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ISOMETRIC COMPOSITION OPERATORS ON THE FOCK–SPACES

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Abstract. In this paper a necessary and sufficient condition for a holomorphic self map ϕ on $\mathbb{C}^N$ to induce an isometric composition operator on the Fock space has been obtained. Some necessary and sufficient conditions for a composition operator C_ϕ to be a quasi-isometric and m-isometric have also been explored.

1. Introduction

Let $z=(z_1,z_2,\ldots,z_N)$ and $w=(w_1,w_2,\ldots,w_N)$ be points in $\mathbb{C}^N$, $\langle z,w\rangle = \sum_{k=1}^{N} z_k\overline{w_k}$ and $|z| = \sqrt{\langle z,z\rangle}$. Let $\mathbb{B}$ denote the open unit ball $\{z : |z| < 1\}$, $S = \partial\mathbb{B}$ the boundary of the unit ball $\mathbb{B}$, $dm(z) = rdrd\theta$ the Lebesgue area measure on $\mathbb{C}$, $dV(z)$ the Lebesgue volume measure on $\mathbb{C}^N$, $V_N = V(\mathbb{B})$, $d\sigma(z)$ the normalized surface measure on S and $H(\mathbb{C}^N)$ the space of all holomorphic functions on $\mathbb{C}^N$ (entire functions). For $p,\alpha \in (0,\infty)$, the Bergman-Fock space[22] $\mathscr{F}^p_\alpha = \mathscr{F}^p_\alpha(\mathbb{C}^N)$ is the space of all entire functions f for which

$$\|f\|^p_{\mathscr{F}^p_\alpha} = \left(\frac{p\alpha}{2\pi}\right)^N \int_{\mathbb{C}^N} |f(z)|^p e^{-\frac{p\alpha}{2}|z|^2} dV(z) < \infty$$

Note that, by using polar coordinates

$$\begin{aligned}\|I\|^p_{\mathscr{F}^p_\alpha} &= \left(\frac{p\alpha}{2\pi}\right)^N V_N \int_0^\infty \int_S \rho^{2N-1} e^{-\frac{\alpha p}{2}\rho^2} d\sigma(\xi)d\rho \\ &= \frac{(p\alpha)^N \int_0^\infty t^{N-1} e^{-\frac{\alpha p}{2}t} dt}{2^N (N-1)!} = 1.\end{aligned}$$

When $1 \leqslant p < \infty$, the space $\mathscr{F}^p_\alpha(\mathbb{C}^N)$ is a Banach space with the norm $\|f\|_{\mathscr{F}^p_\alpha}$, while for $p \in (0,1)$, it is an F-space with the translation-invariant metric $d_{\mathscr{F}^p_\alpha}(f,g) = \|f-g\|^p_{\mathscr{F}^p_\alpha}$.

For $p=2$ the space is reduced to the Fock space, which is a functional Hilbert space with the inner product

$$\langle f,g\rangle = \left(\frac{\alpha}{\pi}\right)^N \int_{\mathbb{C}^N} f(z)\overline{g(z)} e^{-\alpha|z|^2} dV(z).$$

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Let $e_n(z) = \sqrt{\frac{1}{n!}}z^n$ for a positive integer n. The sequence $\{e_n\}_{n\in\mathbb{N}}$ forms an orthonormal basis for $\mathscr{F}_N^2$. Since each point evaluation is a bounded linear functional on $\mathscr{F}_N^2$, for each $w \in \mathbb{C}^N$ there exists a unique function $k_w \in \mathscr{F}_N^2$ such that $\langle f, k_w\rangle = f(w)$ for all $f \in \mathscr{F}_N^2$. The reproducing kernel functions for the Fock-space $\mathscr{F}_N^2$ are given by

$$k_w(z) = e^{\langle z,w\rangle/2}.$$

Carswell et al. [5], Ichiro [21], Grudsky et al. [7], Stroethoff [19], Janson et al. [9] have studied various concrete operators on Fock-spaces, one of them is a composition operators which is defined as follows:

For a given holomorphic mapping $\phi : \mathbb{C}^N \to \mathbb{C}^N$, the composition operator $C_\phi : \mathscr{F}_N^2 \to \mathscr{F}_N^2$ is given by $C_\phi(f) = f\circ\phi \quad f \in \mathscr{F}_N^2$. The multiplication operator M_u induced by an entire function u on $\mathbb{C}^N$ is defined as $M_u f(z) = u(z)f(z)$ for an entire function f.

Let X be a normed linear space. An operator T on X is isometric if $\|Tf\|_X = \|f\|_X$ for $f \in X$, quasi-isometric if $T^{*2}T^2 = T^*T$ [18] and m-isometric if

$$\sum_{p=0}^{m}(-1)^{p\,m}\mathscr{C}_p T^{*m-p}T^{m-p} = 0,$$

where ${}^m\mathscr{C}_p$'s are binomial coefficients [1]. Allen et al. [2–4]; Colonna [6]; Kolaski [10–11], Laitila [12]; Li et al. [13]; Martin et al. [14–16]; Novinger et al. [17]; Zorboska [23] have studied isometric composition operators on various analytic function spaces. Motivated by these we have studied isometric composition operators on the Fock space.

2. Main results

In this section we obtain necessary and sufficient conditions for a composition operator to be an isometry on the Fock space $\mathscr{F}_N^2$. We also find the necessary and sufficient condition for C_ϕ to be a quasi-isometry and m-isometry.

Carswell et al. [5] have proved that if $\phi : \mathbb{C}^N \to \mathbb{C}^N$ is a holomorphic mapping, then C_ϕ is bounded on $\mathscr{F}_N^2$ if and only if $\phi(z) = Az + B$, where A is an $n\times n$ matrix with $\|A\| \leqslant 1$ and B is an $n\times 1$ vector satisfying the condition that $\langle A\xi, B\rangle = 0$ whenever $|A\xi| = |\xi|$ for some ξ in $\mathbb{C}^N$. Also, Carswell et al. [5] have proved the following theorem which is instrumental in obtaining the necessary and sufficient condition for C_ϕ to be an isometry.

THEOREM 2.1. ([5, Theorem 4]) *Suppose $\phi(z) = Az + B$, where either $\|A\| < 1$ and B is arbitrary, or $\|A\| = 1$ and $\langle A\xi, B\rangle = 0$ whenever $|A\xi| = |\xi|$. Then on $\mathscr{F}_N^2$ we have*

$$\|C_\phi\| = \exp\left(\frac{1}{4}(|w_0|^2 - |Aw_0|^2 + |B|^2)\right)$$

where w_0 is any solution to $(I - A^*A)w_0 = A^*B$.

THEOREM 2.2. *Let C_ϕ be a composition operator on $\mathscr{F}_N^2$, where ϕ satisfies the condition of Theorem 2.1. If C_ϕ is an isometry, then $B=0$.*

Before we give the proof of this theorem, we need several preliminary results that will be instrumental in proving Theorem 2.2. First we recall the notion of the singular value decomposition of an $n\times n$ matrix.

THEOREM 2.3. ([8]) *If A is an $n\times n$ matrix of rank k, then A can be written as $A=V\Sigma W$, where V,W are $n\times n$ unitary matrices, and Σ is a diagonal matrix (σ_{ij}) with $\sigma_{11}\geqslant\sigma_{22}\geqslant\cdots\geqslant\sigma_{kk}>\sigma_{k+1,k+1}=\cdots=\sigma_{nn}=0$. The σ_{ii} are the non-negative square roots of the eigenvalues of AA^*, where A^* is the adjoint of A; if we require that they can be listed in decreasing order, then Σ is uniquely determined from A.*

For convenience we shall write σ_i for σ_{ii}, the ith diagonal entry of Σ. Note that if $\|A\|\leqslant 1$, then $\sigma_i\leqslant 1$ for all i and $\sigma_i=1$ for some i if $\|A\|=1$.

Set $j=\max\{r:\sigma_r=1\}$ and $k=\max\{r:\sigma_r>0\}$ so that $k=\operatorname{rank}A$. The singular value decomposition will allow us to perform a normalization.

If $\phi(z)=Az+B$ and $\psi(z)=\Sigma z+B'$, where the singular value decomposition of A is $V\Sigma W$ and $B'=V^*B$ same as given in Theorem 2.3, then we call ψ a normalization of ϕ.

LEMMA 2.4. ([5, Lemma 1]) *Suppose that $\phi(z)=Az+B$ with $\|A\|\leqslant 1$ and $\langle A\xi,B\rangle=0$ whenever $|A\xi|=|\xi|$ and suppose $\psi(z)=\Sigma z+B'$ is a normalization of ϕ. Then the first j coordinates of B' are 0.*

COROLLARY 2.5. ([5, Corollary 1]) *The operator C_ϕ is bounded on $\mathscr{F}_N^2$ if and only if C_ψ is bounded on $\mathscr{F}_N^2$. Moreover, the norm of C_ϕ is equal to the norm of C_ψ.*

Now we shall prove Theorem 2.2.

Proof. By Theorem 2.1

$$\|C_\phi\|_{\mathscr{F}_N^2}=\exp\left(\frac{1}{4}(|w_0|^2-|Aw_0|^2+|B|^2)\right)$$

where w_0 is any solution to $(I-A^*A)w_0=A^*B$.

Since C_ϕ is an isometry, we have $\|C_\phi\|_{\mathscr{F}_N^2}=1$ which implies that

$$\exp\left(\frac{1}{4}(|w_0|^2-|Aw_0|^2+|B|^2)\right)=1.$$

Thus, $|w_0|^2-|Aw_0|^2+|B|^2=0$. Let $\psi(z)=\Sigma z+B'$ be a normalization of ϕ, where Σ and B' are as given in Lemma 2.4. Then $\Sigma=\operatorname{diag}\{\sigma_i\}$ with $\sigma_1=\cdots=\sigma_j=1$ and $\sigma_{j+1},\cdots,\sigma_n<1$ and $B'=(0,\ldots,0,b'_{j+1},b'_{j+2},\ldots,b'_n)^t$. As proved in [5, Theorem 4]

$$|w|^2-|\Sigma w|^2+|B|^2=\sum_{m=j+1}^{n}\frac{|b'_m|^2}{1-\sigma_m^2}$$

where w is any solution to $(I-\Sigma^*\Sigma)w = \Sigma^* B$.

By Corollary 2.5, $\|C_\phi\| = \|C_\psi\|$, and so

$$0 = |w_0|^2 - |Aw_0|^2 + |B|^2 = \sum_{m=j+1}^{n} \frac{|b'_m|^2}{1-\sigma_m^2}.$$

Since $\sigma_m < 1$, therefore, $|b'_m| = 0$ for all $m = j+1,\dots,n$. Therefore, from Lemma 2.4 it follows that

$$B' = (0,0,\dots,0,b'_{j+1},\dots,b'_n)^t = (0,\dots,0)^t_{n\times n}.$$

Thus, $V^*B = 0$, where V is an $n\times n$ unitary matrix. Hence, $B = 0$. □

The following lemma due to Carswell et al. [5] gives the representation of the adjoint of a composition operator in terms of a multiplication operator.

LEMMA 2.6. ([5, Lemma 2]) *If $\phi(z) = Az + B$, where A is an $n\times n$ matrix with $\|A\| \leqslant 1$ and B is an $n\times 1$ vector. If $\langle A\xi, B\rangle = 0$ whenever $|A\xi| = |\xi|$, then $C_\phi^* = M_{k_B} C_\tau$, where $\tau(z) = A^*z$ and M_{k_B} is the multiplication operator with symbol k_B.*

Lemma 2.6 is instrumental with all the hypothesis in all the following results.

THEOREM 2.7. *Let C_ϕ be a composition operator on $\mathscr{F}_N^2$. Then C_ϕ is an isometry if and only if $M_{k_B} C_{\phi\circ\tau} = I$.*

Proof. C_ϕ is an isometry on $\mathscr{F}_N^2$ if and only if $\|C_\phi f\|_{\mathscr{F}_N^2} = \|f\|_{\mathscr{F}_N^2}$ for all $f \in \mathscr{F}_N^2$ which is equivalent to $C_\phi^* C_\phi = I$. By Lemma 2.6, we have $(M_B C_\tau)C_\phi = I$. Hence, $M_{k_B} C_{\phi\circ\tau} = I$. □

We now provide an alternative proof of Theorem 2.2 using the multiplication operator.

Proof. If C_ϕ is an isometry, then by Theorem 2.7 we have $M_{k_B} C_{\phi\circ\tau} = I$ or $(M_{k_B} C_{\phi\circ\tau})f = f$ for all $f \in \mathscr{F}_N^2$. Thus, for all $f \in \mathscr{F}_N^2$ and for all $z \in \mathbb{C}^N$, we have $M_{k_B}(z) f(\phi(\tau(z))) = f(z)$ which implies that

$$k_B(z) f(AA^*z + B) = f(z),$$

since $k_w(z) = e^{\langle z,w\rangle/2}$, we have

$$e^{\langle z,B\rangle/2} f(AA^*z + B) = f(z). \tag{1}$$

Let $f = k_0$ be the point evaluation at 0 on $\mathscr{F}_N^2$. Then

$$f(z) = k_0(z) = e^{\langle z,0\rangle} = 1 \text{ for all } z \in \mathbb{C}^N.$$

From (1), we have

$$e^{\langle z,B\rangle/2} f(AA^*z + B) = f(z) \text{ for all } f \in \mathscr{F}_N^2 \text{ and for all } z \in \mathbb{C}^N.$$

In particular, taking $f = k_0$, we get for all $z \in \mathbb{C}^N$, $e^{\langle z,B\rangle/2} = 1$, and hence, $\langle z,B\rangle = 0$. Therefore, $B = 0$. □

COROLLARY 2.8. *If a composition operator C_ϕ is an isometry on $\mathscr{F}_N^2$, then $\phi(z) = Az$ such that $AA^* = I$.*

Proof. Since C_ϕ is an isometry we have that $M_{k_B}C_{\phi\circ\tau} = I$ and $B = 0$. Now $C_{\phi\circ\tau} = I$ implies that $f(\phi\circ\tau) = f$ for all $f \in \mathscr{F}_N^2$ or $f(\phi(\tau(z)) = f(z)$ for all $f \in \mathscr{F}_N^2$ and $z \in \mathbb{C}^N$. Since $\tau(z) = A^*z$ and $\phi(z) = Az + B$ with $B = 0$, it follows that $f(AA^*(z)) = f(z)$ for all $f \in \mathscr{F}_N^2$ and $z \in \mathbb{C}^N$ or $AA^*z = z$ for all $z \in \mathbb{C}^N$. Hence $AA^* = I$. □

The following results give necessary and sufficient conditions for a composition operator to be a quasi-isometry and m-isometry.

THEOREM 2.9. *A composition operator C_ϕ on Fock-space $\mathscr{F}_N^2$ is quasi-isometry if and only if $M_{k_B\circ\tau}C_{\phi^2\circ\tau^2} = C_{\phi\circ\tau}$.*

Proof. A composition operator C_ϕ is a quasi-isometry on $\mathscr{F}_N^2$ if and only if $C_\phi^{*2}C_\phi^2 = C_\phi^*C_\phi$ which can be re-written as $C_\phi^*(C_\phi^*C_\phi)C_\phi = C_\phi^*C_\phi$.

By Lemma 2.6, we obtain

$$C_\phi^*[(M_{k_B}C_\tau)C_\phi]C_\phi = (M_{k_B}C_\tau)C_\phi .$$

Since $C_\tau \circ C_\phi = C_{\phi\circ\tau}$, we have

$$C_\phi^*(M_{k_B}C_{\phi\circ\tau})C_\phi = M_{k_B}C_{\phi\circ\tau}$$

which is equivalent to $C_\phi^*(M_{k_B}C_{\phi^2\circ\tau}) = M_{k_B}C_{\phi\circ\tau}$. Again by using Lemma 2.6, we have

$$(M_{k_B}C_\tau)(M_{k_B}C_{\phi^2\circ\tau}) = M_{k_B}C_{\phi\circ\tau}.$$

Hence,

$$M_{k_B}(C_\tau M_{k_B})C_{\phi^2\circ\tau} = M_{k_B}C_{\phi\circ\tau}.$$

Since $C_\tau M_{k_B} = M_{k_B\circ\tau}C_\tau$, therefore,

$$M_{k_B}M_{k_B\circ\tau}C_\tau C_{\phi^2\circ\tau} = M_{k_B}C_{\phi\circ\tau} \quad \text{or} \quad M_{k_B}M_{k_B\circ\tau}C_{\phi^2\circ\tau^2} = M_{k_B}C_{\phi\circ\tau}.$$

Hence, $M_{k_B\circ\tau}C_{\phi^2\circ\tau^2} = C_{\phi\circ\tau}$. □

The following theorem can be obtained on the similar lines of the proof of Theorem 2.9.

THEOREM 2.10. *A composition operator on Fock space $\mathscr{F}_N^2$ is an m-isometry if and only if*

$$\sum_{p=0}^{m}(-1)^p\, {}^m\mathscr{C}_p(M_{k_B}M_{k_B\circ\tau}\cdots M_{k_B\circ\tau^{m-p-1}}C_{\phi^{m-p}\tau^{m-p}}) = 0,$$

where ${}^m\mathscr{C}_p$'s are binomial coefficients.

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