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SPECTRA OF LAPLACIANS ON FORMS ON AN INFINITE GRAPH

HÈLA AYADI

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Abstract. In the context of infinite weighted graphs, we consider the discrete Laplacians on 0-forms and 1-forms. Using Weyl's criterion, we prove the relation between the nonzero spectrum of Δ_0 and that of Δ_1. Moreover, we give an extension of the work of John Lott to characterize the 0-spectrum of these two Laplacians.

1. Introduction

In recent years, much attention has been paid to the analysis of discrete Laplacians and elliptic differential operators acting on graphs [13], [5], [6] and [19]. More precisely, authors have intensively studied the spectrum of the discrete Laplacian on an infinite graph in various areas, for example, harmonic analysis on graphs (see [16], [20]), probability theory especially Markov chains (see [8], [12]), potential theory such as electric networks (see [17], [12]), and so on. In this paper, we define two Laplacians, mentioned in [1] and [3], one as an operator acting on functions on vertices denoted by Δ_0 and the other one acting on functions on edges denoted by Δ_1. So, it is a natural question to characterize the relation between their spectrum in terms of a certain geometric property of the graph and properties of the operators. Especially we show that the nonzero spectrum of Δ_0 and Δ_1 are the same, by using Weyl's criterion. Moreover, with suitable weight conditions we prove that 0 is in the spectrum of Δ_1, if the operator Δ_0 is invertible. This result is inspired from J. Lott's work [11] (Proposition 9, p. 12) which proves in the case of a simple graph that 0 is either in the spectrum of the Laplacian on 0-forms, or in the spectrum of the Laplacian on 1-forms. In fact, the major interest of J. Lott concerns the zero-in-the-spectrum question for the Laplace-de Rham operator acting on L^2 differential forms of any degree on a complete connected oriented Riemannian manifold. The article [11] is rather expository and gives some positive answers, in relation with topology, for small dimensions. We finish the paper with examples of constructions of Δ_1-harmonic nonzero square-integrable functions.

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2. Preliminaries

2.1. Definition and notation

- A graph G is a couple $(\mathscr{V},\mathscr{E})$ where $\mathscr{V}$ is a set at most countable whose elements are called vertices and $\mathscr{E}$ is a set of oriented edges, considered as a subset of $\mathscr{V}\times\mathscr{V}$.

- If the graph G has a finite set of vertices, it is called a finite graph. Otherwise, G is called an infinite graph.

- We assume that $\mathscr{E}$ has no self-loops and is symmetric:

$$v\in\mathscr{V}\Rightarrow(v,v)\notin\mathscr{E},\ \ (v_1,v_2)\in\mathscr{E}\Rightarrow(v_2,v_1)\in\mathscr{E}.$$

- Choosing an orientation of G consists of defining a partition of $\mathscr{E}$: $\mathscr{E}^+\sqcup\mathscr{E}^-=\mathscr{E}$

$$(v_1,v_2)\in\mathscr{E}^+\Leftrightarrow(v_2,v_1)\in\mathscr{E}^-.$$

- For $e=(v_1,v_2)$, we denote

$$e^-=v_1,\ e^+=v_2\quad\text{and}\quad -e=(v_2,v_1).$$

- We write $v_1\sim v_2$ for $e=(v_1,v_2)\in\mathscr{E}$.

- The graph G is connected if any two vertices x, y in $\mathscr{V}$ can be joined by a path of edges γ_{xy}, that means $\gamma_{xy}=\{e_k\}_{k=1,\ldots,n}$ such that

$$e_1^-=x,\ e_n^+=y\quad\text{and, if}\quad n\geqslant 2\,,\ \forall j\,;\quad 1\leqslant j\leqslant(n-1)\Rightarrow e_j^+=e_{j+1}^-.$$

- The degree (or valence) of a vertex x is the number of edges emanating from x. We denote

$$\deg(x):=\sharp\{e\in\mathscr{E};\ e^-=x\}.$$

- If $\deg(x)<\infty,\ \forall x\in\mathscr{V}$, we say that G is a locally finite graph.

2.2. Weighted graphs

DEFINITION 2.1. A weighted graph (G,c) is given by a graph $G=(\mathscr{V},\mathscr{E})$ and weights on the edges $c:\mathscr{E}\to[0,\infty[$ such that

- $c(x,x)=0,\ \forall x\in\mathscr{V}$.

- $c(x,y)>0,\ \forall(x,y)\in\mathscr{E}$.

- $c(x,y)=c(y,x),\ \forall(x,y)\in\mathscr{E}$.

If $\sum_{y\sim x} c(x,y) < \infty$ for each $x \in \mathscr{V}$, we can define a weight on $\mathscr{V}$ by

$$\tilde{c}(x) = \sum_{y\sim x} c(x,y),\ x \in \mathscr{V}.$$

REMARK 2.1. If the graph G is locally finite, the weight $\tilde{c}$ on any vertex is well defined.

EXAMPLES.

An infinite electrical network is a weighted graph (G,c) where the weight c on the edges are called conductances and their reciprocals are called resistances. This is the convention used in the study of random walks on weighted graphs, see [12] and [16]. Then, $\tilde{c}(x) = \sum_{y\in\mathscr{V}} c(x,y)$ is the weight associated to the vertex x.

A graph G is called a simple graph if the edge weights are equal to 1. In this case,

$$\tilde{c}(x) = \deg(x),\ \forall x \in \mathscr{V}.$$

All the graphs we shall consider in the sequel will be connected, locally finite and weights c given in Definition 2.1.

2.3. Functional spaces

We denote the set of real functions on $\mathscr{V}$ by:

$$\mathscr{C}(\mathscr{V}) = \{f : \mathscr{V} \to \mathbb{R}\}$$

and the set of functions of finite support by $\mathscr{C}_0(\mathscr{V})$.

Moreover, we denote the set of real skew-symmetric functions on $\mathscr{E}$ by:

$$\mathscr{C}^a(\mathscr{E}) = \{\varphi : \mathscr{E} \to \mathbb{R}\,;\varphi(-e) = -\varphi(e)\}$$

and the set of functions of finite support by $\mathscr{C}_0^a(\mathscr{E})$.

We define on the weighted graph (G,c) the following function spaces endowed with the scalar products.

a)

$$l^2(\mathscr{V}) := \left\{ f \in \mathscr{C}(\mathscr{V});\ \sum_{x\in\mathscr{V}} \tilde{c}(x) f^2(x) < \infty \right\},$$

with the inner product

$$\langle f,g\rangle_{\mathscr{V}} = \sum_{x\in\mathscr{V}} \tilde{c}(x) f(x) g(x)$$

and the norm

$$\|f\|_{\mathscr{V}} = \sqrt{\langle f,f\rangle_{\mathscr{V}}}.$$

b)

$$l^2(\mathscr{E}) := \left\{ \varphi \in \mathscr{C}^a(\mathscr{E});\ \frac{1}{2}\sum_{e\in\mathscr{E}} c(e)\varphi^2(e) < \infty \right\},$$

with the inner product

$$\langle \varphi, \psi \rangle_{\mathscr{E}} = \frac{1}{2}\sum_{e\in\mathscr{E}} c(e)\varphi(e)\psi(e)$$

and the norm

$$\|\varphi\|_{\mathscr{E}} = \sqrt{\langle \varphi, \varphi \rangle_{\mathscr{E}}}.$$

Then, $l^2(\mathscr{V})$ and $l^2(\mathscr{E})$ are separable Hilbert spaces (since $\mathscr{V}$ is countable).

2.4. Operators and properties

The difference operator

$$\mathrm{d} : l^2(\mathscr{V}) \longrightarrow l^2(\mathscr{E}),$$

is given by

$$\mathrm{d}(f)(e) = f(e^+) - f(e^-).$$

The coboundary operator is δ, the formal adjoint of d. Thus it satisfies

$$\langle \mathrm{d}f, \varphi \rangle_{\mathscr{E}} = \langle f, \delta\varphi \rangle_{\mathscr{V}} \tag{2.1}$$

for all $f \in l^2(\mathscr{V})$ and for all $\varphi \in l^2(\mathscr{E})$.

As consequence, we have the following formula characterizing δ:

LEMMA 2.1. *The coboundary operator δ is characterized by the formula*

$$\delta\varphi(x) = \frac{1}{\tilde{c}(x)} \sum_{e,e^+=x} c(e)\varphi(e),$$

for all $\varphi \in l^2(\mathscr{E})$.

Proof. For $f \in l^2(\mathscr{V})$ and $\varphi \in l^2(\mathscr{E})$, using (2.1), we get

$$\begin{aligned} \langle \mathrm{d}f, \varphi \rangle_{\mathscr{E}} &= \frac{1}{2}\sum_{e\in\mathscr{E}} c(e)\mathrm{d}f(e)\varphi(e) \\ &= \frac{1}{2}\sum_{e\in\mathscr{E}} c(e)\left(f(e^+) - f(e^-)\right)\varphi(e) \\ &= \frac{1}{2}\sum_{x\in\mathscr{V}} f(x)\left(\sum_{e,e^+=x} c(e)\varphi(e) - \sum_{e,e^-=x} c(e)\varphi(e)\right). \end{aligned}$$

But $c(-e) = c(e)$ and φ is skew-symmetric, so we have

$$\sum_{e,e^+=x} c(e)\varphi(e) = -\sum_{e,e^-=x} c(e)\varphi(e).$$

Then,

$$\begin{aligned}\langle \mathrm{d}f, \varphi\rangle_{\mathscr{E}} &= \sum_{x\in\mathscr{V}} \tilde{c}(x) f(x) \left(\frac{1}{\tilde{c}(x)} \sum_{e,e^+=x} c(e)\varphi(e)\right)\\ &= \langle f, \delta\varphi\rangle_{\mathscr{V}}\end{aligned}$$

and the formula for $\delta\varphi$ follows. $\square$

DEFINITION 2.2. The Laplacian on 0-forms Δ_0 defined by $\delta\mathrm{d}$ on $l^2(\mathscr{V})$ is given by

$$\Delta_0 f(x) = \frac{1}{\tilde{c}(x)} \sum_{y\sim x} c(x,y)\left(f(x) - f(y)\right).$$

In fact, we have

$$\begin{aligned}\Delta_0 f(x) &= \delta(\mathrm{d}f)(x)\\ &= \frac{1}{\tilde{c}(x)} \sum_{e,e^+=x} c(e)\mathrm{d}f(e)\\ &= \frac{1}{\tilde{c}(x)} \sum_{e,e^+=x} c(e)\left(f(e^+) - f(e^-)\right)\\ &= \frac{1}{\tilde{c}(x)} \sum_{y\sim x} c(x,y)\left(f(x) - f(y)\right).\end{aligned}$$

DEFINITION 2.3. The Laplacian on 1-forms Δ_1 defined by $\mathrm{d}\delta$ on $l^2(\mathscr{E})$ is given by

$$\Delta_1\varphi(e) = \frac{1}{\tilde{c}(e^+)} \sum_{e_1,e_1^+=e^+} c(e_1)\varphi(e_1) - \frac{1}{\tilde{c}(e^-)} \sum_{e_2,e_2^+=e^-} c(e_2)\varphi(e_2).$$

In fact, we have

$$\begin{aligned}\Delta_1\varphi(e) &= \mathrm{d}(\delta\varphi)(e)\\ &= \delta\varphi(e^+) - \delta\varphi(e^-)\\ &= \frac{1}{\tilde{c}(e^+)} \sum_{e_1,e_1^+=e^+} c(e_1)\varphi(e_1) - \frac{1}{\tilde{c}(e^-)} \sum_{e_2,e_2^+=e^-} c(e_2)\varphi(e_2).\end{aligned}$$

PROPOSITION 2.1. *The operator Δ_0 is bounded and self-adjoint.*

Proof. For $f,\ g \in l^2(\mathscr{V})$, we have

$$
\begin{aligned}
|\langle \Delta_0 f, g\rangle_{\mathscr{V}}| &= \left| \sum_x \tilde{c}(x) \frac{1}{\tilde{c}(x)} \sum_{y\sim x} c(x,y)\,(f(x)-f(y))\,g(x) \right| \\
&\leqslant \sum_x \sum_{y\sim x} c(x,y)\,|(f(x)-f(y))|\,|g(x)| \\
&\leqslant \sum_x \sum_{y\sim x} c(x,y)\,|f(x)|\,|g(x)| + \sum_x \sum_{y\sim x} c(x,y)\,|f(y)|\,|g(x)| \\
&= \sum_x \tilde{c}(x)\,|f(x)|\,|g(x)| + \sum_x \sum_{y\sim x} c(x,y)\,|f(y)|\,|g(x)| \\
&\leqslant \|f\|_{\mathscr{V}}\,\|g\|_{\mathscr{V}} + I
\end{aligned} \tag{2.2}
$$

where $I := \sum_x \sum_{y\sim x} c(x,y)\,|f(y)|\,|g(x)|$.

Using the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
I &\leqslant \sum_x \left(\sum_{y\sim x} c(x,y)\,|f(y)|^2 \right)^{\frac{1}{2}} \left(\sum_{y\sim x} c(x,y) \right)^{\frac{1}{2}} |g(x)| \\
&= \sum_x \left(\sum_{y\sim x} c(x,y) f^2(y) \right)^{\frac{1}{2}} (\tilde{c}(x))^{\frac{1}{2}}\,|g(x)| \\
&\leqslant \left(\sum_x \sum_{y\sim x} c(x,y) f^2(y) \right)^{\frac{1}{2}} \left(\sum_x \tilde{c}(x) g^2(x) \right)^{\frac{1}{2}} \\
&= \left(\sum_y \tilde{c}(y) f^2(y) \right)^{\frac{1}{2}} \left(\sum_x \tilde{c}(x) g^2(x) \right)^{\frac{1}{2}} \\
&= \|f\|_{\mathscr{V}}\,\|g\|_{\mathscr{V}}.
\end{aligned} \tag{2.3}
$$

Therefore, (2.2) and (2.3) gives

$$|\langle \Delta_0 f, g\rangle_{\mathscr{V}}| \leqslant 2\,\|f\|_{\mathscr{V}}\,\|g\|_{\mathscr{V}}.$$

But by the definition of the norm of operator, we have

$$\|\Delta_0\| = \sup_{\|f\|=1} \|\Delta_0 f\|_{\mathscr{V}} = \sup_{\|f\|=1} \sup_{\|g\|=1} \langle \Delta_0 f, g\rangle_{\mathscr{V}}$$

So $\|\Delta_0\| \leqslant 2$, which shows that Δ_0 is a bounded operator.

Now, we want to prove the selfadjointess of the operator Δ_0 defined on $l^2(\mathscr{V})$. As Δ_0 is a bounded operator on $l^2(\mathscr{V})$, it remains to show that Δ_0 is symmetric.

As we have $\Delta_0 = \delta \mathrm{d}$ and δ is the adjoint operator of d, we obtain for f and $g \in l^2(\mathscr{V})$

$$
\begin{aligned}
\langle \Delta_0 f, g\rangle_{\mathscr{V}} &= \langle \delta \mathrm{d} f, g\rangle_{\mathscr{V}} \\
&= \langle \mathrm{d} f, \mathrm{d} g\rangle_{\mathscr{E}} \\
&= \langle f, \delta \mathrm{d} g\rangle_{\mathscr{V}} \\
&= \langle f, \Delta_0 g\rangle_{\mathscr{V}}. \quad \square
\end{aligned}
$$

REMARK 2.2.

- The operators d and δ are bounded. Indeed, using the inequality $(a-b)^2 \leqslant 2(a^2+b^2)$ and the definition of the weights on vertices: $\tilde{c}(x) = \sum_{y\sim x} c(x,y)$, we obtain

$$\begin{aligned}
\|\mathrm{d}f\|_{\mathscr{E}}^2 &= \frac{1}{2} \sum_{(x,y)\in\mathscr{E}} c(x,y)(\mathrm{d}f(x,y))^2 \\
&= \frac{1}{2} \sum_{(x,y)\in\mathscr{E}} c(x,y)(f(y)-f(x))^2 \\
&\leqslant \sum_{(x,y)\in\mathscr{E}} c(x,y)(f^2(y)+f^2(x)) \\
&= 2\sum_{x\in\mathscr{V}} f^2(x) \sum_{y\sim x} c(x,y) \\
&= 2\sum_{x\in\mathscr{V}} f^2(x)\tilde{c}(x) \\
&= 2\|f\|_{\mathscr{V}}^2 .
\end{aligned}$$

  So d is bounded, and the same is true for the adjoint δ.

  Notice that since Δ_0 is the composite operator of δ and d; this gives another proof that Δ_0 is bounded.

- It is easy to see that Δ_0 is also positive, since $\langle \Delta_0 f, f\rangle_{\mathscr{V}} = \langle \mathrm{d}f, \mathrm{d}f\rangle_{\mathscr{E}} \geqslant 0$.

COROLLARY 2.1. *As the operator Δ_0 is self-adjoint and positive, its spectrum is real and lies in $[0,2]$.*

2.5. Weyl's criterion

As our operator is bounded and self-adjoint on a Hilbert space, we can use Weyl's criterion [14] to characterize its spectrum.

Weyl's criterion: Let $\mathscr{H}$ be a separable Hilbert space, and let Δ be a bounded self-adjoint operator on $\mathscr{H}$. Then λ is in the spectrum of Δ if and only if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ so that $\|f_n\| = 1$ and $\lim_{n\to\infty} \|(\Delta-\lambda)f_n\| = 0$.

We denote $\sigma(\Delta)$ the spectrum of Δ and we set

- $\sigma_d(\Delta)$ is the set of $\lambda \in \sigma(\Delta)$ which is an isolated point and an eigenvalue with finite multiplicity.
- $\sigma_{ess}(\Delta) := \sigma(\Delta) \setminus \sigma_d(\Delta)$.

3. The relation between the spectrum of Δ_0 and Δ_1

3.1. The nonzero spectrum of Δ_0 and Δ_1

In this section we will prove the relation between the spectrum of Δ_0 and that of Δ_1, by using Weyl's criterion.

Following [15] and [18] we have the next lemma.

LEMMA 3.1. *Let* $\Delta_0 = \delta \mathrm{d}$ *and* $\Delta_1 = \mathrm{d}\delta$. *Then we have*

1. $\mathrm{d}\Delta_0 = \Delta_1 \mathrm{d}$.
2. $\delta\Delta_1 = \Delta_0\delta$.

LEMMA 3.2.

1. $\ker\Delta_0 = \ker \mathrm{d}$.
2. $\ker\Delta_1 = \ker\delta$.

Proof.

1. Clearly, we have $\ker \mathrm{d} \subset \ker\Delta_0$.
   On the other hand, if $\Delta_0 f = 0$ for $f \in l^2(\mathscr{V})$ and $f \neq 0$, we have
   $$0 = \langle \Delta_0 f, f\rangle_{\mathscr{V}} = \langle \mathrm{d}f, \mathrm{d}f\rangle_{\mathscr{E}}.$$
   Then $\mathrm{d}f = 0$ for $f \in l^2(\mathscr{V})$.
2. If $\varphi \in \ker\delta$, then $\varphi \in l^2(\mathscr{E})$ and $\delta\varphi = 0$. Thus, $\mathrm{d}\delta\varphi = 0$ and we obtain $\varphi \in \ker\Delta_1$.
   For the other inclusion, let $\varphi \in l^2(\mathscr{V})$, $\varphi \neq 0$ such that $\Delta_1\varphi = 0$. Then
   $$0 = \langle \Delta_1\varphi, \varphi\rangle_{\mathscr{E}} = \langle \delta\varphi, \delta\varphi\rangle_{\mathscr{E}}.$$
   We get $\delta\varphi = 0$ and as a result $\ker\Delta_1 \subset \ker\delta$. $\square$

We arrive at our main result.

THEOREM 1.
$$\sigma(\Delta_1)\setminus\{0\} = \sigma(\Delta_0)\setminus\{0\}.$$

Proof.

- Let $\lambda \neq 0$ be in the spectrum of Δ_0. By Weyl's criterion, there exists a sequence $(f_n)_n$ of $l^2(\mathscr{V})$ such that
  $$\|f_n\|_{\mathscr{V}} = 1 \quad \text{and} \quad \lim_{n\to\infty} \|(\Delta_0 - \lambda) f_n\|_{\mathscr{V}} = 0.$$
  We want to find a sequence $(\varphi_n)_n$ of $l^2(\mathscr{E})$ such that
  $$\|\varphi_n\|_{\mathscr{E}} = 1 \quad \text{and} \quad \lim_{n\to\infty} \|(\Delta_1 - \lambda)\varphi_n\|_{\mathscr{E}} = 0.$$

We set

$$\varphi_n := \frac{\mathrm{d}f_n}{\|\mathrm{d}f_n\|_{\mathscr{E}}}.$$

First, let us check that $\|\mathrm{d}f_n\|_{\mathscr{E}} \neq 0$. We have

$$\begin{aligned}\|\mathrm{d}f_n\|_{\mathscr{E}}^2 &= \langle \Delta_0 f_n, f_n\rangle_{\mathscr{V}} \\ &= \langle (\Delta_0-\lambda) f_n, f_n\rangle_{\mathscr{V}} + \langle \lambda f_n, f_n\rangle_{\mathscr{V}} \\ &= \underbrace{\langle (\Delta_0-\lambda) f_n, f_n\rangle_{\mathscr{V}}}_{\text{converges to } 0} + \lambda.\end{aligned}$$

Then, $\lim_{n\to\infty} \|\mathrm{d}f_n\|_{\mathscr{E}}^2 = \lambda$. Thus, by positivity of Δ_0, there exists $A>0$ and an integer n_0 such that for all $n \geqslant n_0$, we have $\|\mathrm{d}f_n\|_{\mathscr{E}} > A$. This implies that the sequence $(\varphi_n)_n$ is well defined.

Now, we verify that $\lim_{n\to\infty} \|(\Delta_1-\lambda)\varphi_n\|_{\mathscr{E}} = 0$. By the first assertion of Lemma 3.1 and the fact that the operator d is bounded, we obtain for all n sufficiently large

$$\begin{aligned}\|(\Delta_1-\lambda)\varphi_n\|_{\mathscr{E}} &= \left\|(\Delta_1-\lambda)\frac{\mathrm{d}f_n}{\|\mathrm{d}f_n\|_{\mathscr{E}}}\right\|_{\mathscr{E}} \\ &= \frac{\|(\Delta_1-\lambda)\mathrm{d}f_n\|_{\mathscr{E}}}{\|\mathrm{d}f_n\|_{\mathscr{E}}} \\ &= \frac{\|\mathrm{d}(\Delta_0-\lambda)f_n\|_{\mathscr{E}}}{\|\mathrm{d}f_n\|_{\mathscr{E}}} \\ &\leqslant \frac{\|\mathrm{d}\|}{A}\,\|(\Delta_0-\lambda)f_n\|_{\mathscr{V}}.\end{aligned}$$

But $\lim_{n\to\infty}\|(\Delta_0-\lambda)f_n\|_{\mathscr{V}} = 0$. Therefore, $\lim_{n\to\infty}\|(\Delta_1-\lambda)\varphi_n\|_{\mathscr{E}} = 0$ and we can conclude that λ is in the spectrum of $\Delta_1 \setminus \{0\}$.

- The second part of the proof follows in the same fashion, with the roles of d and δ swapped. □

There is a second method to prove Theorem 1 when 0 is not in the spectrum of Δ_0.

LEMMA 3.3. *If 0 is not in the spectrum of Δ_0, then the operator d defined in $l^2(\mathscr{V})$ has a closed range.*

Proof. Let $\varphi \in \overline{\mathrm{Im}\,\mathrm{d}}$, let us check that $\varphi \in \mathrm{Im}\,\mathrm{d}$, that means we look for a function $f \in l^2(\mathscr{V})$ such that $\varphi = \mathrm{d}f$. We have $\varphi \in \overline{\mathrm{Im}\,\mathrm{d}}$, so there exists a sequence $(\varphi_n)_n$ of $\mathrm{Im}\,\mathrm{d}$ such that $\varphi_n = \mathrm{d}f_n$, for $f_n \in l^2(\mathscr{V})$. Moreover, the sequence $(\varphi_n)_n$ converges to φ in $l^2(\mathscr{E})$. On the other hand, by assumption 0 is not in the spectrum of Δ_0 which implies the existence of a positive constant C such that

$$\|f\|_{\mathscr{V}} \leqslant C\|\Delta_0 f\|_{\mathscr{V}},\ \forall f \in l^2(\mathscr{V}).$$

But by the definition of the operator norm and Remark 2.2, we obtain

$$\|\Delta_0 f\| = \sup_{g,\|g\|_{\mathscr{V}}=1} \langle \Delta_0 f, g\rangle_{\mathscr{V}} \leqslant \|\mathrm{d}f\|_{\mathscr{E}} \sup_{g,\|g\|_{\mathscr{V}}=1} \|\mathrm{d}g\|_{\mathscr{E}} \leqslant \sqrt{2}\,\|\mathrm{d}f\|_{\mathscr{E}}.$$

Then

$$\|f\|_{\mathscr{V}} \leqslant \sqrt{2}C\,\|\mathrm{d}f\|_{\mathscr{E}},\ \forall f \in l^2(\mathscr{V}).$$

Thus

$$\|f_n - f_m\|_{\mathscr{V}} \leqslant \sqrt{2}C\,\|\mathrm{d}f_n - \mathrm{d}f_m\|_{\mathscr{E}},\ f_n,\ f_m \in l^2(\mathscr{V}).$$

And

$$\|f_n - f_m\|_{\mathscr{V}} \leqslant \sqrt{2}C\,\|\varphi_n - \varphi_m\|_{\mathscr{E}},\ f_n,\ f_m \in l^2(\mathscr{V}).$$

As the sequence $(\varphi_n)_n$ converges, so it is a Cauchy sequence and also $(f_n)_n$ is a Cauchy sequence in $l^2(\mathscr{V})$ which is complete. Then, $(f_n)_n$ converges to f. By the boundedness of the operator d, we obtain $\mathrm{d}f_n = \varphi_n$ converges to $\mathrm{d}f$ and by uniqueness of the limit, we have $\mathrm{d}f = \varphi$. So φ is in $\mathrm{Im}\,\mathrm{d}$. $\square$

COROLLARY 3.1. *If* 0 *is not in the spectrum of* Δ_0*, then*

$$\sigma(\Delta_1|_{\mathrm{Im}\,\mathrm{d}}) = \sigma(\Delta_0).$$

Proof. By the first assertion of Lemma 3.1, we obtain

$$\Delta_1 \mathrm{d} = \mathrm{d}\Delta_0.$$

But by assumption 0 is not in the spectrum of Δ_0. Then by the first assertion of Lemma 3.2, the operator d is invertible. So we obtain

$$\Delta_1|_{\mathrm{Im}\,\mathrm{d}} = \mathrm{d}\Delta_0\mathrm{d}^{-1}.$$

Thus,

$$\sigma(\Delta_1|_{\mathrm{Im}\,\mathrm{d}}) = \sigma(\Delta_0). \quad \square$$

3.2. The 0-spectrum of Δ_0 and Δ_1

As the nonzero spectrum of Δ_0 and Δ_1 are the same, we are interested in characterizing the 0-spectrum. We give in the following an extension of a result of John Lott's [11] (Proposition 9, p. 12).

THEOREM 2. *Let* (G,c) *be a connected, locally finite and weighted infinite graph such that the weight on edges* c *is bounded, i.e., there exists a constant* $\alpha > 0$ *such that* $\frac{1}{\alpha} \leqslant c(x,y) \leqslant \alpha$*, for all* $(x,y) \in \mathscr{E}$*. Then*

$$0 \in \sigma(\Delta_1) \quad or \quad 0 \in \sigma(\Delta_0).$$

First, we start with preliminary results.

By [17] (page 44) and [9] (chapter 4) we have the next definition.

DEFINITION 3.1. The graph G verifies *the isoperimetric inequality* if there exists a constant $C > 0$ such that for all finite sub-graphs $G_U = (U, \mathscr{E}_U)$ of G, we have

$$|\partial \mathscr{E}_U| \geqslant C|U|,$$

where

$$|\partial \mathscr{E}_U| = \sum_{x \in U} \sum_{y \notin U} c(x,y) \quad \text{and} \quad |U| = \sum_{x \in U} \tilde{c}(x).$$

LEMMA 3.4. *If Δ_0 is invertible then the isoperimetric inequality holds.*

Proof. Let U a finite sub-graph of G. Let us set $g = \mathbf{1}_U$, meaning that $g(x) = 1$ if $x \in U$ and $g(x) = 0$ if $x \notin U$. Then we obtain

$$|U| = \sum_{x \in U} \tilde{c}(x) = \|g\|^2_{\mathscr{V}}$$

and

$$|\partial \mathscr{E}_U| = \sum_{x \in U} \sum_{y \notin U} c(x,y) = \|\mathrm{d}g\|^2_{\mathscr{E}}.$$

By assumption 0 is not in the spectrum of Δ_0. Then by the first assertion of Lemma 3.2, the operator d is invertible, so there exists a positive constant λ so that

$$\|g\|_{\mathscr{V}} \leqslant \lambda \, \|\mathrm{d}g\|_{\mathscr{E}}, \, \forall g \in l^2(\mathscr{V}).$$

Thus, it follows that

$$|\partial \mathscr{E}_U| \geqslant C|U|, \quad \text{with} \quad C = \frac{1}{\lambda^2}. \quad \square$$

DEFINITION 3.2.

- *A branch B* is a finite sequence of vertices $x_0, x_1, \ldots, x_{m+1}$ such that for all j; $1 \leqslant j \leqslant m$, we have $\deg(x_j) = 2$.
- *The length of a branch B*, denoted $\mathrm{long}(B)$, is the number of vertices in this branch, here, $\mathrm{long}(B) = m + 2$.
- *The interior of the branch B* is the set of vertices x_j of B satisfying the following conditions:

  i) $\deg(x_j) = 2$.

  ii) $\forall y \in \mathscr{V}$; $y \sim x_j \Rightarrow y \in B$.

  See [5] and [19] for the definition of the interior set of vertices.

Instead of the argument of Lott [11] inspired by Gromov [10] (p. 236–237), we use the following lemma:

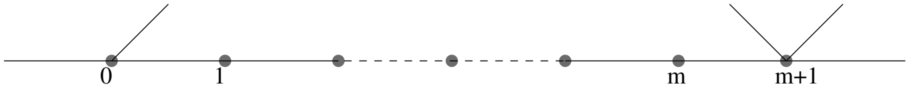

Figure 1: *A branch of length* $m+2$

LEMMA 3.5. *We suppose that the following conditions are satisfied:*

- *The weight on edges* c *is bounded, i.e., there exists a constant* $\alpha > 0$ *such that* $\frac{1}{\alpha} \leqslant c(x,y) \leqslant \alpha$, $\forall (x,y) \in \mathscr{E}$.
- *The operator* Δ_0 *is invertible.*
- *The operator* Δ_1 *is injective.*

Then the graph (G,c) *is a tree which contains branches with uniformly bounded lengths, that means* $\exists M > 0$, $\forall B$ *branch of* G, $\operatorname{long}(B) \leqslant M$.

Proof. On the one hand, the operator Δ_1 is injective which leads to the absence of cycles in the graph, so that G is a tree.

On the other hand, the operator Δ_0 is invertible, then the isoperimetric inequality is checked, by Lemma 3.4 there is a positive constant C such that for all finite sub-graphs U, we have

$$|\partial \mathscr{E}_U| \geqslant C|U|.$$

Let B be a branch with vertices $x_0, x_1, \ldots, x_m, x_{m+1}$. We set $U = \{x_1, \ldots, x_m\}$ the interior of the branch B, then

$$c(x_0,x_1) + c(x_m,x_{m+1}) \geqslant C \sum_{j=1}^{m} \tilde{c}(x_j). \tag{3.4}$$

For the sake of simplicity, we first prove the lemma for the case of the constant weight $c = 1$, before handling the case of general weights.

- If $c = 1$, then we have $\tilde{c}(x) = \sum_{y \sim x} c(x,y) = \sum_{y \sim x} 1 = \deg(x)$, $\forall x \in \mathscr{V}$ (this is J. Lott's case [11]). Therefore, the inequality (3.4) and Definition 3.2 gives

  $$2 \geqslant C \sum_{j=1}^{m} \deg(x_j) = C \sum_{j=1}^{m} 2 = 2Cm.$$

  We set $M := \frac{1}{C} + 2$ (independent of B), then $\operatorname{long}(B) \leqslant M$. Thus the lengths of branches of G are uniformly bounded.

- If $c \neq 1$ but c is bounded, that means there exists $\alpha > 0$ such that $\frac{1}{\alpha} \leqslant c(x,y) \leqslant \alpha$, for all $(x,y) \in \mathscr{E}$. And as we have the weight on the vertices is $\tilde{c}(x) = \sum_{y \sim x} c(x,y)$, we obtain that $\tilde{c}$ is also bounded from below by $\frac{1}{\alpha}$.

By the inequality (3.4) we have

$$2\alpha \geqslant c(x_0,x_1)+c(x_m,x_{m+1}) \geqslant C\sum_{j=1}^{m}\tilde{c}(x_j) \geqslant Cm\frac{1}{\alpha}.$$

Hence,

$$\frac{2\alpha^2}{C} \geqslant m.$$

We set $M = \frac{2\alpha^2}{C}+2$ (independent of B), then $\mathrm{long}(B) \leqslant M$. Thus, the lengths of the branches of G are uniformly bounded. $\square$

Now, we arrive to the proof of Theorem 2.

Proof. Taking the arguments from [11], we argue by contradiction. Suppose that both operators Δ_0 and Δ_1 are invertible. Then, by Lemma 3.5, the graph G is a tree which contains branches with uniformly bounded lengths; see Figure 2 for an example.

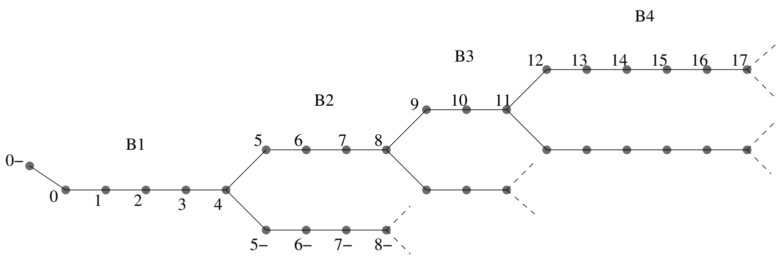

Figure 2: *A branch tree*

But the existence of such tree gives a δ-harmonic nonzero square-integral function φ. Indeed: we consider a part of the branch tree in Figure 2 as an example to simplify the understanding of the construction.
For the sake of simplicity, we first prove the theorem for the case of the constant weight $c=1$, before handling the case of general weights.

First case: $c=1$, we fix a vertex 0 as the origin of the tree and we set 0^- and 1 its different neighbors. Let us take

$$\varphi(0,0^-)=\varphi(0,1)=1.$$

Then, we obtain $\delta\varphi(0)=0$ (the tree is oriented).

Afterwards on the branch B_1, φ is constant, in other words, $\varphi(j,j+1)=1$, for all j, such that $1\leqslant j\leqslant 3$. And at the point 4, we have $\varphi(4,5)=\varphi(4,5^-)=\frac{1}{2}$. It is claimed that $\delta\varphi(4)=0$. And for the points which are in the branch B_2, the function φ is constant and takes the value $\frac{1}{2}$. And so on to the point 8, we have $\varphi(8,9)=\varphi(8,9^-)=\frac{1}{4}$, to obtain $\delta\varphi(8)=0$. And for the points which are in the

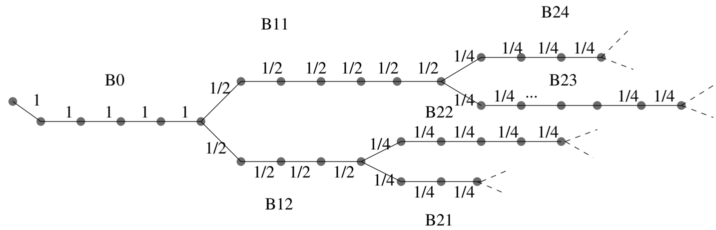

Figure 3: *An example of a branch tree*

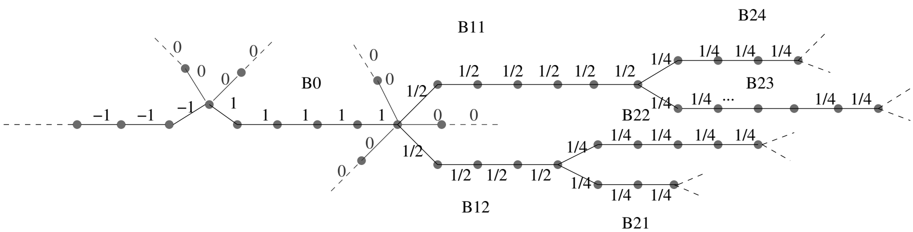

Figure 4: *Another example of a branch tree*

branch B_3, the function φ is constant and takes the value $\frac{1}{4}$. And we continue in this way…

In a general way, G is a tree which contains branches with uniformly bounded lengths and we construct a functions φ in a part of G by selecting always two branches at bifurcation points and at all edges that occur on other branches, φ is set to zero, as in Figure 4. In the Figure 3, the construction of φ is done in the following way: on B_0 the function φ is constant and equals to 1. Then we add a generation, we get two branches $B_{1,1}$ and $B_{1,2}$ such that the function φ takes the value $\frac{1}{2}$. And to the generation m, we have $B_{m,k}$ branches, where $1 \leqslant k \leqslant 2^m$, then the function φ is equal to $\frac{1}{2^m}$. As a result, we show that this construction of φ is in $l^2(\mathscr{E})$. Using the fact that the lengths of the branches of the tree are uniformly bounded by a constant $M > 0$, we obtain

$$\begin{aligned}
\|\varphi\|_{\mathscr{E}}^2 &= \frac{1}{2}\sum_{m\geqslant 0}\sum_{k=1}^{2^m}\sum_{e\in B_{m,k}} (\varphi(e))^2 \\
&= \frac{1}{2}\sum_{m\geqslant 0}\sum_{k=1}^{2^m}\sum_{e\in B_{m,k}} \left(\frac{1}{2^m}\right)^2 \\
&\leqslant \frac{1}{2}\sum_{m\geqslant 0} 2^m M \left(\frac{1}{2^m}\right)^2 \\
&= \frac{M}{2}\sum_{m\geqslant 0}\frac{1}{2^m} < \infty.
\end{aligned}$$

Second case: $c \neq 1$ but c bounded by a positive constant. That means there exists $\alpha > 0$ such that $\frac{1}{\alpha} \leqslant c(x,y) \leqslant \alpha$, $\forall (x,y)$ in $\mathscr{E}$. As in Figure 2, on the branch B_1, the vertex 0 has two neighbors denoted 0^- and 1. We want $\delta\varphi(0) = 0$, so we choose the function φ in the following way $\varphi(0,1) = \frac{c(0,0^-)}{c(0,1)}\varphi(0,0^-)$. And in the interior of B_1, we set $\varphi(j,j+1) = \frac{c(0,0^-)}{c(j,j+1)}\varphi(0,0^-)\,\forall j,\ 1 \leqslant j \leqslant 3$. Next, we look at the point 4 which has two neighbors 5 and 5^-, to obtain $\delta\varphi(4) = 0$ and as we have $\varphi(3,4) = \frac{c(0,0^-)}{c(3,4)}\varphi(0,0^-)$. We choose $\varphi(4,5) = \frac{c(0,0^-)}{2c(4,5)}\varphi(0,0^-)$ and $\varphi(4,5^-) = \frac{c(0,0^-)}{2c(4,5^-)}\varphi(0,0^-)$. Therefore, in the interior of the branch B_2,

$$\varphi(j,j+1) = \frac{c(0,0^-)}{2c(j,j+1)}\varphi(0,0^-)\,\forall j, \quad 5 \leqslant j \leqslant 7.$$

And for the vertex 8, which has two neighbors 9 and 9^-. To have $\delta\varphi(8) = 0$ and by using that $\varphi(7,8) = \frac{c(0,0^-)}{2c(7,8)}\varphi(0,0^-)$. We choose $\varphi(8,9) = \frac{c(0,0^-)}{4c(8,9)}\varphi(0,0^-)$ and $\varphi(8,9^-) = \frac{c(0,0^-)}{4c(8,9^-)}\varphi(0,0^-)$. And in the interior of the branch B_3,

$$\varphi(j,j+1) = \frac{c(0,0^-)}{4c(j,j+1)}\varphi(0,0^-) \quad \text{for} \quad j = 10.$$

And so on... In a general way, see Figure 3, on B_0 the function $\varphi(e_0) = \frac{c(0,0^-)}{c(e_0)}\varphi(0,0^-)$, where e_0 is an edge of B_0. Then we add a generation, we get two branches $B_{1,1}$ and $B_{1,2}$ such that the function φ has a value $\varphi(e_1^k) = \frac{c(0,0^-)}{2c(e_1^k)}\varphi(0,0^-)$, where e_1^k denotes the edges of $B_{1,k}$ for $1 \leqslant k \leqslant 2$. And at generation m, we have $B_{m,k}$ branches, where $1 \leqslant k \leqslant 2^m$, then the function φ equals to $\varphi(e_m^k) = \frac{c(0,0^-)}{2^m c(e_m^k)}\varphi(0,0^-)$, where e_m^k denotes the edges of $B_{m,k}$. And to simplify the formulas, we can suppose that

$$\varphi(0,0^-) = \frac{1}{c(0,0^-)}.$$

Then, we obtain

$$\varphi(e_m^k) = \frac{1}{2^m c(e_m^k)},\ \forall m \geqslant 0 \quad \text{and} \quad 1 \leqslant k \leqslant 2^m.$$

Therefore, this construction gives $\varphi \in l^2(\mathscr{E})$. Using the fact that the lengths of the branches of the tree are uniformly bounded by a constant $M > 0$ and the weight c on the edges is bounded by a positive constant, we obtain

$$\begin{aligned}\|\varphi\|_{\mathscr{E}}^2 &= \frac{1}{2}\sum_e c(e)(\varphi(e))^2 \\ &\leqslant \sum_{m \geqslant 0}\sum_{k=1}^{2^m}\sum_{e \in B_{m,k}} c(e)(\varphi(e))^2\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m \geqslant 0} \sum_{k=1}^{2^m} \sum_{e \in B_{m,k}} c(e) \left(\frac{1}{2^m c(e)} \right)^2 \\
&= \sum_{m \geqslant 0} \sum_{k=1}^{2^m} \sum_{e \in B_{m,k}} \frac{1}{2^{2m} c(e)} \\
&\leqslant \alpha M \sum_{m \geqslant 0} \frac{1}{2^m} \\
&= 2M\alpha < \infty.
\end{aligned}
$$

Finally, we have φ in $l^2(\mathscr{E})$ and δ-harmonic. So, $0 \in \sigma(\Delta_1)$, which contradicts the assumption that 0 is not in the spectrum of Δ_1. $\square$

REMARK 3.1. Any point of the graph can play the role of the first vertex 0 in the previous construction. It is then clear that we can construct an infinite family of independent functions φ which are in $l^2(\mathscr{E})$ and δ-harmonic. Then 0 is an eigenvalue of Δ_1 with infinite multiplicity, so $0 \in \sigma_{ess}(\Delta_1)$.

4. Examples

In this section, we will construct a δ-harmonic function φ in different examples of trees.

1) *Symmetric tree*: Following [7] we introduce the next definition:

DEFINITION 4.1. A tree T_s is symmetric around o with branching numbers $\{m_i\}_{i=0}^{\infty}$ if the degree of each vertex depends only on its distance from o, namely, for each $x \in T_s$, $\deg(x) = m_i$ if $d(o,x) = i$.

Example of a symmetric tree: We fix a vertex o as an origin of the tree. We set $S_n = \{x \in T_s;\ d(o,x) = n\}$. T_s is symmetric around o with branching numbers $\{m_n\}_{n=0}^{\infty}$. In Figure 4, we choose $m_n = 3+n$ for all $n \in \mathbb{N}$ which is an increasing sequence. So, we have $m_0 = 3$ that means $\deg(o) = 3$. And for $x \in S_1$, we obtain $\deg(x) = m_1 = 4$. In the same way, if $x \in S_2$ we have $m_2 = 5$ and so on.

PROPOSITION 4.1. *If the symmetric tree T_s is simple (the edge weights are equal to 1) with* $\deg(x) > 2$ *for all $x \in T_s$, then there is a δ-harmonic function $\varphi \in l^2(\mathscr{E})$.*

Proof. We fix a vertex x_0 as an origin of the tree T_s, we can find an increasing sequence of finite subgraph $\{S_n\}_n$ such that $S_n = \{x \in T_s;\ d(x_0,x) = n\}$ and $T_s = \cup_n S_n$. By the definition of the symmetric tree, we have for all n $\deg(x_n) = m_n$, $\forall x_n \in S_n$. First, we construct a function φ so that $\delta\varphi = 0$ as follows: Let e_0 and b_0 denote two distinct outward edges connecting to the vertex x_0. We define φ to be 0 excepted on these edges where $\varphi(e_0) = 1$ and $\varphi(b_0) = -1$ which gives $\delta\varphi(x_0) = 0$. And denote

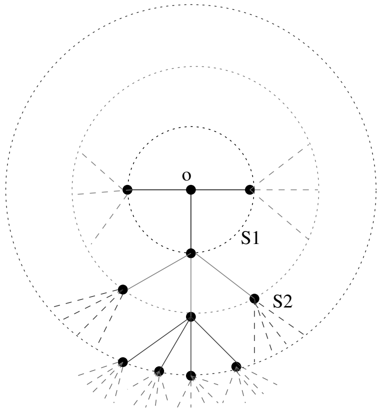

Figure 5: *Symmetric tree*

e_n^k, $n \geqslant 1$, $1 \leqslant k \leqslant \prod_{j=1}^n (m_j - 1)$, resp. b_n^k, $n \geqslant 1$, $1 \leqslant k \leqslant \prod_{j=1}^n (m_j - 1)$, the outward edges emanating from e_0, resp. b_0, of generation n. We define

$$\varphi(e_n^k) = \frac{1}{\prod_{j=1}^n (m_j - 1)} \varphi(e_0),$$

$$\varphi(b_n^k) = \frac{1}{\prod_{j=1}^n (m_j - 1)} \varphi(b_0)$$

and φ takes value 0 on all edges other than e_n^k and b_n^k.

Second, through this construction, we look for $\varphi \in l^2(\mathscr{E})$. Using the fact that $\deg(x_n) = m_n \geqslant 3$, $\forall x_n \in S_n$, $\forall n$, we obtain

$$\begin{aligned}
\|\varphi\|_{\mathscr{E}}^2 &= \frac{1}{2} \sum_{e \in \mathscr{E}} \varphi^2(e) \\
&= \frac{1}{2} \left(2 + \sum_{n \geqslant 1} \sum_{k=1}^{\prod_{j=1}^n (m_j - 1)} \varphi^2(e_n^k) + \varphi^2(b_n^k) \right) \\
&= 1 + \sum_{n \geqslant 1} \sum_{k=1}^{\prod_{j=1}^n (m_j - 1)} \left(\frac{1}{(m_1 - 1)(m_2 - 1) \dots (m_n - 1)} \right)^2 \\
&= 1 + \sum_{n \geqslant 1} \frac{1}{(m_1 - 1)(m_2 - 1) \dots (m_n - 1)} \\
&\leqslant 1 + \sum_{n \geqslant 1} \frac{1}{2^n} \\
&< \infty. \quad \square
\end{aligned}$$

2) *Triadic tree with weights bounded from below*: As [2] (p. 19), we have the following definition of a triadic tree.

DEFINITION 4.2. A tree is a connected graph containing no cycles. The **triadic tree** is a tree such that all the vertices have degree 3.

PROPOSITION 4.2. *If the triadic tree has weights on the edges bounded from below by a positive constant* λ, *then there is a* δ*-harmonic function* $\varphi \in l^2(\mathscr{E})$.

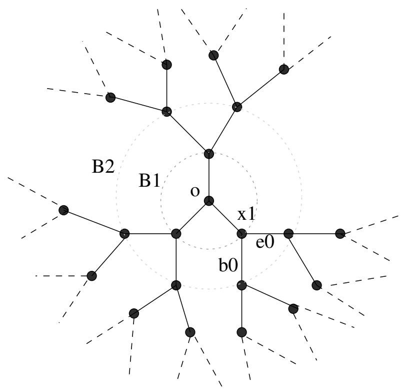

Figure 6: *Triadic tree*

Proof. We fix a vertex o as the origin of the tree T. Define the increasing sequence of finite subgraphs $\{G_n\}_n$, $G_n = \{x \in \mathscr{V};\, d(o,x) \leqslant n\}$ and let $G = \bigcup_n G_n$. Denote $S_n = \{x \in T;\, d(o,x) = n\}$.

We set x_1^1, x_1^2 and x_1^3 the different neighbors of o which are in S_1. We suppose that $\varphi(o,x_1^i) = 0$ for all $i \in \{1,2,3\}$, so we have $\delta\varphi(o) = 0$.

We fix one vertex of S_1 for example $x_1 := x_1^1$, let e_0 and b_0 be the two outward edges of x_1 and define inductively e_m^k, $m \geqslant 1$, $1 \leqslant k \leqslant 2^m$, resp. b_m^k, $m \geqslant 1$, $1 \leqslant k \leqslant 2^m$, to be the outward edges emanating from e_0, resp. b_0, of generation m (the edge are oriented outward). For $m \geqslant 0$, we define φ to be 0 excepted on these edges where

$$\varphi(e_m^k) = \frac{1}{2^m}\frac{1}{c(e_m^k)},\ \forall k; \quad 1 \leqslant k \leqslant 2^m$$

and

$$\varphi(b_m^k) = \frac{-1}{2^m}\frac{1}{c(b_m^k)},\ \forall k; \quad 1 \leqslant k \leqslant 2^m.$$

With this construction, we obtain for each $x_n \in S_n$, $\delta\varphi(x_n) = 0$, $\forall n \geqslant 1$. Moreover, $\varphi \in l^2(\mathscr{E})$. Indeed: by using the assumption that the weights on the edges are bounded from below by a positive constant λ, we obtain

$$\begin{aligned}
\|\varphi\|_{\mathscr{E}}^2 &= \frac{1}{2}\sum_{e\in\mathscr{E}} c(e)\varphi^2(e) \\
&= \frac{1}{2}\left(\sum_{m\geqslant 0}\sum_{k=1}^{2^m} c(e_m^k)\varphi^2(e_m^k) + c(b_m^k)\varphi^2(b_m^k)\right) \\
&= \frac{1}{2}\sum_{m\geqslant 0}\sum_{k=1}^{2^m} c(e_m^k)\frac{1}{2^{2m}}\frac{1}{c^2(e_m^k)} + \frac{1}{2}\sum_{m\geqslant 0}\sum_{k=1}^{2^m} c(b_m^k)\frac{1}{2^{2m}}\frac{1}{c^2(b_m^k)} \\
&= \frac{1}{2}\sum_{m\geqslant 0}\sum_{k=1}^{2^m} \frac{1}{2^{2m}}\frac{1}{c(e_m^k)} + \frac{1}{2}\sum_{m\geqslant 0}\sum_{k=1}^{2^m} \frac{1}{2^{2m}}\frac{1}{c(b_m^k)} \\
&\leqslant \lambda' \sum_{m\geqslant 0}\sum_{k=1}^{2^m}\frac{1}{2^{2m}} \\
&= \lambda' \sum_{m\geqslant 0} 2^m \frac{1}{2^{2m}} \\
&= \lambda' \sum_{m\geqslant 0} \frac{1}{2^m} \\
&= 2\lambda',
\end{aligned}$$

where $\lambda' = \frac{1}{\lambda}$. $\square$

REMARK 4.1.

- The construction of a δ-harmonic nonzero square-integral function depends on the edge weights.
- In the simple triadic tree, 0 is both in the spectrum of Δ_0 [4] and in the spectrum of Δ_1 [2].

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Hèla Ayadi
Unité de recherche Mathématiques et applications (UR/13ES47)
à la faculté des sciences de Bizerte
Laboratoire De Mathématiques Jean Leray, Université de Nantes
e-mail: `halaayadi@yahoo.fr, Hela.Ayadi@univ-nantes.fr`