the description of the stability of the spectrum, the spectral equality

$$\sigma(M_C) = \sigma(A)\cup\sigma(B) \tag{1.1}$$

for bounded operators A,B,C, and $M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ became an interesting research topic. For instance, it was shown that if $AZ - ZB = C$ or $AC = CB$ or $(A-\mu I)C = 0$ or $C(D-\mu I) = 0$ for some $\mu\in\mathbb{C}$, then (1.1) holds [1]; if A,B are normal operators, then (1.1) is satisfied for every $C\in\mathbb{B}(X)$ [2]; if $\sigma(A)\cap\sigma(B)$ has no interior points, then (1.1) holds [4]. However, a number of block operator matrices in theory or applications are not bounded, and the above mentioned conclusions do not hold for unbounded block operator matrices in general. Indeed, let

$$M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \triangleq \begin{bmatrix} I & C \\ 0 & I \end{bmatrix},$$

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where C is an unbounded operator in the Hilbert space X. Then A,B are selfadjoint operators and hence $\sigma(A)\cap\sigma(B)$ has no interior points and $AC=CB$, but $\sigma(M_C)=\mathbb{C}\neq\sigma(A)\cup\sigma(B)=\{1\}$. The main goal of this paper is to investigate the set of unbounded operators A,B and C for which (1.1) holds. Some necessary and sufficient conditions are obtained. For a given upper triangular operator matrix

$$M_C=\begin{bmatrix} A & C \\ 0 & B \end{bmatrix},$$

its natural domain is

$$D(M_C)=D(A)\oplus(D(C)\cap D(B)).$$

However, in the description of E-characteristic operator function (or F-characteristic operator function ([10]) and the perturbation theory for operator matrices always distinguish two classes of block operator matrices, diagonally dominant and upper dominant ([13]). Hence, we investigate the necessary and sufficient conditions for spectral equality (1.1) in diagonally dominant case and upper dominant case in this paper.

Throughout this paper X denotes a complex Hilbert space, $\rho(T)$ $\sigma(T)$ and $N(T)$ $=\{x\in D(T): Tx=0\}$ denote the resolvent set, the spectrum and the null space of a linear operator T on X, respectively. We would like to point out that there are different definitions of the resolvent set $\rho(T)$ and the spectrum $\sigma(T)$ for an unbounded operator T. We adopt the definitions in [5, 6]. We say that $\lambda\in\rho(T)$ if $T-\lambda I$ is injective, $R(T-\lambda I)=X$ and the inverse operator $(T-\lambda I)^{-1}$ is bounded. Therefore the spectrum $\sigma(T)=\mathbb{C}\setminus\rho(T)$ can be divided into the following three disjoint subsets: the point spectrum, the residual spectrum, the continuous spectrum [3]. Precisely they are

$$\begin{aligned}
\sigma_p(T) &= \{\lambda\in\mathbb{C}: T-\lambda I \text{ is not injective}\};\\
\sigma_r(T) &= \{\lambda\in\mathbb{C}: T-\lambda I \text{ is injective, } \overline{R(T-\lambda I)}\neq X\};\\
\sigma_c(T) &= \{\lambda\in\mathbb{C}: T-\lambda I \text{ is injective, } \overline{R(T-\lambda I)}=X, (T-\lambda I)^{-1} \text{ is unbounded}\},
\end{aligned}$$

respectively.

REMARK 1.1. If T is closed, then $\lambda\in\rho(T)$ if and only if $T-\lambda I$ is bijective; $\lambda\in\sigma_c(T)$ if and only if $T-\lambda I$ is injective, $\overline{R(T-\lambda I)}=X$ and $R(T-\lambda I)\neq X$.

For a linear operator T, we define (see [15])

$$\begin{aligned}
\sigma_{p,1}(T) &= \{\lambda\in\sigma_p(T): R(T-\lambda I)=X\};\\
\sigma_{p,2}(T) &= \{\lambda\in\sigma_p(T): R(T-\lambda I)\neq X, \overline{R(T-\lambda I)}=X\};\\
\sigma_{p,3}(T) &= \{\lambda\in\sigma_p(T): \overline{R(T-\lambda I)}=R(T-\lambda I)\neq X\};\\
\sigma_{p,4}(T) &= \{\lambda\in\sigma_p(T): \overline{R(T-\lambda I)}\neq R(T-\lambda I), \overline{R(T-\lambda I)}\neq X\};\\
\sigma_{r,1}(T) &= \{\lambda\in\sigma_r(T): R(T-\lambda I) \text{ is closed}\};\\
\sigma_{r,2}(T) &= \{\lambda\in\sigma_r(T): R(T-\lambda I) \text{ is not closed}\},
\end{aligned}$$

and

$$\sigma_{ap}(T)=\{\lambda\in\mathbb{C}: \exists\{x_n\}_{n=1}^{+\infty}\subset D(T), \|x_n\|=1, n=1,2,\cdots,(T-\lambda I)x_n\to 0, n\to+\infty\}.$$

The set $\Gamma(T)=\mathbb{C}\setminus\sigma_{ap}(T)$ is called the set of points of regular type. A linear operator T is said to be bounded below ([3]) if there exists a constant $M>0$ such that

$$\|Tx\|\geqslant M\|x\|,\quad \forall x\in D(T).$$

For a densely defined closed operator T in X, $\lambda\in\Gamma(T)\iff T-\lambda I$ is bounded below $\iff (T-\lambda I)^{-1}$ is bounded (see Problem 73 of [3]) and we also have

$$\begin{aligned}\sigma(T)&=\sigma_{ap}(T)\cup\sigma_{r,1}(T);\\ \Gamma(T)&=\rho(T)\cup\sigma_{r,1}(T).\end{aligned}$$

The following lemma will be useful in the forthcoming discussion.

LEMMA 1.2. Let T be densely defined closed operator in X. Then

(i) $\lambda\in\sigma_{p,1}(T)$ if and only if $\overline{\lambda}\in\sigma_{r,1}(T^*)$;

(ii) $\lambda\in\sigma_{p,2}(T)$ if and only if $\overline{\lambda}\in\sigma_{r,2}(T^*)$;

(iii) $\lambda\in\sigma_{p,3}(T)$ if and only if $\overline{\lambda}\in\sigma_{p,3}(T^*)$;

(iv) $\lambda\in\sigma_{p,4}(T)$ if and only if $\overline{\lambda}\in\sigma_{p,4}(T^*)$;

(v) If $\sigma_{r,1}(T)$ is not empty, then it is an open set, and hence $\sigma_{p,1}(T)$ is also open.

Proof. For a densely defined closed operator T, $R(T)$ is closed if and only if $R(T^*)$ is closed, so the proofs of (i), (ii), (iii) and (iv) are trivial. Next we will prove that $\sigma_{r,1}(T)$ is an open set when it is not empty. Let $\lambda\in\Gamma(T)$. Then there exists a constant $M>0$ such that

$$\|(T-\lambda I)x\|\geqslant M\|x\|,\ \forall x\in D(T).$$

For all $|\lambda-\lambda'|<\frac{M}{2}$, and $x\in D(T)$, $\|x\|=1$ we have

$$\|(T-\lambda' I)x\|\geqslant\|(T-\lambda I)x\|-|\lambda-\lambda'|\geqslant\frac{M}{2},$$

which implies that $\Gamma(T)$ is open. Since $\rho(T)$ is also open, $\overline{\rho(T)}\cap\overline{\sigma_{r,1}(T)}=\emptyset$ and

$$\begin{aligned}\Gamma(T)\setminus\overline{\rho(T)}&=(\rho(T)\cup\sigma_{r,1}(T))\setminus\overline{\rho(T)}\\ &=\sigma_{r,1}(T)\setminus\overline{\rho(T)}\\ &=\sigma_{r,1}(T),\end{aligned}$$

and thus $\sigma_{r,1}(T)$ is open. By (i) we have $\sigma_{r,1}(T^*)$ and $\sigma_{p,1}(T)$ are symmetric with respect to the real axis, so $\sigma_{p,1}(T)$ is open and the proof is complete. □

The main results of this paper are the following:

THEOREM 1.3. Let $M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : D(A) \times D(B) \to X \times X$ be a densely defined upper triangular operator matrix, where A, B are densely defined closed and C is closable. Then $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ if and only if $(\sigma_{r,1}(A) \cap \sigma_{p,1}(B)) \cup (\rho(A) \cap \sigma_{p,1}(B)) = \emptyset$ or any $\lambda \in (\sigma_{r,1}(A) \cap \sigma_{p,1}(B)) \cup (\rho(A) \cap \sigma_{p,1}(B))$ satisfies one of the following:

(i) $N(C) \cap N(B - \lambda I) \neq \{0\}$;

(ii) $CN(B - \lambda I) \cap R(A - \lambda I) \neq \{0\}$;

(iii) $CN(B - \lambda I) + R(A - \lambda I) \neq X$.

REMARK 1.4. Under the additional condition $D(A^*) \subset D(C^*)$, any $\lambda \in \rho(A) \cap \sigma_{p,1}(B)$, let $y_0 \in N(B - \lambda I)$. Then $\overline{(A - \lambda I)^{-1}C} y_0 \in D(A)$ and

$$\begin{bmatrix} A - \lambda I & C \\ 0 & B - \lambda I \end{bmatrix} \begin{bmatrix} -\overline{(A - \lambda I)^{-1}C} y_0 \\ y_0 \end{bmatrix} = 0.$$

Thus $CN(B - \lambda I) \cap R(A - \lambda I) \neq \{0\}$. Therefore, Theorem 1.3 can be stated as $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ if and only if $\sigma_{r,1}(A) \cap \sigma_{p,1}(B) = \emptyset$ or any $\lambda \in \sigma_{r,1}(A) \cap \sigma_{p,1}(B)$ satisfies one of the three conditions (i), (ii) and (iii).

In general, the spectral property $\sigma(M_C) \subset \sigma(A) \cup \sigma(B)$ does not hold in the upper dominant case. Therefore, to obtain the necessary and sufficient condition for

$$\sigma(M_C) = \sigma(A) \cup \sigma(B),$$

it needs to impose additional conditions on A, B and C.

THEOREM 1.5. Let $M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : D(A) \times D(C) \to X \times X$ be a densely defined closed upper triangular operator matrix, where A, B and C are densely defined closed. If $D(A^*) \subset D(C^*)$, $\rho(A) \cap \rho(B) \neq \emptyset$ and $\overline{R((B - \mu I)|_{D(C)})} = X$ for some $\mu \in \rho(A) \cap \rho(B)$. Then $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ if and only if $(\sigma_{r,1}(B^*) \cap \sigma_{p,1}(A^*)) \cup (\rho(B^*) \cap \sigma_{p,1}(A^*)) = \emptyset$ or any $\lambda \in (\sigma_{r,1}(B^*) \cap \sigma_{p,1}(A^*)) \cup (\rho(B^*) \cap \sigma_{p,1}(A^*))$ satisfies one of the followings:

(i) $N(C^*) \cap \mathbb{N}(A^* - \lambda I) \neq \{0\}$;

(ii) $C^*\mathbb{N}(A^* - \lambda I) \cap R(B^* - \lambda I) \neq \{0\}$;

(iii) $C^*\mathbb{N}(A^* - \lambda I) + R(B^* - \lambda I) \neq X$.

When A, B and C are everywhere defined bounded operators, the results of Han, Du and Barraa's can be deduced as a corollary.

COROLLARY 1.6. (see Theorem 1 in [1]) If the everywhere defined bounded operators A, B and C satisfy one of the followings conditions:

(i) There exists everywhere defined bounded operator Z such that $C = AZ - ZB$,

(ii) $AC = CB$,

(iii) $(A-\mu I)C = 0$ or $C(B-\mu I) = 0$ for some $\mu \in \mathbb{C}$,

then $\sigma(M_C) = \sigma(A) \cup \sigma(B)$.

Proof. Since

$$(\sigma(A)\cup\sigma(B)) \setminus (\sigma_{r,1}(A)\cap\sigma_{p,1}(B)) \subset \sigma(M_C) \subset \sigma(A)\cup\sigma(B),$$

under the condition $(\sigma_{r,1}(A)\cap\sigma_{p,1}(B)) = \emptyset$, the proof is trivial, so we assume $(\sigma_{r,1}(A)\cap\sigma_{p,1}(B)) \neq \emptyset$.

(i) We prove this statement by using Theorem 1.3, rather than applying the similarity of M_C and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. If there exists everywhere defined bounded operator Z such that $C = AZ - ZB$, then for any $\lambda \in (\sigma_{r,1}(A)\cap\sigma_{p,1}(B))$ we have

$$\begin{aligned} CN(B-\lambda I) &= ((A-\lambda I)Z - Z(B-\lambda I))N(B-\lambda I) \\ &= (A-\lambda I)ZN(B-\lambda I) \subset R(A-\lambda I), \end{aligned}$$

which implies $CN(B-\lambda I) + R(A-\lambda I) \neq X$. Hence by Theorem 1.3 and Remark 1.4, we have $\sigma(M_C) = \sigma(A)\cup\sigma(B)$.

(ii) When $AC = CB$, let $(\sigma_{r,1}(A)\cap\sigma_{p,1}(B)) \neq \emptyset$ and $\lambda \in (\sigma_{r,1}(A)\cap\sigma_{p,1}(B))$. Then $(A-\lambda I)C = C(B-\lambda I)$ and $(A-\lambda I)Cy = 0$ for any $y \in N(B-\lambda I)$. Since $A-\lambda I$ is injective, it follows that $Cy = 0$. Therefore, $CN(B-\lambda I) + R(A-\lambda I) = R(A-\lambda I) \neq X$. Hence by Theorem 1.3 and Remark 1.4, we have $\sigma(M_C) = \sigma(A)\cup\sigma(B)$.

(iii) When $(A-\mu I)C = 0$, for some $\mu \in \mathbb{C}$, let $\lambda \in (\sigma_{r,1}(A)\cap\sigma_{p,1}(B))$. Then we have $(A-\lambda I)C = (\mu-\lambda)C$. If $\mu = \lambda$, then $(A-\lambda I)C = 0$ and thus $C = 0$, hence $\sigma(M_C) = \sigma(A)\cup\sigma(B)$. If $\mu \neq \lambda$, then $C = \frac{1}{\mu-\lambda}(A-\lambda I)C$. Thus

$$R(C|_{N(B-\lambda I)}) \subset R(A-\lambda I).$$

Hence $CN(B-\lambda I) + R(A-\lambda I) \neq X$, and by Theorem 1.3 and Remark 1.4, we have $\sigma(M_C) = \sigma(A)\cup\sigma(B)$.

When $C(B-\mu I) = 0$ for some $\mu \in \mathbb{C}$, let $\lambda \in (\sigma_{r,1}(A)\cap\sigma_{p,1}(B))$. Then

$$C(B-\lambda I) = (\mu-\lambda)C.$$

If $\mu = \lambda$, then $C(B-\lambda I) = 0$. In view of $R(B-\lambda I) = X$, we have $C = 0$. Thus $\sigma(M_C) = \sigma(A)\cup\sigma(B)$. If $\mu \neq \lambda$, then $C = \frac{1}{\mu-\lambda}C(B-\lambda I)$. Thus

$$CN(B-\lambda I) = \{0\},$$

which implies $\mathbb{N}(C)\cap\mathbb{N}(B-\lambda I) \neq \{0\}$, and thus $\sigma(M_C) = \sigma(A)\cup\sigma(B)$. □

COROLLARY 1.7. (see Corollary 8–11 in [4]) If the everywhere defined bounded operators A and B satisfy one of the followings conditions:

(i) $\sigma(A)\cap\sigma(B)$ has no interior points,

(ii) $\rho_\sigma^l(A)\cap\rho_\sigma^r(B)=\emptyset$,

(iii) either A is cohyponormal (i.e., A^* is hyponormal or $AA^*\geqslant A^*A$) or B is hyponormal (i.e., $B^*B\geqslant BB^*$),

then $\sigma(M_C)=\sigma(A)\cup\sigma(B)$, where $\rho_\sigma^l(A)=\{\lambda\in\sigma(A): A-\lambda I \text{ is injective}\}$, $\rho_\sigma^r(B)=\{\lambda\in\sigma(B): B-\lambda I \text{ is surjective}\}$.

Proof. (i) Note that $((\sigma_{r,1}(A)\cap\sigma_{p,1}(B))\subset\sigma(A)\cap\sigma(B)$ and by Lemma 1.2 $(\sigma_{r,1}(A)\cap\sigma_{p,1}(B))$ is composed of interior points. Therefore, if $\sigma(A)\cap\sigma(B)$ has no interior points, then

$$(\sigma_{r,1}(A)\cap\sigma_{p,1}(B))=\emptyset.$$

Thus by Theorem 1.3 and Remark 1.4, we have $\sigma(M_C)=\sigma(A)\cup\sigma(B)$.

(ii) Note that $\rho_\sigma^l(A)\cap\rho_\sigma^r(B)=\emptyset$ implies $\sigma_{r,1}(A)\cap\sigma_{p,1}(B)=\emptyset$. By Theorem 1.3 and Remark 1.4, we have $\sigma(M_C)=\sigma(A)\cup\sigma(B)$.

(iii) If A is cohyponormal, then we can claim that $\sigma_r(A)=\emptyset$. In fact, let $\lambda\in\sigma_r(A)$, then $\overline{\lambda}\in\sigma_p(A^*)$. Then there exists $x_0\neq 0$ such that $(A^*-\lambda I)x_0=0$. Since $A^*-\overline{\lambda}I$ is hyponormal, so

$$\|(A^*-\overline{\lambda}I)x_0\|\geqslant\|(A-\lambda I)x_0\|,$$

which contradicts with $\lambda\in\sigma_r(A)$. Hence $\sigma_r(A)=\emptyset$ and thus $\sigma_{r,1}(A)\cap\sigma_{p,1}(B)=\emptyset$. Furthermore, $D(A^*)\subset D(C^*)$ holds naturally, by Theorem 1.3 and Remark 1.4, we have $\sigma(M_C)=\sigma(A)\cup\sigma(B)$. When B is hyponormal, it is easy to show that $\sigma_{p,1}(B)=\emptyset$ and we also have $\sigma(M_C)=\sigma(A)\cup\sigma(B)$. $\square$

COROLLARY 1.8. (see [2]) If A,B are normal operators, then $\sigma(M_C)=\sigma(A)\cup\sigma(B)$.

Proof. If A,B are normal operators, then $\sigma_r(A),\sigma_r(B)$ are empty, so $\sigma_{r,1}(A)=\emptyset$ and $\sigma_{p,1}(B)=\emptyset$. Thus $\sigma_{r,1}(A)\cap\sigma_{p,1}(B)=\emptyset$, and by Remark 1.4 we have $\sigma(M_C)=\sigma(A)\cup\sigma(B)$. $\square$

By definition of the point spectrum, it is easy to show that $M_C=\begin{bmatrix}A & C\\ 0 & B\end{bmatrix}$ is injective if and only if

(i) A is injective,

(ii) $N(C)\cap N(B)=\{0\}$,

(iii) $R(A)\cap CN(B)=\{0\}$.

Thus the following proposition follows.

PROPOSITION 1.9. Let $M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : D(A) \times (D(C) \cap D(B)) \to X \times X$ be an upper triangular operator matrix, where A, B, C are linear operators. Then $\sigma_p(M_C) = \sigma_p(A) \cup \sigma_p(B)$ if and only if $\sigma_p(B|_{D(C)\cap D(B)}) \subset \sigma_p(A)$ or for any $\lambda \in \sigma_p(B|_{D(C)\cap D(B)}) \setminus \sigma_p(A)$ such that $N(C) \cap N(B|_{D(C)\cap D(B)} - \lambda I) \neq \{0\}$ or $CN(B|_{D(C)\cap D(B)} - \lambda I) \cap R(A - \lambda I) \neq \{0\}$ holds.

As an application of Proposition 1.9, the following corollary illustrates that the conditions of [1] are sufficient to obtain the equality of point spectrum.

COROLLARY 1.10. If the everywhere defined bounded operators A, B and C satisfy one of the followings conditions:

(i) there exists everywhere defined bounded operator Z such that $C = AZ - ZB$,

(ii) $AC = CB$,

(iii) $(A - \mu I)C = 0$ for some $\mu \in \mathbb{C}$,

(iv) $C(B - \mu I) = 0$ for some $\mu \in \mathbb{C}$, $\mu \notin \sigma_{p,3}(B) \cup \sigma_{p,4}(B)$,

then $\sigma_p(M_C) = \sigma_p(A) \cup \sigma_p(B)$.

Proof. (i) This statement can be proved by the similarity of M_C and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. However, we would like to give another method by Proposition 1.9. Without loss of generality, let $\lambda \in \sigma_p(B) \setminus \sigma_p(A)$. Then

$$C = (A - \lambda I)Z - Z(B - \lambda I),$$

and

$$CN(B - \lambda I) = (A - \lambda I)ZN(B - \lambda I).$$

Thus $CN(B - \lambda I) \subset R(A - \lambda I)$, and $CN(B - \lambda I) \cap R(A - \lambda I) = CN(B - \lambda I)$. If we assume $N(C) \cap N(B - \lambda I) = \{0\}$. Then $CN(B - \lambda I) \neq \{0\}$, and thus $CN(B - \lambda I) \cap R(A - \lambda I) \neq \{0\}$. By Proposition 1.9, we have $\sigma_p(M_C) = \sigma(A)_p \cup \sigma_p(B)$. Similarly, if we assume $CN(B - \lambda I) \cap R(A - \lambda I) = \{0\}$, then we have $N(C) \cap N(B - \lambda I) \neq \{0\}$. By Proposition 1.9, we also have $\sigma_p(M_C) = \sigma(A)_p \cup \sigma_p(B)$.

(ii) Let $\lambda \in \sigma_p(B) \setminus \sigma_p(A)$. Then

$$(A - \lambda I)C = C(B - \lambda I)$$

and

$$(A - \lambda I)CN(B - \lambda I) = 0.$$

Since $\lambda \notin \sigma_p(A)$, $CN(B - \lambda I) = 0$ and thus $N(C) \cap N(B - \lambda I) \neq \{0\}$. By Proposition 1.9, we have $\sigma_p(M_C) = \sigma(A)_p \cup \sigma_p(B)$.

(iii) When $(A-\mu I)C=0$ for some $\mu \in \mathbb{C}$, let $\lambda \in \sigma_p(B) \setminus \sigma_p(A)$. Then in the case of $\mu=\lambda$, the conclusion is trivial. Hence we assume that $\lambda \neq \mu$. In view of $(A-\lambda I)C=(\mu-\lambda)C$, we have $(A-\lambda I)CN(B-\lambda I)=(\mu-\lambda)CN(B-\lambda I)$. Thus

$$CN(B-\lambda I) \cap R(A-\lambda I)=CN(B-\lambda I).$$

If $CN(B-\lambda I) \neq \{0\}$, by Proposition 1.9, we have $\sigma_p(M_C)=\sigma(A)_p \cup \sigma_p(B)$. If $CN(B-\lambda I)=\{0\}$, then $N(C) \cap N(B-\lambda I) \neq \{0\}$, and also have $\sigma_p(M_C)=\sigma(A)_p \cup \sigma_p(B)$.

(iv) When $C(B-\mu I)=0$ for some $\mu \in \mathbb{C}, \mu \notin \sigma_{p,3}(B) \cup \sigma_{p,4}(B)$, let $\lambda \in \sigma_p(B) \setminus \sigma_p(A)$. Then

$$C(B-\lambda I)=(\mu-\lambda)C.$$

Case 1: $\mu \neq \lambda$. Then

$$\frac{1}{\mu-\lambda}C(B-\lambda I)N(B-\lambda I)=CN(B-\lambda I)=\{0\}.$$

Thus $N(C) \cap N(B-\lambda I) \neq \{0\}$, and by Proposition 1.9, we have $\sigma_p(M_C)=\sigma(A)_p \cup \sigma_p(B)$.

Case 2: $\mu=\lambda$. Then $C(B-\lambda I)=0$, and we can claim that $C=0$. Thus $\sigma_p(M_C)=\sigma(A)_p \cup \sigma_p(B)$. Indeed, $\lambda \notin \sigma_{p,3}(B) \cup \sigma_{p,4}(B)$. Thus the range $R(B-\lambda I)$ is dense in X. In view of C is everywhere defined bounded, we have $C=0$. $\square$

The following example illustrates that Theorem 1.3 and Proposition 1.9 are also useful to characterize the spectra of unbounded operator matrices.

EXAMPLE 1.11. Consider the PDE of rectangular plate with two opposite edges simply supported

$$E\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 w=f(x,y), \tag{1.2}$$

where the boundary conditions for the simply supported edges are

$$w=0, \frac{\partial^2 w}{\partial y^2}=0, \text{ for } y=0 \text{ and } y=1.$$

Set that $\theta=\frac{\partial w}{\partial x}$, $q=E(\frac{\partial^3 w}{\partial x^3}+\frac{\partial^3 w}{\partial x \partial^2 y})$, $p=-E(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2})$. Then the Hamiltonian system (see [8]) of (1.2) is

$$\frac{\partial}{\partial x}\begin{bmatrix} w \\ \theta \\ p \\ q \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{\partial^2}{\partial y^2} & 0 & 0 & -\frac{1}{E} \\ 0 & 0 & 0 & \frac{\partial^2}{\partial y^2} \\ 0 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} w \\ \theta \\ p \\ q \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ f(x,y) \\ 0 \end{bmatrix},$$

and the corresponding Hamiltonian operator in $L^2[0,1] \times L^2[0,1] \times L^2[0,1] \times L^2[0,1]$ is

$$H=\begin{bmatrix} A & C \\ 0 & B \end{bmatrix},$$

where $A = \begin{bmatrix} 0 & I \\ -\frac{d^2}{dy^2} & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{E} \end{bmatrix}$, $B = \begin{bmatrix} 0 & \frac{d^2}{dy^2} \\ -I & 0 \end{bmatrix}$. In view of the boundary conditions, we can see $B = -A^*$ and

$$\sigma(A) = \sigma(B) = \sigma_p(A) = \{k\pi : k = \pm 1, \pm 2, \cdots\}.$$

that is, $\sigma_{r,1}(A) = \emptyset$ and thus $\sigma(H) = \sigma(A) \cup \sigma(B) = \{k\pi : k = \pm 1, \pm 2, \cdots\}$.

2. The proof of main results

To prove the main results, we start with the following lemma:

LEMMA 2.1. *Let $M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} : D(A) \times D(B) \to X \times X$ be a densely defined upper triangular operator matrix, where A, B are densely defined closed and C is closable. Then $0 \in \rho(M_C)$ if and only if*

(i) A is bounded below,

(ii) B is surjective,

(iii) $N(C) \cap N(B) = \{0\}$,

(iv) $CN(B) \oplus R(A) = X$.

Proof. Since A, B are densely defined closed, C is closable and $D(B) \subset D(C)$, we can claim that M_C is closed. In fact, let $\left\{ \begin{bmatrix} x_n \\ y_n \end{bmatrix} \right\}_{n=1}^{\infty} \subset D(M_C)$, $\begin{bmatrix} x_n \\ y_n \end{bmatrix} \to \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ and

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} \to \begin{bmatrix} f_0 \\ g_0 \end{bmatrix}.$$

In view of closedness of B we have $y_0 \in D(B)$. In addition, C closable and $D(B) \subset D(C)$ imply that C is B-bounded (see Remark 1.5 in [9]). Thus $\{Cy_n\}_{n=1}^{\infty}$ is a Cauchy sequence, and hence $\{Cy_n\}_{n=1}^{\infty}$ is convergent. Let $Cy_n \to h_0$. Then

$$Ax_n \to f_0 - h_0$$

and $x_0 \in D(A)$, which imply $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in D(M_C)$. Thus M_C is closed. When $0 \in \rho(M_C)$, the proofs of (i), (ii) and (iii) are trivial. Next we will prove $CN(B) \oplus R(A) = X$. Let $y_0 \in CN(B) \cap R(A)$. Then there exist $x_1 \in D(A)$, $x_2 \in N(B)$ such that

$$Ax_1 = Cx_2 = y_0,$$

and

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix} = 0,$$

which implies $x_1 = x_2 = 0$, and thus $CN(B) \cap R(A) = \{0\}$. Moreover, in view of $0 \in \rho(M_C)$ and M_C is closed, for any $f \in X$ there exist $x_1 \in D(A), x_2 \in N(B)$ such that

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix},$$

which implies $CN(B) \oplus R(A) = X$.

To prove the necessity part, it suffices to show that M_C is bijective. Let

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

Then $Ax = B(-y) \in R(A) \cap CN(B)$, and $Ax = 0$ and thus $x = 0$. Furthermore, in view of $N(C) \cap N(B) = \{0\}$, we also have $y = 0$. Hence M_C is injective. For any $\begin{bmatrix} f \\ g \end{bmatrix} \in X \times X$, considering the surjectiveness of B, there exists $x_3 \in D(B)$ such that

$$Bx_3 = g.$$

Furthermore, since $CN(B) \oplus R(A) = X$, there exist $x_1 \in D(A), x_2 \in N(B)$ such that

$$Ax_1 + Cx_2 = f - Cx_3.$$

Thus

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 + x_3 \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix},$$

which implies that M_C is surjective. The proof is complete. $\square$

We are going to give a proof of Theorem 1.3.

Proof of Theorem 1.3. When $\sigma(M_C) = \sigma(A) \cup \sigma(B)$, without loss of generality, let $(\sigma_{r,1}(A) \cap \sigma_{p,1}(B)) \cup (\rho(A) \cap \sigma_{p,1}(B)) \neq \emptyset$. Then

$$(\sigma_{r,1}(A) \cap \sigma_{p,1}(B)) \cup (\rho(A) \cap \sigma_{p,1}(B)) \subset \sigma(M_C).$$

Suppose that $\lambda \in ((\sigma_{r,1}(A) \cap \sigma_{p,1}(B)) \cup (\rho(A) \cap \sigma_{p,1}(B))$ satisfies $N(C) \cap N(B - \lambda) = \{0\}$, $CN(B - \lambda) \cap R(A - \lambda) = X$ and $CN(B - \lambda) + R(A - \lambda) = X$. Then by Lemma 2.1 we have $\lambda \in \rho(M_C)$, which contradicts with $\lambda \in \sigma(M_C)$.

We are now going to prove the necessity part. When $(\sigma_{r,1}(A) \cap \sigma_{p,1}(B)) \cup (\rho(A) \cap \sigma_{p,1}(B)) = \emptyset$, to prove $\sigma(M_C) = \sigma(A) \cup \sigma(B)$, it suffices to show that $\lambda \in \rho(M_C)$ implies $\lambda \in \rho(A) \cap \rho(D)$. Let $\lambda \in \rho(M_C)$. Then by Lemma 2.1 we have $\lambda \in (\rho(A) \cup \sigma_{r,1}(A)) \cap (\rho(B) \cup \sigma_{p,1}(B))$. Suppose $\lambda \in \sigma_{r,1}(A) \cap \rho(B)$. Then in view of $D(B) \subset D(C)$ we have

$$\begin{bmatrix} A - \lambda I & C \\ 0 & B - \lambda I \end{bmatrix} = \begin{bmatrix} I & C(B - \lambda I)^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - \lambda I & 0 \\ 0 & B - \lambda I \end{bmatrix}.$$

In view of $\lambda \in \rho(M_C) \cap \rho(B)$, we have $\lambda \in \rho(A)$, which contradicts with $\lambda \in \sigma_{r,1}(A)$. Hence $\lambda \in \rho(A) \cap \rho(B)$.

When $(\sigma_{r,1}(A)\cap\sigma_{p,1}(B))\cup(\rho(A)\cap\sigma_{p,1}(B))\neq\emptyset$, let $\lambda\in\rho(M_C)$. Then by Lemma 2.1 we have $A-\lambda I$ is bounded below and $B-\lambda I$ is surjective, that say

$$\lambda\in(\rho(A)\cup\sigma_{r,1}(A))\cap(\sigma_{p,1}(B)\cup\rho(B)).$$

Suppose $\lambda\in(\sigma_{r,1}(A)\cap\sigma_{p,1}(B))\cup(\rho(A)\cap\sigma_{p,1}(B))$. Then in view of given condition, we have $\lambda\in\sigma(M_C)$, which contradicts with $\lambda\in\rho(M_C)$. Similarly, $\lambda\notin(\sigma_{r,1}(A)\cap\rho(B))$ so $\lambda\in\rho(A)\cap\rho(B)$ and proof is complete. $\square$

To prove the Theorem 1.5, we introduce following lemma.

LEMMA 2.2. Let $M_C=\begin{bmatrix}A & C\\ 0 & B\end{bmatrix}: D(A)\times D(C)\to X\times X$ be a densely defined closed upper triangular operator matrix, where A,B and C are densely defined closed. If $\rho(A)\cap\rho(B)\neq\emptyset$ and $\overline{R((B-\mu I)|_{D(C)})}=X$ for some $\mu\in\rho(A)\cap\rho(B)$. Then $M_C^*=\begin{bmatrix}A^* & 0\\ C^* & B^*\end{bmatrix}$.

Proof. Since $\begin{bmatrix}A^* & 0\\ C^* & B^*\end{bmatrix}\subset M_C^*$ is trivial, to prove $M_C^*=\begin{bmatrix}A^* & 0\\ C^* & B^*\end{bmatrix}$, it suffices to show that $D(M_C^*)\subset D\left(\begin{bmatrix}A^* & 0\\ C^* & B^*\end{bmatrix}\right)$. Let $\begin{bmatrix}x^*\\ y^*\end{bmatrix}\in D(M_C^*)$. Then in view of $R(A^*-\overline{\mu}I)=X$ and $R(B^*-\overline{\mu}I)=X$, we have

$$R\left(\begin{bmatrix}A^*-\overline{\mu}I & 0\\ C^* & B^*-\overline{\mu}I\end{bmatrix}\right)=X\times X.$$

Thus there exists $\begin{bmatrix}x\\ y\end{bmatrix}\in D(A^*)\times D(B^*)$ such that

$$\begin{bmatrix}A^*-\overline{\mu}I & 0\\ C^* & B^*-\overline{\mu}I\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=(M_C^*-\overline{\mu}I)\begin{bmatrix}x^*\\ y^*\end{bmatrix}. \tag{2.1}$$

Furthermore, from $R(A-\mu I)=X$, $\overline{R((B-\mu I)|_{D(C)})}=X$ and the closedness of M_C we can claim that $M_C^*-\overline{\mu}I$ is injective. Indeed, let $\begin{bmatrix}f\\ g\end{bmatrix}\in X\times X$. Then there exists $\{y_n\}\subset D(C)$ such that

$$(B-\mu I)y_n\to g.$$

Since $R(A-\mu I)=X$, there exists $x_n\in D(A)$, $n=1,2,\cdots$ such that

$$(A-\mu I)x_n=g-Cy_n,\quad n=1,2,\cdots,$$

which implies $\begin{bmatrix}A-\mu I & C\\ 0 & B-\mu I\end{bmatrix}\begin{bmatrix}x_n\\ y_n\end{bmatrix}\to\begin{bmatrix}f\\ g\end{bmatrix}$. Thus $\overline{R(M_C-\mu I)}=X\times X$. Suppose $M_C^*-\overline{\mu}I$ is not injective. Then there exists $V_0\in D(M_C^*)$, $V_0\neq 0$ such that

$$((M_C^*-\overline{\mu}I)V_0,U)=0$$

holds for all $U \in D(M_C)$ and hence

$$(V_0, (M_C - \mu I)U) = 0,$$

which contradicts with $\overline{R(M_C - \mu I)} = X \times X$. Thus $M_C^* - \overline{\mu} I$ is injective. Since $\begin{bmatrix} A^* & 0 \\ C^* & B^* \end{bmatrix}$ $\subset M_C^*$, by equation (2.1), we have

$$(M_C^* - \overline{\mu}) \left(\begin{bmatrix} x^* \\ y^* \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix} \right) = 0.$$

Thus $\begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \in D\left(\begin{bmatrix} A^* & 0 \\ C^* & B^* \end{bmatrix} \right)$ and so $M_C^* = \begin{bmatrix} A^* & 0 \\ C^* & B^* \end{bmatrix}$. $\square$

Finally we give a proof of Theorem 1.5.

Proof of Theorem 1.5. By Lemma 2.1 $M_C^* = \begin{bmatrix} A^* & 0 \\ C^* & B^* \end{bmatrix}$, and similar to the proof of Theorem 1.3, we have $\sigma(M_C^*) = \sigma(A^*) \cup \sigma(B^*)$ if and only if $(\sigma_{r,1}(B^*) \cap \sigma_{p,1}(A^*)) \cup (\rho(B^*) \cap \sigma_{p,1}(A^*)) = \emptyset$ or for any $\lambda \in (\sigma_{r,1}(B^*) \cap \sigma_{p,1}(A^*)) \cup (\rho(B^*) \cap \sigma_{p,1}(A^*))$ satisfies one of the followings:

(i) $N(C^*) \cap N(A^* - \lambda I) \neq \{0\}$;

(ii) $C^* N(A^* - \lambda I) \cap R(B^* - \lambda I) \neq \{0\}$;

(iii) $C^* N(A^* - \lambda I) + R(B^* - \lambda I) \neq X$.

Furthermore, since M_C is closed, $\lambda \in \rho(M_C)$ if and only if $\overline{\lambda} \in \rho(M_C^*)$, $\lambda \in \rho(A) \cap \rho(B)$ if and only if $\overline{\lambda} \in \rho(A^*) \cap \rho(B^*)$. Hence $\sigma(M_C) = \sigma(A) \cup \sigma(B)$ if and only if $\sigma(M_C^*) = \sigma(A^*) \cup \sigma(B^*)$, and the proof is complete. $\square$

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