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ANALYSIS ON THE TIME–VARYING GAP OF DISCRETE TIME–VARYING LINEAR SYSTEMS

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Abstract. This paper is devoted to give some analysis on the time-varying (TV) gap of discrete time-varying linear systems in the frame work of nest algebra. It is shown that the TV gap has no advantage over gap metric in stability and robustness analysis on the single-sided discrete time-axis $\mathbb{N}$, while it stands out on the whole time-axis $\mathbb{Z}$.

1. Introduction

Gap metric comes from the concept of angles between closed manifolds of Hilbert space. Its origin is in functional analysis, where it was used in perturbation theory of linear operators. In 1980s, Zames and El-Sakkary defined gap metric on possibly unstable but stabilizable systems based on the graphs of the systems in [1]. The gap metric has been established as sensible and useful measure of distance between linear systems from a geometric perspective, it induces the weakest topology in which closed-loop stability (relative to continuity of closed-loop operators) is a robust property (see [2, 3, 4, 5]). Now, the gap metric and its variants have been powerful tools in solving the feedback stability and robust control problems ([6, 7]).

As the development of H^∞ control theory, a lot of insight has been obtained by considering its time-varying analogue, the nest algebra of causal stable operators on an appropriate complex Hilbert space of input-output signals. The control theory for infinite-dimensional time-varying linear systems (i.e., systems with an infinite number states) was sparked and developed based on the nest algebra approach in 1980s. Since then, there has been significant interest in this issue which generated numerous papers (see [8-12]). The stability theory for time-varying linear systems over singly infinite time-axis $\mathbb{N}$ has been well established from the input-output point of view. In such setting the gap metric also plays an important role in closed-loop stability analysis ([13]).

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It is shown that the closed-loop stability is equivalent to the gap metric related to the plant and the controller less than 1. The time-varying gap was introduced as an extension of the gap metric to study stability robustness of time-varying systems in [14], it is characterized in terms of every instant of time. For a class of time-varying linear systems, the TV gap metric also induces a correct topology in dealing with the closed stability and robust stability. Feintuch presented that the computation of the TV gap between two plants can be reduced to a two-block optimization problem, and the optimum is shown be equal to the norm of a time-varying Hankel operator defined on the space of causal Hilbert-Schmidt operators in [15]. The authors gave a characterization of TV gap between the plant and controller based on the commutant lifting theorem of nest algebra in [16].

Much of modern control focuses on the use of double-sided signals, such as communication systems and feedback systems in signal processing. However, the introduction of double-sided signals has been coupled with the inconsistencies in the corresponding mathematical formulism. It is well known that the closedness of the plant is a necessary condition for it being stabilizable. Unfortunately, the graph of a causal linear system on time-axis $\mathbb{Z}$ may possibly not be closed, this is a potential pitfalls of study with the double-infinite time-axis [17, 18]. While the setting in this paper depends on the resolution topology as in [13], it differs significantly from [17, 18]. Specifically, every causal time-varying linear system over the whole time-axis considered subsequently is a closed operator. In the framework of nest algebra, there have been numerous attempts in the literature to generalize results about linear systems on single sided signal space to double case [19, 20, 21]. It turns out that some results about stabilization of time-varying systems on $\mathbb{N}$ have nice generalization to those on $\mathbb{Z}$, such as the strong representation approach and Youla parametrization. However, some results on single-sided signal space can not be extended to the double-sided case. The gap metric is one of these situations. More precisely, the gap metric can not be used to judge the stability of closed-loop system on the double-sided signal space as it plays on the single-sided case, while the TV gap can handle this.

In this paper, we are interested in the analysis of TV gap of discrete time-varying linear systems defined on the singly and doubly infinite time-axis, respectively. First, we will show that the TV gap and gap are equivalent in study of stability and stability robustness over singly infinite time-axis $\mathbb{N}$. Moreover, the concepts of gap and TV gap are extended to the time-varying linear systems on doubly infinite time-axis $\mathbb{Z}$, and some analysis on TV gap is shown. At last, we derive that gap and TV-gap are still equivalent in measuring the distance between two linear plants on time-axis $\mathbb{Z}$. While, the TV gap and gap related to the plant and controller have distinct difference on double-sided signal space.

This paper is organized as follows. In Section 2, we introduce the notations, together with some definitions and preliminary results. The TV gap between plants defined on the whole time axis is considered in Section 3. The relationships between the gap and TV gap concerned with the plant and controller are carried out in Section 4. The paper ends with a conclusion.

2. Preliminaries

The symbols $\mathbb{Z}$ and $\mathbb{N}$ denote respectively the integer and natural numbers. Let $\mathscr{H}$ and $\mathscr{K}$ be two separable Hilbert spaces. The direct sum of $\mathscr{H}$ and $\mathscr{K}$ is defined by

$$\mathscr{H} \oplus \mathscr{K} = \left\{ \begin{bmatrix} h \\ k \end{bmatrix} : h \in \mathscr{H}, k \in \mathscr{K} \right\}.$$

$\mathscr{H} \oplus \mathscr{K}$ is also a Hilbert space with the inner product:

$$\left\langle \begin{bmatrix} h_1 \\ k_1 \end{bmatrix}, \begin{bmatrix} h_2 \\ k_2 \end{bmatrix} \right\rangle = \langle h_1, h_2 \rangle_{\mathscr{H}} + \langle k_1, k_2 \rangle_{\mathscr{K}}.$$

$\mathscr{B}(\mathscr{H}, \mathscr{K})$ denotes the Banach space of bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$ with the usual operator norm

$$||T|| = \sup_{x \in \mathscr{H}, ||x||_{\mathscr{H}} \leqslant 1} ||Tx||_{\mathscr{K}}.$$

The image and kernel of $T \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ are denoted, respectively, $\mathrm{Im}T = \{y \in \mathscr{K} : y = Tx, x \in \mathscr{H}\}$ and $\mathrm{Ker}T = \{x \in \mathscr{H} : Tx = 0\}$. The restriction of T to the closed subspace $\mathscr{V} \subseteq \mathscr{H}$ is denoted by $T|\mathscr{V}$. Let $P_{\mathscr{V}}$ denote the orthogonal projection with image $\mathscr{V}$. T^* stands for the adjoint of the operator T. Denote $\mathscr{B}(\mathscr{H}) := \mathscr{B}(\mathscr{H}, \mathscr{H})$.

If $\mathscr{M}_1$ and $\mathscr{M}_2$ are two closed linear subspaces of $\mathscr{H}$, the orthogonal difference of $\mathscr{M}_1$ and $\mathscr{M}_2$ is denoted by

$$\mathscr{M}_1 \ominus \mathscr{M}_2 = \mathscr{M}_1 \cap \mathscr{M}_2^{\perp} = \{h \in \mathscr{M}_1 : \langle h, k \rangle_{\mathscr{H}} = 0, \forall k \in \mathscr{M}_2\}.$$

The directed gap from $\mathscr{M}_1$ to $\mathscr{M}_2$ is given by $\vec{\delta}(\mathscr{M}_1, \mathscr{M}_2) = \|P_{\mathscr{M}_2^{\perp}} P_{\mathscr{M}_1}\|$. The gap between $\mathscr{M}_1$ and $\mathscr{M}_2$ is

$$\delta(\mathscr{M}_1, \mathscr{M}_2) = \max\{\vec{\delta}(\mathscr{M}_1, \mathscr{M}_2), \vec{\delta}(\mathscr{M}_2, \mathscr{M}_1)\}.$$

LEMMA 2.1. [13] *Let $\mathscr{M}_1, \mathscr{M}_2$ be the closed subspaces of Hilbert space $\mathscr{H}$. Then the following statements hold:*

1. $\delta(\mathscr{M}_1, \mathscr{M}_2) = \|P_{\mathscr{M}_1} - P_{\mathscr{M}_2}\|$.
2. $0 \leqslant \delta(\mathscr{M}_1, \mathscr{M}_2) \leqslant 1$.
3. If $\delta(\mathscr{M}_1, \mathscr{M}_2) < 1$, *then* $\vec{\delta}(\mathscr{M}_1, \mathscr{M}_2) = \vec{\delta}(\mathscr{M}_2, \mathscr{M}_1)$.
4. $\delta(\mathscr{M}_1, \mathscr{M}_2) < 1$ *if and only if* $\mathscr{M}_1 + \mathscr{M}_2^{\perp} = \mathscr{H}$ *and* $\mathscr{M}_1 \cap \mathscr{M}_2^{\perp} = \{0\}$.

As it is well known ([13], Chapter 5) the physical notion of causality for linear systems is formulated for linear transformations in terms of leaving invariant a totally ordered set of closed subspaces of $\mathscr{H}$. In this paper, we will deal with two situations in terms of the chains of orthogonal projections associated with these subspaces:

(1) Singly infinite chains: $\mathscr{P}_I = \{0 = P_0 < P_1 < \cdots < P_n < P_{n+1} < \cdots; I\}$.

(2) Doubly infinite chains: $\mathscr{P}_{II} = \{0; \cdots < P_{-m} < P_{-m+1} < \cdots < P_0 < P_1 < \cdots < P_n < P_{n+1} < \cdots; I\}$.

Assume that $\lim_{n\to\infty} P_n = I$ in the strong operator topology and in case (2) $\lim_{n\to-\infty} P_n = 0$ in the strong operator topology. Note that in both two cases the corresponding nest of subspaces is complete (see the general definition in [13], p. 47). Let $\mathscr{P}$ denote either $\mathscr{P}_I$ or $\mathscr{P}_{II}$. Having fixed the complete nest $\mathscr{P}$, for each $P_i (P_i \neq I)$, the seminorm on $\mathscr{H}$ is defined by

$$\|x\|_i = \|P_i x\|, \;\; x \in \mathscr{H}.$$

It is clear that $\bigcap_{P_i \in \mathscr{P}, P_i \neq I} \{x \in \mathscr{H} : \|x\|_i = 0\} = \{0\}$, $\mathscr{H}$ is a locally convex topology space whose topology is defined by $\{\|\cdot\|_i : P_i \in \mathscr{P}, P_i \neq I\}$, called the resolution topology. $\mathscr{H}$ is metrizable (see [22], p. 105). Let $\mathscr{H}_e$ denote the completion of the metric space $\mathscr{H}$ with respect to the resolution topology.

A linear transformation T on $\mathscr{H}_e$ is causal if $P_i T P_i = P_i T$ for all $P_i \in \mathscr{P}$. A (time-varying) linear system on $\mathscr{H}_e$ is a causal linear transformation on $\mathscr{H}_e$, which is continuous with respect to the resolution topology. If the linear transformations S, T on $\mathscr{H}_e$ are causal and continuous under the resolution topology, so are ST and $\alpha S + \beta T$, $\alpha, \beta \in \mathbb{C}$, it follows that the set of (time-varying) linear systems on $\mathscr{H}_e$ is an algebra. A time-varying linear system is stable if its restriction to $\mathscr{H}$ is a bounded linear operator. The set of stable time-varying linear systems on $\mathscr{H}_e$, denoted by $\mathscr{S}$, is a weakly closed algebra containing the identity, referred to in the operator theory literature as a nest algebra ([23]), i.e.,

$$\begin{aligned}\mathscr{S} &= \{T \in \mathscr{B}(\mathscr{H}) : T(I - P_i)\mathscr{H} \subseteq (I - P_i)\mathscr{H}, \forall P_i \in \mathscr{P}\} \\ &= \{T \in \mathscr{B}(\mathscr{H}) : P_i T P_i = P_i T, \forall P_i \in \mathscr{P}\}.\end{aligned}$$

In terms of the coordinate spaces determined by $\mathscr{P}$, the stable operator is the bounded operator whose matrix representations is lower triangular.

In this paper, we will consider two types of signal spaces: the double-sided signal space $\ell^2(\mathbb{Z})$ and single-sided signal space $\ell^2(\mathbb{N})$.

The double-sided signal space $\ell^2(\mathbb{Z})$ is the space of double-sided complex square summable sequences

$$\ell^2(\mathbb{Z}) := \left\{ x = (\cdots, x_{-1}, x_0, x_1, \cdots) : x_i \in \mathbb{C}, \sum_{i=-\infty}^{+\infty} |x_i|^2 < \infty \right\}.$$

The doubly infinite chains of $\ell^2(\mathbb{Z})$ is $\mathscr{P}_{II} = \{P_n : n \in \mathbb{Z} \cup \{\pm\infty\}\}$, where $P_{-\infty} = 0$, $P_{+\infty} = I$ and

$$P_n(\cdots, x_{-1}, x_0, \cdots, x_{n-1}, x_n, \cdots) = (\cdots, x_{-1}, x_0, \cdots, x_{n-1}, 0, \cdots), \;\; n \in \mathbb{Z}.$$

The extended space of $\ell^2(\mathbb{Z})$ is $\ell_e^2(\mathbb{Z}) := \{x = (\cdots, x_{-1}, x_0, x_1, \cdots) : P_n x \in \ell^2(\mathbb{Z}), n \in \mathbb{Z}\}$, and the algebra of time-varying linear systems on $\ell_e^2(\mathbb{Z})$ is denoted by $\mathscr{L}(\mathbb{Z})$, and $\mathscr{S}(\mathbb{Z})$ stands for the set of stable ones.

The single-sided signal space $\ell^2(\mathbb{N})$ is the space of signals on single-sided time axis $\mathbb{N}$,

$$\ell^2(\mathbb{N}) = \left\{ (x_0, x_1, \cdots) : x_n \in \mathbb{C}, \sum_{n=0}^{+\infty} |x_n|^2 < \infty \right\}.$$

In this case, the orthogonal projection P_n in the singly infinite chains $\mathscr{P}_I$ is

$$P_n(x_0,\cdots,x_{n-1},x_n,\cdots) = (x_0,\cdots,x_{n-1},0,\cdots), \quad n \geqslant 1,$$

and $P_0 = 0$.

Let $\ell_e^2(\mathbb{N})$, $\mathscr{L}(\mathbb{N})$ and $\mathscr{S}(\mathbb{N})$ be the corresponding definitions when choosing the signal space as $\ell^2(\mathbb{N})$. In fact, $\ell^2(\mathbb{N})$ can be seen as a subspace of $\ell^2(\mathbb{Z})$ by extending $x \in \ell^2(\mathbb{N})$ to $\ell^2(\mathbb{Z})$, in which the sequence is defined to be zero outside $\mathbb{N}$.

An important subalgebra of $\mathscr{S}(\mathbb{Z})$ is the algebra consisting of stable time-invariant systems on $\ell^2(\mathbb{Z})$. This algebra consists of the doubly infinite lower-triangular Laurant matrices, the stable one is H^∞.

For the consistency, by $\mathscr{L}$ we mean $\mathscr{L}(\mathbb{Z})$ or $\mathscr{L}(\mathbb{N})$, likewise, $\mathscr{S}$ stands for either $\mathscr{S}(\mathbb{Z})$ or $\mathscr{S}(\mathbb{N})$, ℓ^2 stands for either $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{N})$, and $\mathscr{P}$ stands for $\mathscr{P}_{II}$ or $\mathscr{P}_I$.

Consider the standard feedback configuration contributed by the plant $L \in \mathscr{L}$ and the controller $C \in \mathscr{L}$, and the closed-loop equation is

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} I & C \\ L & I \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

The closed-loop system $\{L,C\}$ is stable if each entry of the operator matrix from external input $e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ to internal input $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ belongs to $\mathscr{S}$, equivalently, $\begin{bmatrix} I & C \\ L & I \end{bmatrix} : \mathscr{D}(L) \oplus \mathscr{D}(C) \to \ell^2 \oplus \ell^2$ has a bounded causal inverse defined on $\ell^2 \oplus \ell^2$. This inverse is given by the transfer matrix

$$H(L,C) = \begin{bmatrix} (I-CL)^{-1} & -C(I-LC)^{-1} \\ -L(I-CL)^{-1} & (I-LC)^{-1} \end{bmatrix}. \tag{2.1}$$

L is stabilizable if there exists a $C \in \mathscr{L}$ such that $\{L,C\}$ is stable.

REMARK 2.1. It is known that the causal discrete linear system on semi-infinite axis defined by the resolution topology is closed. In the context of this paper, every linear system L is a causal linear transformation from $\ell_e^2(\mathbb{Z})$ to $\ell_e^2(\mathbb{Z})$, which implies that $\mathscr{D}(P_nL) = \ell^2(\mathbb{Z})$. Moreover, it can be easily checked that any linear system $L \in \mathscr{L}(\mathbb{Z})$ is a closed linear operator (see the proof of Theorem 5.3.4 in [13], p. 83).

DEFINITION 2.1. Let $L \in \mathscr{L}$, M, N, $\hat{M}$, $\hat{N} \in \mathscr{S}$, and $\mathscr{G}(L)$ denotes the graph of L.

1. $\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$ is a strong left representation of L if $\mathscr{G}(L) = \operatorname{Ker}\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$ and there exist $\hat{X}, \hat{Y} \in \mathscr{S}$ such that $\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix} \begin{bmatrix} \hat{X} \\ \hat{Y} \end{bmatrix} = I$. It is normalized if it is a co-isometry, i.e., $\hat{M}\hat{M}^* + \hat{N}\hat{N}^* = I$.

2. $\begin{bmatrix} M \\ N \end{bmatrix}$ is a strong right representation of L if $\mathscr{G}(L) = \mathrm{Im}\begin{bmatrix} M \\ N \end{bmatrix}$ and there exist $X, Y \in \mathscr{S}$ such that $\begin{bmatrix} Y & X \end{bmatrix}\begin{bmatrix} M \\ N \end{bmatrix} = I$. It is is normalized if it is an isometry, i.e., $M^*M + N^*N = I$.

In fact, the strong representation is an alternative, but equivalent, approach to the coprime factorization of linear systems. More precisely, $\begin{bmatrix} M \\ N \end{bmatrix}$ is a strong right representation of $L \in \mathscr{L}$ if and only if $L = NM^{-1}$ with $M^{-1} \in \mathscr{L}$ and M, N are right coprime in $\mathscr{S}$. Similarly, $\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$ is a strong left representation of $L \in \mathscr{L}$ if and only if $L = \hat{M}^{-1}\hat{N}$ with $\hat{M}^{-1} \in \mathscr{L}$ and $\hat{M}$, $\hat{N}$ are left coprime in $\mathscr{S}$.

The stabilizability of linear systems is closely related to the existence of strong right and strong left representations. It is pointed out by Dale and Smith that a time-varying linear system is stabilizable if and only if it has strong left and strong right representations in [24]. The equivalence between the existences of a strong left or right representation is derived by the complete finiteness of the nest algebra in [11].

THEOREM 2.1. [11] *Let $L \in \mathscr{L}$. Then the following statements are equivalent:*
1. L is stabilizable.
2. L has a (normalized) strong right representation $\begin{bmatrix} M \\ N \end{bmatrix}$.
3. L has a (normalized) strong left representation $\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$.

If this is the case the representations can be chosen such that the double Bezout identity

$$\begin{bmatrix} Y & X \\ -\hat{N} & \hat{M} \end{bmatrix}\begin{bmatrix} M & -\hat{X} \\ N & \hat{Y} \end{bmatrix} = \begin{bmatrix} M & -\hat{X} \\ N & \hat{Y} \end{bmatrix}\begin{bmatrix} Y & X \\ -\hat{N} & \hat{M} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tag{2.2}$$

hold for some X, Y, $\hat{X}$, $\hat{Y} \in \mathscr{S}$.

LEMMA 2.2. [13] *Let $\begin{bmatrix} M \\ N \end{bmatrix}$ be a strong right representation of $L \in \mathscr{L}$, $\begin{bmatrix} V \\ U \end{bmatrix}$ and $\begin{bmatrix} -\hat{U} & \hat{V} \end{bmatrix}$ be strong right and left representations of $C \in \mathscr{L}$, respectively. Then the following statements are equivalent:*
1. $\{L, C\}$ is stable
2. $\begin{bmatrix} M & U \\ N & V \end{bmatrix}$ is invertible in $\mathscr{S}$.
3. $\hat{V}M - \hat{U}N$ is invertible in $\mathscr{S}$.

The strong representation serves as a tool for the stability analysis of time-varying linear systems. Another basic tool is gap metric. The gap metric between two plans in $\mathscr{L}$ is defined as the gap metric between the associated graphs of those.

For the systems defined on fixed singly infinite discrete time support, the inverse of a causal linear systems is causal whenever it exists. Moreover, the stability of the closed-loop system is equivalent to requiring that $\ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ can be decomposed as the algebraic direct sum of $\mathscr{G}(L)$ and $\mathscr{G}^{-1}(C)$. This gives a stability criteria characterized in terms of gap metric.

LEMMA 2.3. *$L,C \in \mathscr{L}(\mathbb{N})$, then*

$$\vec{\delta}\left(\mathscr{G}(L),\mathscr{G}^{-1}(C)^{\perp}\right) \leqslant \vec{\delta}\left(\mathscr{G}^{-1}(C)^{\perp},\mathscr{G}(L)\right) = \delta\left(\mathscr{G}^{-1}(C)^{\perp},\mathscr{G}(L)\right).$$

Moreover, $\{L,C\}$ is stable if and only if

$$\vec{\delta}\left(\mathscr{G}^{-1}(C)^{\perp},\mathscr{G}(L)\right) = \left\|P_{\mathscr{G}(L)^{\perp}}P_{\mathscr{G}^{-1}(C)^{\perp}}\right\| < 1.$$

3. TV gap and gap between plants over $\mathbb{N}$ and $\mathbb{Z}$

The use of gap metric for formulating and studying stabilization was begun in the work by El-Sakkary. The TV gap was introduced by Feintuch to generalize the use of gap metric to time-varying linear systems. In this section, we will extend and formulate the TV gap to the double-sided signal space $\ell^2(\mathbb{Z})$. The key result in this section is that the gap and TV gap are equivalent in measuring the distance between two stabilizable plants over $\ell^2(\mathbb{Z})$.

Let $G_i := \begin{bmatrix} M_i \\ N_i \end{bmatrix}$, $\hat{G}_i := \begin{bmatrix} -\hat{N}_i \ \hat{M}_i \end{bmatrix}$ be the normalized strong right and left representations of $L_i \in \mathscr{L}$, respectively, $i = 1,2$. It is easily checked that

$$\vec{\delta}(L_1,L_2) = \|-\hat{N}_2M_1 + \hat{M}_2N_1\|.$$

For each $P_n \in \mathscr{P}$, the restriction of the causal operator $M \in \mathscr{S}$ to the invariant space $h_n^2 := (I-P_n)\ell^2$, denoted by $M(n)$, is

$$M|h_n^2 = (I-P_n)M(I-P_n)|h_n^2.$$

Since $\begin{bmatrix} M_i \\ N_i \end{bmatrix}$ is an isometry, by the causality, $\begin{bmatrix} M_i \\ N_i \end{bmatrix}(I-P_n)$ is an isometry acting on h_n^2. Then the orthogonal projection onto its image in $h_n^2 \oplus h_n^2$ is given by

$$\Pi_{in} = \left(\begin{bmatrix} M_i \\ N_i \end{bmatrix}(I-P_n)\begin{bmatrix} M_i^* \ N_i^* \end{bmatrix}\right)|h_n^2.$$

DEFINITION 3.1. The directed TV gap from L_1 to L_2 is defined by

$$\vec{\alpha}(L_1,L_2) = \sup_{P_n \in \mathscr{P}} \vec{\delta}_n(L_1,L_2),$$

where $\vec{\delta}_n(L_1,L_2) = \left\|\left(\begin{bmatrix} I(n) & 0 \\ 0 & I(n) \end{bmatrix} - \Pi_{1n}\right)\Pi_{2n}\right\|$. The TV gap between L_1 and L_2 is

$$\alpha(L_1,L_2) = \max\{\vec{\alpha}(L_1,L_2),\vec{\alpha}(L_2,L_1)\}.$$

It is noted that $\alpha(\cdot,\cdot)$ defines a metric on the stabilizable systems in $\mathscr{L}$.

The following result generalizes the single-sided system counterpart proposed by Feintuch in [14]. The proof is similar to that of Proposition 6.1 in [14], here it is omitted.

LEMMA 3.1. $\vec{\alpha}(L_1,L_2) = \inf_{R\in\mathscr{S}} \left\| \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} R \right\|$.

LEMMA 3.2. [25] *Assume that $A,B\in\mathscr{S}$. If AB is invertible in $\mathscr{S}$, then A and B are both invertible in $\mathscr{S}$.*

A very important result about TV gap defined on the whole time axis $\mathbb{Z}$ is presented in the following. This result means that TV gap defined a "correct" topology for the robustness of closed-loop stability over double-sided signal space.

THEOREM 3.1. *Let L, C, $L_n \in \mathscr{L}(\mathbb{Z})$, $n\geqslant 1$. Assume that $\{L,C\}$ and $\{L_n,C\}$ are stable for all $n\geqslant 1$. Then the following statements are equivalent:*

1. $\vec{\alpha}(L,L_n)\to 0$ as $n\to+\infty$.

2. $\|H(L_n,C)-H(L,C)\|\to 0$ as $n\to+\infty$.

Proof. Assume that $\begin{bmatrix} M \\ N \end{bmatrix}$ and $\begin{bmatrix} M_n \\ N_n \end{bmatrix}$ are normalized strong right representations of L and L_n, respectively. Choose a strong left representation $\begin{bmatrix} -\hat{U} & \hat{V} \end{bmatrix}$ of C with $\hat{V}M-\hat{U}N=I$. It follows from Lemma 2.2 that $(\hat{V}M_n-\hat{U}N_n)^{-1}\in\mathscr{S}$. Note that

$$H(L,C) = \begin{bmatrix} M(\hat{V}M-\hat{U}N)^{-1}\hat{V} & -M(\hat{V}M-\hat{U}N)^{-1}\hat{U} \\ -N(\hat{V}M-\hat{U}N)^{-1}\hat{V} & I+N(\hat{V}M-\hat{U}N)^{-1}\hat{U} \end{bmatrix}$$
$$= \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} (\hat{V}M-\hat{U}N)^{-1} \begin{bmatrix} \hat{V} & -\hat{U} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

Thus

$$\begin{aligned} &\|H(L_n,C)-H(L,C)\| \\ &= \left\| \begin{bmatrix} M_n \\ N_n \end{bmatrix} (\hat{V}M_n-\hat{U}N_n)^{-1} \begin{bmatrix} \hat{V} & -\hat{U} \end{bmatrix} - \begin{bmatrix} M \\ N \end{bmatrix} (\hat{V}M-\hat{U}N)^{-1} \begin{bmatrix} \hat{V} & -\hat{U} \end{bmatrix} \right\| \\ &= \left\| \left(\begin{bmatrix} M_n \\ N_n \end{bmatrix} (\hat{V}M_n-\hat{U}N_n)^{-1} - \begin{bmatrix} M \\ N \end{bmatrix} \right) \begin{bmatrix} \hat{V} & -\hat{U} \end{bmatrix} \right\|. \end{aligned} \tag{3.1}$$

It follows from Lemma 3.1 that

$$\begin{aligned} \vec{\alpha}(L,L_n) &\leqslant \left\| \begin{bmatrix} M_n \\ N_n \end{bmatrix} (\hat{V}M_n-\hat{U}N_n)^{-1} - \begin{bmatrix} M \\ N \end{bmatrix} \right\| \\ &\leqslant \left\| \left(\begin{bmatrix} M_n \\ N_n \end{bmatrix} (\hat{V}M_n-\hat{U}N_n)^{-1} - \begin{bmatrix} M \\ N \end{bmatrix} \right) \begin{bmatrix} \hat{V} & -\hat{U} \end{bmatrix} \right\| \cdot \left\| \begin{bmatrix} M \\ N \end{bmatrix} \right\| \\ &= \|H(L_n,C)-H(L,C)\|. \end{aligned}$$

Therefore $\|H(L_n,C)-H(L,C)\| \to 0$ implies that $\vec{\alpha}(L_n,L) \to 0$.

Conversely, $\vec{\alpha}(L_n,L) \to 0$ implies that

$$\left\| \begin{bmatrix} M \\ N \end{bmatrix} - \begin{bmatrix} M_n \\ N_n \end{bmatrix} R_n \right\| \to 0 \tag{3.2}$$

holds for some $R_n \in \mathscr{S}(\mathbb{Z})$ as, $n \to 0$. Then there exists $\alpha > 0$ such that

$$\sup_{n\geqslant 1} \left\| \begin{bmatrix} M_n \\ N_n \end{bmatrix} R_n \right\| \leqslant \alpha. \tag{3.3}$$

Note (3.2),

$$\left\| I - (\hat{V}M_n - \hat{U}N_n)R_n \right\| = \left\| \begin{bmatrix} \hat{V} & -\hat{U} \end{bmatrix} \left(\begin{bmatrix} M \\ N \end{bmatrix} - \begin{bmatrix} M_n \\ N_n \end{bmatrix} R_n \right) \right\| \to 0, \;\; n \to +\infty.$$

This implies that $(\hat{V}M_n - \hat{U}N_n)R_n$ is invertible in $\mathscr{S}(\mathbb{Z})$ for n large enough, by Lemma 3.2, $((\hat{V}M_n - \hat{U}N_n)R_n)^{-1} = R_n^{-1}(\hat{V}M_n - \hat{U}N_n)^{-1}$ for n large enough, and

$$\left\| I - R_n^{-1}(\hat{V}M_n - \hat{U}N_n)^{-1} \right\| \to 0, \;\; n \to +\infty. \tag{3.4}$$

Let $k := \left\| \begin{bmatrix} \hat{V} & -\hat{U} \end{bmatrix} \right\|$. For n large enough, it follows from (3.1) and (3.3) that

$$\begin{aligned}
&\|H(L_n,C) - H(L,C)\| \\
&\leqslant k \left\| \begin{bmatrix} M_n \\ N_n \end{bmatrix} (\hat{V}M_n - \hat{U}N_n)^{-1} - \begin{bmatrix} M \\ N \end{bmatrix} \right\| \\
&\leqslant k \left\| \begin{bmatrix} M_n \\ N_n \end{bmatrix} R_n R_n^{-1} (\hat{V}M_n - \hat{U}N_n)^{-1} - \begin{bmatrix} M_n \\ N_n \end{bmatrix} R_n \right\| + k \left\| \begin{bmatrix} M_n \\ N_n \end{bmatrix} R_n - \begin{bmatrix} M \\ N \end{bmatrix} \right\| \\
&\leqslant k \cdot \alpha \left\| I - R_n^{-1}(\hat{V}M_n - \hat{U}N_n)^{-1} \right\| + k \left\| \begin{bmatrix} M_n \\ N_n \end{bmatrix} R_n - \begin{bmatrix} M \\ N \end{bmatrix} \right\|.
\end{aligned}$$

Combining the facts (3.2) and (3.4), ones can get that $\|H(L_n,C)-H(L,C)\| \to 0$ as $n \to +\infty$. □

One purpose of this section is showing that the TV gap between two stabilizable plants in $\mathscr{L}$ is equal to the gap metric between them. This means that the TV gap metric is the same as the gap metric in measuring the distance between stabilizable systems. Before demonstrating this result, we have to look at some notations and lemmas.

Let $\mathscr{C}_2$ be the Hilbert space of Hilbert-Schmidt operators on ℓ^2 with the inner product,

$$\langle f, g \rangle = tr(g^* f),$$

where $tr(\cdot)$ denotes the trace of its argument. $\mathscr{A}_2$ denotes the space of causal Hilbert-Schmidt operators on ℓ^2, that is, $\mathscr{A}_2 = \mathscr{S} \cap \mathscr{C}_2$. $\mathscr{A}_2$ can be viewed as the time-varying counterpart of the standard Hardy space H^2 in H^∞ theory. Define the orthogonal projection $P_{\mathscr{A}_2}$ of $\mathscr{C}_2$ onto $\mathscr{A}_2$, which is analogous to the standard positive Riesz projection for the time-invariant case.

In [15] and [10], the optimum of the special time-varying two-block problem is shown to be equal to the norm of a time-varying Hankel operator defined on the space of causal, Hilbert-Schmidt operators, and the existence of an optimal solution is obtained by applying the nest algebra's commutant lifting theorem. These results can be summarized as the following lemma.

LEMMA 3.3. *There exists at least one operator $R_0 \in \mathscr{S}$ such that*

$$\inf_{R\in\mathscr{S}} \|G_1 - G_2R\| = \|\Pi L_{G_1}\|,$$

where the operator $L_{G_1} : \mathscr{A}_2 \to (\mathscr{A}_2 \oplus \mathscr{A}_2)$ is the left multiplication with symbol G_1, that is,

$$L_{G_1} f = G_1 f, \quad f \in \mathscr{A}_2,$$

and Π is the orthogonal projection on $(\mathscr{A}_2 \oplus \mathscr{A}_2)$ with the image $(\mathscr{A}_2 \oplus \mathscr{A}_2) \ominus G_2\mathscr{A}_2$.

By "L_G", we shall always mean the left multiplication with the symbol G defined on specific space.

LEMMA 3.4. $\vec{\delta}\,(L_1, L_2) = \inf_{R\in\mathscr{B}(\ell^2)} \|G_1 - G_2R\|$.

Proof. Since G_2, $\hat{G}_2^*$ are isometric and $\mathrm{Im} G_2 = \mathrm{Ker}\hat{G}_2$, $\begin{bmatrix} G_2 & \hat{G}_2^* \end{bmatrix}$ is an isometry. In addition,

$$\mathrm{Ker}\begin{bmatrix} G_2^* \\ \hat{G}_2 \end{bmatrix} = \mathrm{Ker}\hat{G}_2 \cap \mathrm{Ker}G_2^* = \{0\},$$

this is because $(\mathrm{Ker}G_2^*)^\perp = \mathrm{Im}G_2 = \mathrm{Ker}\hat{G}_2$. Thus, $\begin{bmatrix} G_2 & \hat{G}_2^* \end{bmatrix}$ is surjective and isometric, and then $\begin{bmatrix} G_2 & \hat{G}_2^* \end{bmatrix}$ is a unitary operator on $\ell^2 \oplus \ell^2$. It follows from the fact $\hat{G}_2 G_2 = 0$ that

$$\begin{aligned}
\inf_{R\in\mathscr{B}(\ell^2)} \|G_1 - G_2R\| &= \inf_{R\in\mathscr{B}(\ell^2)} \left\| \begin{bmatrix} G_2^* \\ \hat{G}_2 \end{bmatrix} (G_1 - G_2R) \right\| \\
&= \inf_{R\in\mathscr{B}(\ell^2)} \left\| \begin{bmatrix} G_2^*G_1 - R \\ \hat{G}_2 G_1 \end{bmatrix} \right\| \\
&= \|\hat{G}_2 G_1\|. \quad \square
\end{aligned}$$

LEMMA 3.5. $\mathrm{Ker}L_{\hat{G}_2} \cap (\mathscr{A}_2 \oplus \mathscr{A}_2) = G_2\mathscr{A}_2$,

Proof. It sufficient to prove the case of $\ell^2 = \ell^2(\mathbb{Z})$, the single-sided case is similar. For any $\begin{bmatrix} f \\ g \end{bmatrix} \in \mathrm{Ker}L_{\hat{G}_2} \cap (\mathscr{A}_2 \oplus \mathscr{A}_2)$, and $n \in \mathbb{Z}$, let $\begin{bmatrix} f_n \\ g_n \end{bmatrix}$ denotes the $n-th$ column of $\begin{bmatrix} f \\ g \end{bmatrix}$. Then the following three facts can be immediately obtained,

1. $f_n,\ g_n \in \ell^2(\mathbb{Z})$,
2. $P_n f_n = 0,\ P_n g_n = 0$,
3. $\hat{G}_2 \begin{bmatrix} f_n \\ g_n \end{bmatrix} = 0$.

Since $\mathrm{Ker}\hat{G}_2 = \mathrm{Im} G_2 = \mathscr{G}(L_2)$, thus there exists a vector $h_n \in \ell^2(\mathbb{Z})$ such that

$$\begin{bmatrix} f_n \\ g_n \end{bmatrix} = G_2 h_n.$$

It follows that

$$0 = P_n f_n = P_n \left(\begin{bmatrix} I & 0 \end{bmatrix} G_2 \right) P_n h_n.$$

By the fact that an operator $T \in \mathscr{S}(\mathbb{Z})$ is invertible in $\mathscr{L}(\mathbb{Z})$ if and only if $P_n T P_n$ is invertible on $P_n \ell^2(\mathbb{Z})$ for all $n \in \mathbb{Z}$, $P_n \left(\begin{bmatrix} I & 0 \end{bmatrix} G_2 \right) P_n$ is invertible on $P_n \ell^2(\mathbb{Z})$, so $P_n h_n = 0$. Define the linear transformation h as

$$h = [\cdots, h_{-1}, h_0, h_1 \cdots].$$

From the discussion above, ones can conclude that

1. $\begin{bmatrix} f \\ g \end{bmatrix} = G_2 h$,
2. $h \in \mathscr{L}(\mathbb{Z})$.

Since G_2 is an isometry, $\|h\|_{\mathscr{C}_2} = \|G_2^* G_2 h\|_{\mathscr{C}_2} = \left\| G_2^* \begin{bmatrix} f \\ g \end{bmatrix} \right\|_{\mathscr{C}_2} \leqslant \left\| \begin{bmatrix} f \\ g \end{bmatrix} \right\|_{\mathscr{C}_2 \oplus \mathscr{C}_2}$.

Hence $h \in \mathscr{A}_2$. This in turn implies that $\begin{bmatrix} f \\ g \end{bmatrix} \in G_2 \mathscr{A}_2$. Therefore

$$\mathrm{Ker} L_{\hat{G}_2} \cap (\mathscr{A}_2 \oplus \mathscr{A}_2) \subseteq G_2 \mathscr{A}_2.$$

The opposite inclusion is obvious, valid for $\hat{G}_2 G_2 = 0$. The proof is completed. □

LEMMA 3.6. $(\mathscr{A}_2 \oplus \mathscr{A}_2) \ominus G_2 \mathscr{A}_2 = \begin{bmatrix} P_{\mathscr{A}_2} & 0 \\ 0 & P_{\mathscr{A}_2} \end{bmatrix} \hat{G}_2^* \mathscr{A}_2$.

Proof. For any $f, g \in \mathscr{A}_2$, we have

$$\begin{aligned} &\left\langle \begin{bmatrix} P_{\mathscr{A}_2} & 0 \\ 0 & P_{\mathscr{A}_2} \end{bmatrix} \hat{G}_2^* f, G_2 g \right\rangle_{\mathscr{C}_2 \oplus \mathscr{C}_2} \\ &= \langle P_{\mathscr{A}_2} \begin{bmatrix} I & 0 \end{bmatrix} \hat{G}_2^* f, \begin{bmatrix} I & 0 \end{bmatrix} G_2 g, \rangle_{\mathscr{C}_2} + \langle P_{\mathscr{A}_2} \begin{bmatrix} 0 & I \end{bmatrix} \hat{G}_2^* f, \begin{bmatrix} 0 & I \end{bmatrix} G_2 g, \rangle_{\mathscr{C}_2} \\ &= \langle f, \hat{G}_2 \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} G_2 g, \rangle_{\mathscr{C}_2} + \langle f, \hat{G}_2 \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} G_2 g, \rangle_{\mathscr{C}_2} \\ &= \langle f, \hat{G}_2 G_2 g \rangle_{\mathscr{C}_2} = 0. \end{aligned}$$

Hence $\begin{bmatrix} P_{\mathscr{A}_2} & 0 \\ 0 & P_{\mathscr{A}_2} \end{bmatrix} \hat{G}_2^* A_2 \subseteq (\mathscr{A}_2 \oplus \mathscr{A}_2) \ominus G_2 \mathscr{A}_2$. The proof of the opposite inclusion can be reduced to that of the following inclusion relationship

$$(\mathscr{A}_2 \oplus \mathscr{A}_2) \ominus \begin{bmatrix} P_{\mathscr{A}_2} & 0 \\ 0 & P_{\mathscr{A}_2} \end{bmatrix} \hat{G}_2^* \mathscr{A}_2 \subseteq G_2 \mathscr{A}_2. \tag{3.5}$$

To this aim, take any k belonging to the left set of the preceding equation (3.5) and $h \in \mathscr{A}_2$, we have

$$0 = \left\langle k, \begin{bmatrix} P_{\mathscr{A}_2} & 0 \\ 0 & P_{\mathscr{A}_2} \end{bmatrix} \hat{G}_2^* h \right\rangle_{\mathscr{C}_2 \oplus \mathscr{C}_2} = \langle \hat{G}_2 k, h \rangle_{\mathscr{C}_2}.$$

This implies that $\hat{G}_2 k \in \mathscr{C}_2 \ominus \mathscr{A}_2$. Combining the fact $\hat{G}_2 k \in \mathscr{A}_2$, ones can deduce that

$$k \in \mathrm{Ker} L_{\hat{G}_2} \cap (\mathscr{A}_2 \oplus \mathscr{A}_2).$$

Thus

$$(\mathscr{A}_2 \oplus \mathscr{A}_2) \ominus \begin{bmatrix} P_{\mathscr{A}_2} & 0 \\ 0 & P_{\mathscr{A}_2} \end{bmatrix} \hat{G}_2^* \mathscr{A}_2 \subseteq \mathrm{Ker} L_{\hat{G}_2} \cap (\mathscr{A}_2 \oplus \mathscr{A}_2).$$

By Lemma 3.5, (3.5) can be obtained. $\square$

With the preceding preparations, we present one of our main results in the following.

THEOREM 3.2. *Assume that $L_1, L_2 \in \mathscr{L}$ are stabilizable. Then*

$$\vec{\alpha}(L_1, L_2) = \vec{\delta}(L_1, L_2).$$

Proof. According to Lemma 3.1 and Lemma 3.4, it is obvious that $\vec{\alpha}(L_1, L_2) \geqslant \vec{\delta}(L_1, L_2)$. For the other direction, it follows from Lemma 3.1 and Lemma 3.3 that $\vec{\alpha}(L_1, L_2) = \|\Pi L_{G_1}\|$. By Lemma 3.6,

$$\mathrm{Im}\,\Pi = \begin{bmatrix} P_{\mathscr{A}_2} & 0 \\ 0 & P_{\mathscr{A}_2} \end{bmatrix} \hat{G}_2^* \mathscr{A}_2 \subseteq \hat{G}_2^* \mathscr{C}_2.$$

It follows that

$$\vec{\alpha}(L_1, L_2) \leqslant \left\| P_{\hat{G}_2^* \mathscr{C}_2} L_{G_1} \right\| = \left\| L_{\hat{G}_2^* \hat{G}_2} L_{G_1} \right\| \leqslant \left\| L_{\hat{G}_2^* \hat{G}_2 G_1} | \mathscr{C}_2 \right\|.$$

The basic property of the left multiplication on $\mathscr{C}_2$ is that its norm is equal to the norm of its symbol. Therefore

$$\vec{\alpha}(L_1, L_2) \leqslant \|\hat{G}_2^* \hat{G}_2 G_1\| = \|\hat{G}_2 G_1\| = \vec{\delta}(L_1, L_2). \quad \square$$

Theorem 3.2 implies that all the results about gap metric concerned to plants go through unchanged for the TV gap concerned the class of plants which admit the normalized strong representation over double-sided signal space. In other words, the TV gap and gap between two stabilizable plants coincide on the whole time-axis $\mathbb{Z}$. In next section, we will point out the difference between the gap and TV gap in dealing with closed stability of linear systems on the whole time-axis $\mathbb{Z}$.

4. TV gap and gap between plant and controller over $\mathbb{N}$ and $\mathbb{Z}$

The key result of this section concerns the difference between the TV gap and gap in terms of closed-loop stability criteria for linear systems on the whole time-axis $\mathbb{Z}$.

Let $\begin{bmatrix} V \\ U \end{bmatrix}$ and $\begin{bmatrix} -\hat{U} & \hat{V} \end{bmatrix}$ be normalized strong right and left representations of $C \in \mathscr{L}$, and $\begin{bmatrix} M \\ N \end{bmatrix}$, $\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$ be the same for the plant $L \in \mathscr{L}$, respectively. Denote

$$\vec{\alpha}\left(\mathscr{G}^{-1}(C)^{\perp},\mathscr{G}(L)\right) = \sup_{P_n \in \mathscr{P}} \left\| \left(\begin{bmatrix} I(n) & 0 \\ 0 & I(n) \end{bmatrix} - \Pi_{Cn} \right) \left(\begin{bmatrix} I(n) & 0 \\ 0 & I(n) \end{bmatrix} - \Pi_{Ln} \right) \right\|$$

and

$$\vec{\alpha}\left(\mathscr{G}(L),\mathscr{G}^{-1}(C)^{\perp}\right) = \sup_{P_n \in \mathscr{P}} \left\| \Pi_{Cn} \Pi_{Ln} \right\|,$$

where $\Pi_{Cn} = \begin{bmatrix} U \\ V \end{bmatrix} (I - P_n) \begin{bmatrix} U^* & V^* \end{bmatrix}$ and $\Pi_{Ln} = \begin{bmatrix} M \\ N \end{bmatrix} (I - P_n) \begin{bmatrix} M^* & N^* \end{bmatrix}$ are the orthogonal projections on $h_n^2 \oplus h_n^2$ having the images $\begin{bmatrix} U \\ V \end{bmatrix} h_n^2$ and $\begin{bmatrix} M \\ N \end{bmatrix} h_n^2$, respectively. The TV gap between the plant L and the controller C is defined by

$$\alpha\left(\mathscr{G}^{-1}(C)^{\perp},\mathscr{G}(L)\right) = \max\left\{ \vec{\alpha}\left(\mathscr{G}^{-1}(C)^{\perp},\mathscr{G}(L)\right), \vec{\alpha}\left(\mathscr{G}^{-1}(C)^{\perp},\mathscr{G}(L)\right) \right\}.$$

REMARK 4.1. It is clear that

$$\begin{aligned} \vec{\alpha}\left(\mathscr{G}^{-1}(C)^{\perp},\mathscr{G}(L)\right) &\geqslant \left\| \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} U \\ V \end{bmatrix} \begin{bmatrix} U^* & V^* \end{bmatrix} \right) \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} M \\ N \end{bmatrix} \begin{bmatrix} M^* & N^* \end{bmatrix} \right) \right\| \\ &= \left\| P_{\mathscr{G}(L)^{\perp}} P_{\mathscr{G}^{-1}(C)^{\perp}} \right\| \\ &= \vec{\delta}\left(\mathscr{G}^{-1}(C)^{\perp},\mathscr{G}(L)\right). \end{aligned}$$

Note that

$$\begin{aligned} \vec{\alpha}\left(\mathscr{G}(L),\mathscr{G}^{-1}(C)^{\perp}\right) &= \sup_{P_n \in \mathscr{P}} \left\| \begin{bmatrix} U \\ V \end{bmatrix} (I - P_n) \begin{bmatrix} U^* & V^* \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} (I - P_n) \begin{bmatrix} M^* & N^* \end{bmatrix} \right\| \\ &= \left\| U^* M + V^* N \right\| \\ &= \vec{\delta}\left(\mathscr{G}(L),\mathscr{G}^{-1}(C)^{\perp}\right) \end{aligned}$$

It is concluded that

$$\alpha\left(\mathscr{G}^{-1}(C)^{\perp},\mathscr{G}(L)\right) \geqslant \delta\left(\mathscr{G}^{-1}(C)^{\perp},\mathscr{G}(L)\right). \tag{4.1}$$

As mentioned above, $\vec{\alpha}(L_1,L_2)$ and $\vec{\delta}(L_1,L_2)$ has close connection with the two-block problem. We now consider the relationships among $\vec{\delta}\left(\mathscr{G}^{-1}(C)^{\perp},\mathscr{G}(L)\right)$, $\vec{\alpha}\left(\mathscr{G}^{-1}(C)^{\perp},\mathscr{G}(L)\right)$ and two-block problem. Some lemmas are

LEMMA 4.1. [13] *Let A,B,C and $D \in \mathscr{B}(\ell^2)$. Then*

$$\inf_{Q\in\mathscr{S}}\left\|\begin{bmatrix} A-Q & B \\ C & D \end{bmatrix}\right\| = \sup_{P_n\in\mathscr{P}}\left\|\begin{bmatrix} P_n & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} I-P_n & 0 \\ 0 & I \end{bmatrix}\right\|$$

LEMMA 4.2. $\inf_{R\in\mathscr{S}}\left\|\begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} - \begin{bmatrix} U \\ V \end{bmatrix} R\right\| = \sup_{P_n\in\mathscr{P}}\left\|\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \Pi_{Cn}\right)\begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix}(I-P_n)\right\|.$

Proof. Since $\begin{bmatrix} U^* & V^* \\ -\hat{V} & \hat{U} \end{bmatrix}$ is a unitary operator on $\ell^2 \oplus \ell^2$, applying Arveson's distance formula in Lemma 4.1,

$$\begin{aligned}
&\inf_{R\in\mathscr{S}}\left\|\begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} - \begin{bmatrix} U \\ V \end{bmatrix} R\right\| \\
&= \inf_{R\in\mathscr{S}}\left\|\begin{bmatrix} U^* & V^* \\ -\hat{V} & \hat{U} \end{bmatrix}\left(\begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} - \begin{bmatrix} U \\ V \end{bmatrix} R\right)\right\| \\
&= \inf_{R\in\mathscr{S}}\left\|\begin{bmatrix} V^*\hat{M}^* - U^*\hat{N}^* - R \\ \hat{V}\hat{N}^* + \hat{U}\hat{M}^* \end{bmatrix}\right\| \\
&= \sup_{P_n\in\mathscr{P}}\left\|\begin{bmatrix} P_n(V^*\hat{M}^* - U^*\hat{N}^*)(I-P_n) \\ (\hat{V}\hat{N}^* + \hat{U}\hat{M}^*)(I-P_n) \end{bmatrix}\right\|. \qquad (4.2)
\end{aligned}$$

For any $x \in \ell^2$, ones can compute that

$$\begin{aligned}
&\left\|\begin{bmatrix} P_n(V^*\hat{M}^* - U^*\hat{N}^*)(I-P_n) \\ (\hat{V}\hat{N}^* + \hat{U}\hat{M}^*)(I-P_n) \end{bmatrix} x\right\|^2 \\
&= \left\|P_n(V^*\hat{M}^* - U^*\hat{N}^*)(I-P_n)x\right\|^2 + \left\|(\hat{V}\hat{N}^* + \hat{U}\hat{M}^*)(I-P_n)x\right\|^2 \\
&= \left\langle \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix}(I-P_n)x, \begin{bmatrix} U \\ V \end{bmatrix} P_n \begin{bmatrix} U^* & V^* \end{bmatrix}\begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix}(I-P_n)x \right\rangle \\
&\quad + \left\langle \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix}(I-P_n)x, \begin{bmatrix} -\hat{V}^* \\ \hat{U}^* \end{bmatrix}\begin{bmatrix} -\hat{V} & \hat{U} \end{bmatrix}\begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix}(I-P_n)x \right\rangle.
\end{aligned}$$

Using $\begin{bmatrix} -\hat{V}^* \\ \hat{U}^* \end{bmatrix}\begin{bmatrix} -\hat{V} & \hat{U} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} U \\ V \end{bmatrix}\begin{bmatrix} U^* & V^* \end{bmatrix}$, it follows that

$$\begin{aligned}
&\left\|\begin{bmatrix} P_n(V^*\hat{M}^* - U^*\hat{N}^*)(I-P_n) \\ (\hat{V}\hat{N}^* + \hat{U}\hat{M}^*)(I-P_n) \end{bmatrix} x\right\|^2 \\
&= \left\langle \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix}(I-P_n)x, \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \Pi_{Cn}\right)\begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix}(I-P_n)x \right\rangle \\
&= \left\|\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \Pi_{Cn}\right)\begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix}(I-P_n)x\right\|^2, x \in \ell^2,
\end{aligned}$$

which implies that

$$\left\| \begin{bmatrix} P_n(V^*\hat{M}^* - U^*\hat{N}^*)(I-P_n) \\ (\hat{V}\hat{N}^* + \hat{U}\hat{M}^*)(I-P_n) \end{bmatrix} \right\| = \left\| \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \Pi_{Cn} \right) \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} (I-P_n) \right\|. \tag{4.3}$$

Combining (4.2) and (4.3), the desired result is obtained. □

COROLLARY 4.1. $\vec{\alpha}\left(\mathscr{G}^{-1}(C)^{\perp}, \mathscr{G}(L)\right) \leqslant \inf_{R \in \mathscr{S}} \left\| \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} - \begin{bmatrix} U \\ V \end{bmatrix} R \right\|.$

Proof. It is easily checked that $\begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} (I-P_n) \begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix}$ is the orthogonal projection on $\ell^2 \oplus \ell^2$ having the image $\begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} h_n^2$. If it were known that

$$(h_n^2 \oplus h_n^2) \ominus \begin{bmatrix} M \\ N \end{bmatrix} h_n^2 = (I-P_n) \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} \ell^2, \tag{4.4}$$

combining the fact $(I-P_n) \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} \ell^2 = (I-P_n)\left(\begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} h_n^2 \right) \subseteq \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} h_n^2$, it would be

$$\begin{aligned} \vec{\alpha}\left(\mathscr{G}^{-1}(C)^{\perp}, \mathscr{G}(L)\right) &\leqslant \sup_{P_n \in \mathscr{P}} \left\| \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \Pi_{Cn} \right) \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} (I-P_n) \begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix} \right\|^2 \\ &= \sup_{P_n \in \mathscr{P}} \left\| \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \Pi_{Cn} \right) \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} (I-P_n) \right\|^2. \end{aligned}$$

Then, the desired result is obtained by Lemma 4.2. The rest of the proof is to demonstrate (4.4). For any $f \in (h_n^2 \oplus h_n^2) \ominus (I-P_n) \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} \ell^2$ and $g \in \ell^2$, we have

$$0 = \left\langle f, (I-P_n) \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} g \right\rangle = \langle \begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix} f, g \rangle,$$

this implies $\begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix} f = 0$. According to Theorem 2.1, there exist $X, Y, \hat{X}, \hat{Y} \in \mathscr{S}$ such that

$$\begin{bmatrix} Y & X \\ -\hat{N} & \hat{M} \end{bmatrix} \begin{bmatrix} M & -\hat{X} \\ N & \hat{Y} \end{bmatrix} = \begin{bmatrix} M & -\hat{X} \\ N & \hat{Y} \end{bmatrix} \begin{bmatrix} Y & X \\ -\hat{N} & \hat{M} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Note that

$$\begin{aligned} f &= \begin{bmatrix} M \\ N \end{bmatrix} \begin{bmatrix} Y & X \end{bmatrix} f + \begin{bmatrix} -\hat{X} \\ \hat{Y} \end{bmatrix} \begin{bmatrix} -\hat{N} & \hat{M} \end{bmatrix} f \\ &= \begin{bmatrix} M \\ N \end{bmatrix} ((I-P_n) \begin{bmatrix} Y & X \end{bmatrix} f) \in \begin{bmatrix} M \\ N \end{bmatrix} h_n^2. \end{aligned}$$

Then

$$\left(h_n^2 \oplus h_n^2\right) \ominus (I - P_n)\begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} \ell^2 \subseteq \begin{bmatrix} M \\ N \end{bmatrix} h_n^2,$$

which is equivalent to

$$\left(h_n^2 \oplus h_n^2\right) \ominus \begin{bmatrix} M \\ N \end{bmatrix} h_n^2 \subseteq (I - P_n)\begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} \ell^2.$$

Consider the other inclusion relationship. For any $x, y \in h_n^2$. It follows from the fact $\begin{bmatrix} -\hat{N} \ \hat{M} \end{bmatrix}\begin{bmatrix} M \\ N \end{bmatrix} = 0$ and the causality that

$$\begin{aligned} \left\langle (I - P_n)\begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} x, \begin{bmatrix} M \\ N \end{bmatrix} y \right\rangle &= \left\langle x, \begin{bmatrix} -\hat{N} \ \hat{M} \end{bmatrix} (I - P_n)\begin{bmatrix} M \\ N \end{bmatrix} y \right\rangle \\ &= \left\langle x, \begin{bmatrix} -\hat{N} \ \hat{M} \end{bmatrix}\begin{bmatrix} M \\ N \end{bmatrix} (I - P_n) y \right\rangle = 0. \end{aligned}$$

Therefore, $(I - P_n)\begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} \ell^2 \subseteq \left(h_n^2 \oplus h_n^2\right) \ominus \begin{bmatrix} M \\ N \end{bmatrix} h_n^2$. The proof is completed. □

Recently, the authors use the commutant lifting theorem of nest algebra to derive the minimum of the four-block problem of time-varying system in [16]. Applying Theorem 4.1 in [16], we can give an abstract characterization for the optimum of the two-block problem in terms of the norm of a time-varying Hankel operator analogous to the Hankel operator that is well known in the optimal standard H^∞ problem.

PROPOSITION 4.1. $\inf_{R \in \mathscr{S}} \left\| \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} - \begin{bmatrix} U \\ V \end{bmatrix} R \right\| = \|\tilde{\Pi} L_H\|,$

where L_H is the left multiplication from $\mathscr{A}_2$ to $\mathscr{C}_2 \oplus \mathscr{C}_2$ with the symbol $H := \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix}$, and $\tilde{\Pi}$ is the orthogonal projection from $\mathscr{C}_2 \oplus \mathscr{C}_2$ to $(\mathscr{C}_2 \oplus \mathscr{C}_2) \ominus \begin{bmatrix} U \\ V \end{bmatrix} \mathscr{A}_2$.

The following result provides a characterization of gap between the plant L and controller C in terms of every instant of time.

THEOREM 4.1. *Let $L, C \in \mathscr{L}(\mathbb{N})$. If $\begin{bmatrix} -\hat{N} \ \hat{M} \end{bmatrix}$ is a normalized strong left representation of L, then*

$$\delta\left(\mathscr{G}^{-1}(C)^\perp, \mathscr{G}(L)\right) = \sup_{P_k \in \mathscr{P}_I} \left\| \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \Pi_{Ck} \right) \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} (I - P_k) \right\|.$$

Proof. First, consider the case of $\{L, C\}$ being stable. By Lemma 2.3,

$$\vec{\delta}\left(\mathscr{G}(L), \mathscr{G}^{-1}(C)^\perp\right) = \delta\left(\mathscr{G}^{-1}(C)^\perp, \mathscr{G}(L)\right) < 1. \tag{4.5}$$

For each $x \in \ell^2(\mathbb{N})$, note that

$$\begin{aligned}
&\left\| \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \Pi_{Ck} \right) \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} (I-P_k)x \right\|^2 \\
&= \left\| \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} (I-P_k)x \right\|^2 - \left\| \begin{bmatrix} U \\ V \end{bmatrix} (I-P_k) \begin{bmatrix} U^* & V^* \end{bmatrix} \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} (I-P_k)x \right\|^2 \\
&= \|(I-P_k)x\|^2 - \left\| (I-P_k) \begin{bmatrix} U^* & V^* \end{bmatrix} \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} (I-P_k)x \right\|^2 .
\end{aligned}$$

Hence we get that

$$\begin{aligned}
&\left\| \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \Pi_{Ck} \right) \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} (I-P_k) \right\|^2 \\
&= \sup_{\|(I-P_k)x\|=1} \left(\|(I-P_k)x\|^2 - \left\| (I-P_k) \begin{bmatrix} U^* & V^* \end{bmatrix} \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} (I-P_k)x \right\|^2 \right) \\
&= 1 - \inf_{\|(I-P_k)x\|=1} \left\| (I-P_k)(-U^*\hat{N}^* + V^*\hat{M}^*)x \right\|^2 \\
&= 1 - \inf_{\|(I-P_k)x\|=1} \left\| (-\hat{N}U + \hat{M}V)(I-P_k)x \right\|^2 \\
&\leqslant 1 - \inf_{\|x\|=1} \left\| (-\hat{N}U + \hat{M}V)x \right\|^2 .
\end{aligned} \tag{4.6}$$

The third equation holds because $(-\hat{N}U + \hat{M}V)(I-P_k)$ is an invertible operator restricting on $(I-P_k)\ell^2(\mathbb{N})$. Since $\begin{bmatrix} -\hat{N} & \hat{M} \\ M^* & N^* \end{bmatrix}$ is unitary, $\begin{bmatrix} -\hat{N}U + \hat{M}V \\ M^*U + N^*V \end{bmatrix}$ is an isometry, so

$$\|x\|^2 = \left\| \begin{bmatrix} -\hat{N}U + \hat{M}V \\ M^*U + N^*V \end{bmatrix} x \right\|^2 = \|(-\hat{N}U + \hat{M}V)x\|^2 + \|(M^*U + N^*V)x\|^2.$$

It follows that

$$1 - \inf_{\|x\|=1} \left\| (-\hat{N}U + \hat{M}V)x \right\|^2 = \|M^*U + N^*V\|^2 = \vec{\delta}\left(\mathscr{G}(L), \mathscr{G}^{-1}(C)^{\perp}\right)^2 . \tag{4.7}$$

Combining (4.5), (4.6) and (4.7), we deduce that

$$\sup_{P_k \in \mathscr{P}_I} \left\| \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \Pi_{Ck} \right) \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} (I-P_k) \right\| \leqslant \delta\left(\mathscr{G}(L), \mathscr{G}^{-1}(C)^{\perp}\right). \tag{4.8}$$

Now consider the other case, suppose that $\{L, C\}$ is not stable. Again applying Lemma 2.3, $\delta\left(\mathscr{G}^{-1}(C)^{\perp}, \mathscr{G}(L)\right) = 1$. Then

$$\begin{aligned}
\sup_{P_k \in \mathscr{P}_I} \left\| \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \Pi_{Ck} \right) \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} (I-P_k) \right\| &\leqslant \sup_{P_k \in \mathscr{P}_I} \left\| \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \Pi_{Ck} \right\| \cdot \left\| \begin{bmatrix} -\hat{N}^* \\ \hat{M}^* \end{bmatrix} \right\| \\
&\leqslant 1 = \delta\left(\mathscr{G}(L), \mathscr{G}^{-1}(C)^{\perp}\right). \quad \square
\end{aligned}$$

As a consequence of Lemma 4.2, Corollary 4.1 and Theorem 4.1, we can obtain the following result.

COROLLARY 4.2. *Let $L,C \in \mathscr{L}(\mathbb{N})$. Then*

$$\delta\left(\mathscr{G}^{-1}(C)^{\perp},\mathscr{G}(L)\right)=\alpha\left(\mathscr{G}^{-1}(C)^{\perp},\mathscr{G}(L)\right)=\inf_{R\in\mathscr{S}(\mathbb{N})}\left\|\begin{bmatrix}-\hat{N}^*\\ \hat{M}^*\end{bmatrix}-\begin{bmatrix}U\\ V\end{bmatrix}R\right\|$$

All these discussions show that TV gap offers no advantage over gap metric on the analysis of stability and robustness on single-sided signal space. We now turn to the question of finding the connection between TV gap and closed-loop stability on the whole time-axis $\mathbb{Z}$. We will show that the standard gap defined on the case of $\ell^2(\mathbb{Z})$ can not directly apply to judge the stability of closed-loop system as it plays over the single sided signal space. This is an instinct difference between singly and doubly infinite time-axis. We construct the following example to illustrate this fact.

EXAMPLE 4.1. Consider the time-invariant plant $L = I - S$ and controller $C = I$ on $\ell^2(\mathbb{Z})$, where S is the bilateral shift opertor defined on $\ell^2(\mathbb{Z})$.

It is easily checked that $\begin{bmatrix}\frac{I}{\sqrt{2}}\\ \frac{I}{\sqrt{2}}\end{bmatrix}$ and $\begin{bmatrix}-\frac{I}{\sqrt{2}} & \frac{I}{\sqrt{2}}\end{bmatrix}$ are the normalized strong right and left representations of C, respectively. L has a normalized strong left representation $\begin{bmatrix}-A(I-S) & A\end{bmatrix}$ and a normalized strong right representation $\begin{bmatrix}A\\ (I-S)A\end{bmatrix}$, where $A = \sqrt{\frac{2}{3-\sqrt{5}}}\left(I-\frac{3-\sqrt{5}}{2}S\right)^{-1} = \sqrt{\frac{3-\sqrt{5}}{2}}\cdot\sum_{n=0}^{+\infty}\left(\frac{3-\sqrt{5}}{2}S\right)^n$.

We compute that

$$\begin{aligned}\left\|P_{\mathscr{G}(L)^{\perp}}P_{\mathscr{G}^{-1}(C)^{\perp}}\right\| &= \|\hat{M}\hat{U}^*-\hat{N}\hat{V}^*\|\\ &=\left\|A\cdot\frac{I}{\sqrt{2}}-A(I-S)\cdot\frac{I}{\sqrt{2}}\right\|\\ &=\left\|\sqrt{2}A-\frac{1}{\sqrt{2}}AS\right\|\\ &=\frac{3\sqrt{10}}{10}\end{aligned}$$

and

$$\begin{aligned}\left\|P_{\mathscr{G}(L)}P_{\mathscr{G}^{-1}(C)}\right\| &= \|M^*U-N^*V\|\\ &=\left\|A^*\cdot\frac{I}{\sqrt{2}}-(I-S^*)A^*\cdot\frac{I}{\sqrt{2}}\right\|\\ &=\frac{3\sqrt{10}}{10}.\end{aligned}$$

Therefore

$$\delta\left(\mathscr{G}^{-1}(C)^{\perp},\mathscr{G}(L)\right)=\max\left\{\left\|P_{\mathscr{G}(L)^{\perp}}P_{\mathscr{G}^{-1}(C)^{\perp}}\right\|,\left\|P_{\mathscr{G}(L)}P_{\mathscr{G}^{-1}(C)}\right\|\right\}=\frac{3\sqrt{10}}{10}.$$

It is clear that $\{I-S,I\}$ is not stable because

$$\begin{bmatrix} I & I \\ I-S & I \end{bmatrix}^{-1}=\begin{bmatrix} S^{-1} & -S^{-1} \\ I-S^{-1} & S^{-1} \end{bmatrix}=\begin{bmatrix} S^{*} & -S^{*} \\ I-S^{*} & S^{*} \end{bmatrix}$$

is not causal.

From the preceding example, the gap metric can not facilitate the study of closed-loop stability over doubly infinite time axis, even when C and L are both time-invariant linear systems. Fortunately, we find out that the TV gap is a suitable tool for the purpose of determining closed-loop stability of linear systems on $\mathbb{Z}$. To demonstrate the main result, we first show some useful lemmas.

LEMMA 4.3. *If* $\begin{bmatrix} M \\ N \end{bmatrix}$ *is a normalized strong right representation of* $L\in\mathscr{L}$*, then* $\begin{bmatrix} M(n) \\ N(n) \end{bmatrix}$ *is a normalized strong right representation of* $L(n):=L|h^2_{n,e}$ *for all* $P_n\in\mathscr{P}$*.*

Proof. Suppose that $YM+XN=I$ holds for some $X,Y\in\mathscr{S}$. Since M is invertible in $\mathscr{L}$, it follows from the causality that

$$M(n)^{-1}=((I-P_n)M^{-1}(I-P_n))|h^2_{n,e},$$

and then $L(n)=N(n)M(n)^{-1}$. Thus, we deduce that $\mathrm{Im}\begin{bmatrix} M(n) \\ N(n) \end{bmatrix}\subseteq\mathscr{G}(L(n))$. For any $x\in\mathscr{D}(L(I-P_n))\cap h^2_n$, we have

$$\begin{bmatrix} M \\ N \end{bmatrix}((I-P_n)M^{-1}(I-P_n)x)=\begin{bmatrix} I \\ L \end{bmatrix}(I-P_n)x=\begin{bmatrix} (I-P_n) \\ (I-P_n)L(I-P_n) \end{bmatrix}x\in h^2_n\oplus h^2_n.$$

This implies that $(I-P_n)M^{-1}(I-P_n)x=\begin{bmatrix} Y & X \end{bmatrix}\begin{bmatrix} (I-P_n) \\ (I-P_n)L(I-P_n) \end{bmatrix}x\in h^2_n$. Hence, $\mathscr{G}(L(n))\subseteq\mathrm{Im}\begin{bmatrix} M(n) \\ N(n) \end{bmatrix}$. It concludes that $\begin{bmatrix} M(n) \\ N(n) \end{bmatrix}$ is a strong right representation of $L(n)$. Since $\begin{bmatrix} M \\ N \end{bmatrix}$ is a normalized representation,

$$\begin{bmatrix} M(n)^{*} & N(n)^{*} \end{bmatrix}\begin{bmatrix} M(n) \\ N(n) \end{bmatrix}=\left((I-P_n)\begin{bmatrix} M^{*} & N^{*} \end{bmatrix}\right)\left(\begin{bmatrix} M \\ N \end{bmatrix}(I-P_n)\right)|h^2_n=I(n).$$

Then, $\begin{bmatrix} M(n) \\ N(n) \end{bmatrix}$ is a normalized strong right representation of $L(n)$. $\square$

REMARK 4.2. By the preceding result, we give a geometrical interpretation for the TV gap between two linear plants on $\mathbb{Z}$:

$$\vec{\alpha}(L_1,L_2) = \sup_{-\infty\leqslant n\leqslant+\infty} \vec{\delta}(L_1(n),L_2(n)), \tag{4.9}$$

and the TV gap between the plant and controller:

$$\vec{\alpha}\left(\mathscr{G}^{-1}(C)^{\perp},\mathscr{G}(L)\right) = \sup_{-\infty\leqslant n\leqslant+\infty} \vec{\delta}\left(\mathscr{G}^{-1}(C(n))^{\perp},\mathscr{G}(L(n))\right), \tag{4.10}$$

PROPOSITION 4.2. *If $\begin{bmatrix}-\hat{N} & \hat{M}\end{bmatrix}$ is a strong left representation of $L \in \mathscr{L}$, then $\begin{bmatrix}-\hat{N}(n) & \hat{M}(n)\end{bmatrix}$ is a strong left representation of $L(n)$ for all $P_n \in \mathscr{P}$.*

Proof. By Theorem 2.1, there exists a strong right representation $\begin{bmatrix}M\\ N\end{bmatrix}$ of L such that

$$\begin{bmatrix}Y & X\\ -\hat{N} & \hat{M}\end{bmatrix}\begin{bmatrix}M & -\hat{X}\\ N & \hat{Y}\end{bmatrix} = \begin{bmatrix}M & -\hat{X}\\ N & \hat{Y}\end{bmatrix}\begin{bmatrix}Y & X\\ -\hat{N} & \hat{M}\end{bmatrix} = \begin{bmatrix}I & 0\\ 0 & I\end{bmatrix} \tag{4.11}$$

hold for some $X,Y,\hat{X},\hat{Y} \in \mathscr{S}$. By the causality, we have

$$\begin{bmatrix}M(n) & -\hat{X}(n)\\ N(n) & \hat{Y}(n)\end{bmatrix}\begin{bmatrix}Y(n) & X(n)\\ -\hat{N}(n) & \hat{M}(n)\end{bmatrix} = \begin{bmatrix}I(n) & 0\\ 0 & I(n)\end{bmatrix}. \tag{4.12}$$

For any $x \in h_n^2$, we have

$$\begin{bmatrix}-\hat{N}(n) & \hat{M}(n)\end{bmatrix}\begin{bmatrix}M(n)\\ N(n)\end{bmatrix}x = \begin{bmatrix}-\hat{N} & \hat{M}\end{bmatrix}\begin{bmatrix}M\\ N\end{bmatrix}(I-P_n)x = 0,$$

thus, $\operatorname{Ker}\begin{bmatrix}-\hat{N}(n) & \hat{M}(n)\end{bmatrix} \subseteq \operatorname{Im}\begin{bmatrix}M(n)\\ N(n)\end{bmatrix}$. Conversely, for any $\begin{bmatrix}x\\ y\end{bmatrix} \in \operatorname{Ker}\begin{bmatrix}-\hat{N}(n) & \hat{M}(n)\end{bmatrix}$, we have

$$\begin{aligned}\begin{bmatrix}x\\ y\end{bmatrix} &= \begin{bmatrix}M(n) & -\hat{X}(n)\\ N(n) & \hat{Y}(n)\end{bmatrix}\begin{bmatrix}Y(n) & X(n)\\ -\hat{N}(n) & \hat{M}(n)\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}\\ &= \begin{bmatrix}M(n)\\ N(n)\end{bmatrix}\begin{bmatrix}Y(n) & X(n)\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}-\hat{X}(n)\\ \hat{Y}(n)\end{bmatrix}\begin{bmatrix}-\hat{N}(n) & \hat{M}(n)\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}\\ &= \begin{bmatrix}M(n)\\ N(n)\end{bmatrix}\begin{bmatrix}Y(n) & X(n)\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} \in \operatorname{Im}\begin{bmatrix}M(n)\\ N(n)\end{bmatrix}.\end{aligned}$$

According to Lemma 4.3, $\operatorname{Ker}\begin{bmatrix}-\hat{N}(n) & \hat{M}(n)\end{bmatrix} = \operatorname{Im}\begin{bmatrix}M(n)\\ N(n)\end{bmatrix} = \mathscr{G}(L(n))$. The proof is completed. □

THEOREM 4.2. *Suppose $L \in \mathscr{L}(\mathbb{Z})$ and $C \in \mathscr{L}(\mathbb{Z})$ are both stabilizable. If*

$$\alpha\left(\mathscr{G}^{-1}(C)^{\perp}, \mathscr{G}(L)\right) < 1,$$

then $\{L, C\}$ is stable.

Proof. Suppose that $\begin{bmatrix} M \\ N \end{bmatrix}$ and $\begin{bmatrix} V \\ U \end{bmatrix}$ are the normalized strong right representations of L and C, respectively. According to Lemma 4.3, $\begin{bmatrix} M(n) \\ N(n) \end{bmatrix}$ is a normalized strong right representation of $L(n)$ and $\begin{bmatrix} V(n) \\ U(n) \end{bmatrix}$ is a normalized strong right representation of $C(n)$ for all $n \in \mathbb{Z}$. By Remark 4.2, we have $\vec{\delta}\left(\mathscr{G}^{-1}(C(n))^{\perp}, \mathscr{G}(L(n))\right) < 1$. It follows from Lemma 2.3 that $\{L(n), C(n)\}$ is stable, and then for each $n \in \mathbb{Z}$, there exist bounded causal linear operators X_n, Y_n, Z_n, W_n on $(I - P_n)\ell^2(\mathbb{Z})$ such that

$$\begin{aligned} \begin{bmatrix} M(n) & U(n) \\ N(n) & V(n) \end{bmatrix} \begin{bmatrix} Y_n & X_n \\ Z_n & W_n \end{bmatrix} &= \begin{bmatrix} Y_n & X_n \\ Z_n & W_n \end{bmatrix} \begin{bmatrix} M(n) & U(n) \\ N(n) & V(n) \end{bmatrix} \\ &= \begin{bmatrix} I(n) & 0 \\ 0 & I(n) \end{bmatrix}. \end{aligned} \tag{4.13}$$

Since $\alpha\left(\mathscr{G}^{-1}(C)^{\perp}, \mathscr{G}(L)\right) < 1$, by Remark 4.1, we have $\vec{\delta}\left(\mathscr{G}^{-1}(C)^{\perp}, \mathscr{G}(L)\right) < 1$. It follows from Lemma 2.1 that $\mathscr{G}^{-1}(C) + \mathscr{G}(L) = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})$ and $\mathscr{G}^{-1}(C) \cap \mathscr{G}(L) = \{0\}$. Then it can be easily checked that $\begin{bmatrix} M & U \\ N & V \end{bmatrix}$ is bijective in $\mathscr{B}(\ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}))$, thus there exist $X, Y, Z, W \in \mathscr{B}(\ell^2(\mathbb{Z}))$ such that

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix} \begin{bmatrix} Y & X \\ Z & W \end{bmatrix} = \begin{bmatrix} Y & X \\ Z & W \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{4.14}$$

It follows from the casuality that

$$\begin{aligned} \begin{bmatrix} Y(I-P_n) & X(I-P_n) \\ Z(I-P_n) & W(I-P_n) \end{bmatrix} &= \begin{bmatrix} Y & X \\ Z & W \end{bmatrix} \left(\begin{bmatrix} M(n) & U(n) \\ N(n) & V(n) \end{bmatrix} \begin{bmatrix} Y_n & X_n \\ Z_n & W_n \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} Y & X \\ Z & W \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} \right) \begin{bmatrix} Y_n & X_n \\ Z_n & W_n \end{bmatrix} \\ &= \begin{bmatrix} Y_n & X_n \\ Z_n & W_n \end{bmatrix}, \end{aligned}$$

which implies that $(I - P_n)X(I - P_n) = X(I - P_n)$ holds for all $P_n \in \mathscr{P}$. Hence $X, \in \mathscr{S}(\mathbb{Z})$, and so are $Y, Z, W \in \mathscr{S}(\mathbb{Z})$. It follows from Lemma 2.2 that $\{L, C\}$ is stable. $\square$

5. Conclusion

In this paper, we have analyzed the TV gap defined on the single and double-sided signal space, respectively. The TV-gap and gap has distinct difference in studying the stability and robustness of closed-loop linear systems on the whole time axis $\mathbb{Z}$. This result does not coincide with that on single-infinite time axis $\mathbb{N}$.

We have worked only with discrete-time time-varying linear systems on $\ell^2(\mathbb{Z})$, but it is not clear how to extend these results to continuous nest and continuous-time time-varying systems over $L^2(-\infty,+\infty)$. In our opinion, the goal of rigorously establishing stabilization theory for the continuous-time time-varying system on full time axis is a challenging problem for future research.

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