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REMARKS ON "WEAK LIMITS OF ALMOST INVARIANT PROJECTIONS" BY FOIAS, PASNICU AND VOICULESCU

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Abstract. Ultraproducts of operators are used to give simpler proofs of certain results in the paper "Weak limits of almost invariant projections" by Foias, Pasnicu and Voiculescu.

1. Introduction

Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space. The algebra of bounded linear operators on $\mathscr{H}$ is denoted by $\mathscr{B}(\mathscr{H})$, and the ideal of compact operators in $\mathscr{B}(\mathscr{H})$ is denoted by $\mathscr{K}(\mathscr{H})$. Let p be the quotient map from $\mathscr{B}(\mathscr{H})$ onto $\mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$.

In [3], Foias, Pasnicu and Voiculescu established the following characterizations of an operator Q being the weak limit of projections that are almost invariant under an algebra $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$.

THEOREM 1.1. *Let $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$ be a norm-separable norm closed algebra containing I, and $Q \in \mathscr{B}(\mathscr{H})$, $0 \leqslant Q \leqslant I$. Then the following statements are equivalent.*

(i) There exists a sequence $(P_n)_{n=1}^{\infty}$ of projections in $\mathscr{B}(\mathscr{H})$ such that $\lim_{n\to\infty} \|(I-P_n)TP_n\| = 0$ for all $T \in \mathscr{A}$ and $w\lim_{n\to\infty} P_n = Q$.

(ii) There exists a sequence $(R_n)_{n=1}^{\infty}$ of projections in $\mathscr{B}(\mathscr{H})$ such that $w\lim_{n\to\infty}(I-R_n)TR_n = 0$ for all $T \in \mathscr{A}$ and $w\lim_{n\to\infty} R_n = Q$.

(iii) There exists a representation ρ of $p(C^*(\mathscr{A}))$ on some separable Hilbert space $\mathscr{H}'$ and a subspace $L \subset \mathscr{H} \oplus \mathscr{H}'$ invariant under $(\mathrm{id} \oplus (\rho \circ p))(\mathscr{A})$ such that

$$P_{\mathscr{H}\oplus 0} P_L|_{\mathscr{H}\oplus 0} = Q.$$

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Note that in both (i) and (ii), the projections P_n and R_n are almost invariant under the algbera $\mathscr{A}$, whereas in (iii), L is an (exactly) invariant under the algebra $(\mathrm{id} \oplus (\rho \circ p))(\mathscr{A})$ instead.

In the same paper, they obtain as a consequence the following characterization of strong reductivity.

THEOREM 1.2. *Let $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$ be a norm-separable commutative algebra containing I. The following properties are equivalent:*

(i) $\mathscr{A}$ is strongly reductive,

(ii) the norm-closure of $\mathscr{A}$ is a C^*-algebra,

(iii) for every representation ρ of $\mathscr{A}$ in the norm-closed unitary orbit of the identity representation of $\mathscr{A}$ on $\mathscr{H}$, the algebra $\rho(\mathscr{A})$ is reductive.

Note that in Theorem 1.2, (ii)$\Rightarrow$(i) is obvious, and (i)$\Rightarrow$(iii) is simple and elementary but slightly technical (see [3, page 92]). The main part of Theorem 1.2 is (iii)$\Rightarrow$(ii). They ask whether there is a simple, direct proof of (iii)$\Rightarrow$(i). The purpose of this paper is to provide such a proof as well as an alternative proof of the nontrivial implication (ii)$\Rightarrow$(iii) in Theorem 1.1.

In Section 2, we recall some definitions, a construction of Calkin and Voiculescu's noncommutative Weyl-von Neumann Theorem which are needed in the rest of this paper. In Section 3, we give a direct proof of (iii)$\Rightarrow$(i) in Theorem 1.2. In Section 4, we give an alternative proof of (ii)$\Rightarrow$(iii) in Theorem 1.1.

2. Prelimiaries

An algebra $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$ is *reductive* if every subspace of $\mathscr{H}$ invariant under $\mathscr{A}$ reduces $\mathscr{A}$; $\mathscr{A}$ is *strongly reductive* (see [4] and [1]) if for every sequence $(P_n)_{n=1}^{\infty}$ of projections in $\mathscr{B}(\mathscr{H})$ satisfying

$$\lim_{n\to\infty} \|(I-P_n)TP_n\| = 0, \quad T \in \mathscr{A},$$

we have

$$\lim_{n\to\infty} \|TP_n - P_nT\| = 0, \quad T \in \mathscr{A}.$$

Let $\psi_1, \psi_2 : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ be two representations of an algebra $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$. We say that ψ_2 *is in the norm-closed unitary orbit of* ψ_1, if there exists a sequence $(U_n)_{n=1}^{\infty}$ of unitary operators such that:

$$\lim_{n\to\infty} \|\psi_2(T) - U_n\psi_1(T)U_n^{-1}\| = 0,$$

for all $T \in \mathscr{A}$.

Let $\mathscr{U}$ be a free ultrafilter on $\mathbb{N}$. If $(a_n)_{n\geqslant 1}$ is a bounded sequence in $\mathbb{C}$, then its ultralimit through $\mathscr{U}$ is denoted by $\lim\limits_{n,\mathscr{U}} a_n$. Consider the Banach space

$$\mathscr{H}^{\mathscr{U}} := \ell^{\infty}(\mathscr{H}) \Big/ \left\{ (x_n)_{n\in\mathbb{N}} \in \ell^{\infty}(\mathscr{H}) : \lim_{n,\mathscr{U}} \|x_n\| = 0 \right\}.$$

If $(x_n)_{n\mathbb{N}} \in \ell^\infty(\mathscr{H})$ then its image in $\mathscr{H}^{\mathscr{U}}$ is denoted by $(x_n)_{\mathscr{U}}$, and it can be easily checked that

$$\|(x_n)_{\mathscr{U}}\| = \lim_{n,\mathscr{U}} \|x_n\|.$$

Moreover, $\mathscr{H}^{\mathscr{U}}$ is, in fact, a Hilbert space with inner product

$$\langle (x_n)_{\mathscr{U}}, (y_n)_{\mathscr{U}} \rangle = \lim_{n,\mathscr{U}} \langle x_n, y_n \rangle.$$

But $\mathscr{H}^{\mathscr{U}}$ is nonseparable.

If $(T_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathscr{B}(\mathscr{H})$, then its *ultraproduct* $(T_1, T_2, \ldots)_{\mathscr{U}} \in \mathscr{B}(\mathscr{H}^{\mathscr{U}})$ is defined by $(x_n)_{\mathscr{U}} \mapsto (T_n x_n)_{\mathscr{U}}$. If $T \in \mathscr{B}(\mathscr{H})$ then its *ultrapower* $T^{\mathscr{U}} \in \mathscr{B}(\mathscr{H}^{\mathscr{U}})$ is defined by $(x_n)_{\mathscr{U}} \mapsto (T x_n)_{\mathscr{U}}$. It is easy to see that

$$\|(T_1, T_2, \ldots)_{\mathscr{U}}\| = \lim_{n,\mathscr{U}} \|T_n\|,$$

$$(T_1, T_2, \ldots)^*_{\mathscr{U}} = (T_1^*, T_2^*, \ldots)_{\mathscr{U}},$$

and in particular, $(T^{\mathscr{U}})^* = (T^*)^{\mathscr{U}}$.

Consider the subspace

$$\widehat{\mathscr{H}} := \left\{ (x_n)_{\mathscr{U}} \in \mathscr{H}^{\mathscr{U}} : w \lim_{n,\mathscr{U}} x_n = 0 \right\}.$$

Here $w \lim_{n,\mathscr{U}} x_n$ is the weak limit of $(x_n)_{n\in\mathbb{N}}$ through $\mathscr{U}$, i.e., the unique element $x \in \mathscr{H}$ such that

$$\langle x, y \rangle = \lim_{n,\mathscr{U}} \langle x_n, y \rangle, \quad y \in \mathscr{H}. \tag{2.1}$$

Consider also the (closed) subspace $\{(x)_{\mathscr{U}} = (x, x, \ldots)_{\mathscr{U}} : x \in \mathscr{H}\}$ of $\mathscr{H}^{\mathscr{U}}$. The projection from $\mathscr{H}^{\mathscr{U}}$ onto this subspace is given by $(x_n)_{\mathscr{U}} \mapsto (w \lim_{k,\mathscr{U}} x_k)_{\mathscr{U}}$, and so $\{(x)_{\mathscr{U}} : x \in \mathscr{H}\}^\perp = \widehat{\mathscr{H}}$. We shall identify $\{(x)_{\mathscr{U}} : x \in \mathscr{H}\}$ with $\mathscr{H}$. So we have $\mathscr{H}^{\mathscr{U}} = \mathscr{H} \oplus \widehat{\mathscr{H}}$.

For $T \in \mathscr{B}(\mathscr{H})$, $\widehat{\mathscr{H}}$ is a reducing subspace for $T^{\mathscr{U}}$ and thus we can define $\widehat{T} \in \mathscr{B}(\widehat{\mathscr{H}})$ by

$$\widehat{T} := T^{\mathscr{U}}|_{\widehat{\mathscr{H}}}.$$

Hence we have

$$T^{\mathscr{U}} = T \oplus \widehat{T} \tag{2.2}$$

with respect to the decomposition $\mathscr{H}^{\mathscr{U}} = \mathscr{H} \oplus \widehat{\mathscr{H}}$.

Note that $\widehat{K} = 0$ for $K \in \mathscr{K}(\mathscr{H})$. (The proof of this uses the topological definition of weak ultralimit rather than (2.1) above and uses also the fact that every sequence in a compact Hausdorff space converges to an element through $\mathscr{U}$.) Throughout this paper, the map $f : \mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H}) \to \mathscr{B}(\widehat{\mathscr{H}})$ is defined by $f(p(T)) = \widehat{T}$.

THEOREM 2.1. ([2], Theorem 5.5) *The map f is an isometric $*$-isomorphism into $\mathscr{B}(\widehat{\mathscr{H}})$.*

Let us recall the definition of approximate unitary equivalence of representations and a result of Voiculescu.

Let $\psi_1, \psi_2 : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ be two representations of an algebra $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$. Then ψ_1 and ψ_2 are *approximately unitarily equivalent* [6], denoted by $\psi_1 \sim_a \psi_2$, if there is a sequence $(U_n)_{n=1}^{\infty}$ of unitary operators such that

$$\psi_2(T) - U_n \psi_1(T) U_n^{-1} \in \mathscr{K}(\mathscr{H}), \quad n \geqslant 1,$$

and

$$\lim_{n\to\infty} \|\psi_2(T) - U_n \psi_1(T) U_n^{-1}\| = 0$$

for all $T \in \mathscr{A}$. Note that if $\psi_1 \sim_a \psi_2$ then ψ_2 is in the norm-closed unitary orbit of ψ_1.

THEOREM 2.2. ([6], Theorem 1.3) *Let $\mathscr{A}$ be a separable C^*-algebra with unit and ρ a representation of $\mathscr{A}$ on $\mathscr{H}$. Let π be a representation of $p(\rho(\mathscr{A}))$ on a separable Hilbert space $\mathscr{H}_\pi$. Then $\rho \sim_a \rho \oplus \pi \circ p \circ \rho$.*

Suppose now that $\mathscr{A} \subset \mathscr{B}(\mathscr{H})$. Take ρ to be the identity representation id of $\mathscr{A}$ on $\mathscr{H}$. If $\mathscr{M}$ is a separable subspace of $\widehat{\mathscr{H}}$ that reduces $(f \circ p)(\mathscr{A})$, then we define a representation $f_{\mathscr{M}}$ of $p(\mathscr{A})$ on $\mathscr{M}$ by $f_{\mathscr{M}}(p(S)) = \widehat{S}|_{\mathscr{M}}$. Taking π to be this representation in Theorem 2.2 with $\mathscr{H}_\pi = \mathscr{M}$, we obtain

COROLLARY 2.3. *Let $\mathscr{A}$ be a separable C^*-subalgebra of $\mathscr{B}(\mathscr{H})$ containing I. Let $\mathscr{M}$ be a separable subspace of $\widehat{\mathscr{H}}$ that reduces $(f \circ p)(\mathscr{A})$. Then $\mathrm{id} \sim_a \mathrm{id} \oplus (f_{\mathscr{M}} \circ p \circ \mathrm{id})$.*

3. Proof of (iii)$\Rightarrow$(i) in Theorem 1.2

PROPOSITION 3.1. *If (iii) in Theorem 1.2 holds then the algebra $\{T^{\mathscr{U}} : T \in \mathscr{A}\}$ in $\mathscr{B}(\mathscr{H}^{\mathscr{U}})$ is reductive.*

Proof. By Corollary 2.3, for every separable reducing subspace $\mathscr{M}$ of $\widehat{H}$ that reduces $(f \circ p)(\mathscr{A})$, we have $\mathrm{id} \sim_a \mathrm{id} \oplus (f_{\mathscr{M}} \circ p \circ \mathrm{id})$, and so by assumption,

$$(\mathrm{id} \oplus (f_{\mathscr{M}} \circ p \circ \mathrm{id}))(\mathscr{A}) = \{T \oplus [f(p(T))|_{\mathscr{M}}] : T \in \mathscr{A}\}$$

is reductive. But $T^{\mathscr{U}}|_{\mathscr{H} \oplus \mathscr{M}} = T \oplus (\widehat{T}|_{\mathscr{M}}) = T \oplus [f(p(T))|_{\mathscr{M}}]$. Therefore, $\{T^{\mathscr{U}}|_{\mathscr{H} \oplus \mathscr{M}} : T \in \mathscr{A}\}$ is reductive.

For every separable subspace $\mathscr{N}$ of $\mathscr{H}^{\mathscr{U}}$, there is a separable reducing subspace $\mathscr{M}$ for $(f \circ p)(\mathscr{A})$ such that $\mathscr{N} \subset \mathscr{H} \oplus \mathscr{M}$. (Take, for example, $\mathscr{M}$ to be the smallest subspace of $\widehat{\mathscr{H}}$ that contains $P_{\widehat{H}}\mathscr{N}$ and reduces $(f \circ p)(\mathscr{A})$.) Thus, if $\mathscr{N}$ is invariant under $\{T^{\mathscr{U}} : T \in \mathscr{A}\}$, then $\mathscr{N}$ is invariant under $\{T^{\mathscr{U}}|_{\mathscr{H} \oplus \mathscr{M}} : T \in \mathscr{A}\}$. Since $\{T^{\mathscr{U}}|_{\mathscr{H} \oplus \mathscr{M}} : T \in \mathscr{A}\}$ is reductive, this implies that $\mathscr{N}$ reduces $\{T^{\mathscr{U}}|_{\mathscr{H} \oplus \mathscr{M}} : T \in \mathscr{A}\}$ and thus reduces $\{T^{\mathscr{U}} : T \in \mathscr{A}\}$. Therefore, every separable subspace of $\mathscr{H}^{\mathscr{U}}$ that is invariant under $\{T^{\mathscr{U}} : T \in \mathscr{A}\}$ reduces $\{T^{\mathscr{U}} : T \in \mathscr{A}\}$.

Suppose now that $\mathscr{N}$ is a subspace of $\mathscr{H}^{\mathscr{U}}$ invariant under $\{T^{\mathscr{U}} : T \in \mathscr{A}\}$ but $\mathscr{N}$ is not necessarily separable. Let $z \in \mathscr{N}$. Then $\vee\{T^{\mathscr{U}} z : T \in \mathscr{A}\}$ is a separable subspace of $\mathscr{H}^{\mathscr{U}}$ that is invariant under $\{T^{\mathscr{U}} : T \in \mathscr{A}\}$. So by the conclusion of the previous paragraph, $\vee\{T^{\mathscr{U}} z : T \in \mathscr{A}\}$ reduces $\{T^{\mathscr{U}} : T \in \mathscr{A}\}$. Thus, $(T^{\mathscr{U}})^* z \in \vee\{T^{\mathscr{U}} z : T \in \mathscr{A}\}$ for all $T \in \mathscr{A}$. Since $\mathscr{N}$ is invariant under $\{T^{\mathscr{U}} : T \in \mathscr{A}\}$, this implies that $(T^{\mathscr{U}})^* z \in \mathscr{N}$ for all $T \in \mathscr{A}$ and $z \in \mathscr{N}$. Therefore, $\mathscr{N}$ reduces $\{T^{\mathscr{U}} : T \in \mathscr{A}\}$. It follows that $\{T^{\mathscr{U}} : T \in \mathscr{A}\}$ is reductive. $\square$

We are now ready to complete the proof of (iii)$\Rightarrow$(i) in Theorem 1.2. Suppose that (iii) is true and (i) is not true. Then there exist $\varepsilon > 0$, $T_0 \in \mathscr{A}$ and a sequence $(P_n)_{n\geqslant}$ of projections in $\mathscr{B}(\mathscr{H})$ such that $\lim_{n\to\infty} \|(I-P_n)TP_n\| = 0$ for all $T \in \mathscr{A}$ but $\|T_0P_n - P_nT_0\| \geqslant \varepsilon$ for all $n \in \mathbb{N}$.

Note that $(P_1, P_2, \ldots)_{\mathscr{U}}$ is a projection in $\mathscr{B}(\mathscr{H}^{\mathscr{U}})$ and

$$(I - (P_1, P_2, \ldots)_{\mathscr{U}}) T^{\mathscr{U}} (P_1, P_2, \ldots)_{\mathscr{U}} = ((I-P_1)TP_1, (I-P_2)TP_2, \ldots)_{\mathscr{U}} = 0$$

for all $T \in \mathscr{A}$. So by Proposition 3.1, $T^{\mathscr{U}}(P_1, P_2, \ldots)_{\mathscr{U}} = (P_1, P_2, \ldots)_{\mathscr{U}} T^{\mathscr{U}}$ for all $T \in \mathscr{A}$. This means that

$$\lim_{n,\mathscr{U}} \|TP_n - P_nT\| = 0, \quad T \in \mathscr{A}.$$

But $\|T_0P_n - P_nT_0\| \geqslant \varepsilon$ for all $n \geqslant 1$ which is a contradiction. Therefore, (iii)$\Rightarrow$(i).

4. Proof of (ii)$\Rightarrow$(iii) in Theorem 1.1

PROPOSITION 4.1. *Let $\mathscr{A}$ be a norm-separable algebra containing I and let $Q \in \mathscr{B}(\mathscr{H})$. If there exists a bounded sequence $(R_n)_{n=1}^\infty$ in $\mathscr{B}(\mathscr{H})$ such that $w\lim_{n\to\infty}(I - R_n^*)TR_n = 0$ for all $T \in \mathscr{A}$ and $w\lim_{n\to\infty} R_n = Q$, then there is a separable subspace L of $\mathscr{H}^{\mathscr{U}}$ invariant under $\{T^{\mathscr{U}} : T \in \mathscr{A}\}$ such that*

$$P_{\mathscr{H}\oplus 0} P_L|_{\mathscr{H}\oplus 0} = Q.$$

Proof. Take

$$L = \vee\{(TR_ny)_{\mathscr{U}} : T \in \mathscr{A}, y \in \mathscr{H}\}.$$

Then L is a separable subspace of $\mathscr{H}^{\mathscr{U}}$ that is invariant under $T^{\mathscr{U}}$ for every $T \in \mathscr{A}$. It remains to show that

$$P_{\mathscr{H}\oplus 0} P_L|_{\mathscr{H}\oplus 0} = Q.$$

For every $x, y \in \mathscr{H}$,

$$\begin{aligned}\langle (x)_{\mathscr{U}} - (R_nx)_{\mathscr{U}}, (TR_ny)_{\mathscr{U}} \rangle &= \langle ((I-R_n)x)_{\mathscr{U}}, (TR_ny)_{\mathscr{U}} \rangle \\ &= \lim_{n,\mathscr{U}} \langle (I-R_n)x, TR_ny \rangle \\ &= \lim_{n,\mathscr{U}} \langle x, (I-R_n^*)TR_ny \rangle \\ &= 0 \quad \text{by assumption.}\end{aligned}$$

Thus, $((x)_{\mathscr{U}} - (R_n x)_{\mathscr{U}}) \perp L$ for every $x \in \mathscr{H}$. But $(R_n x)_{\mathscr{U}} \in L$. Therefore, by the definition of orthogonal projection onto L,

$$P_L(x)_{\mathscr{U}} = (R_n x)_{\mathscr{U}}.$$

Taking $P_{\mathscr{H}\oplus 0}$ on both sides, we obtain

$$P_{\mathscr{H}\oplus 0} P_L(x)_{\mathscr{U}} = P_{\mathscr{H}\oplus 0}(R_n x)_{\mathscr{U}} = w \lim_{n,\mathscr{U}} R_n x = Qx. \quad \square$$

We are now ready to complete the proof of (ii)$\Rightarrow$(iii) in Theorem 1.1.

Assume (ii). Applying Proposition 4.1, we obtain a separable subspace L of $\mathscr{H}^{\mathscr{U}}$ invariant under $\{T^{\mathscr{U}} : T \in \mathscr{A}\}$ such that

$$P_{\mathscr{H}\oplus 0} P_L|_{\mathscr{H}\oplus 0} = Q.$$

By (2.2),

$$T^{\mathscr{U}} = T \oplus \widehat{T} = T \oplus f(p(T)) = (\mathrm{id} \oplus (f \circ p))(T).$$

Take $\mathscr{H}'$ to be the smallest subspace of $\widehat{\mathscr{H}}$ that contains $P_{\widehat{\mathscr{H}}}L$ and reduces $(f \circ p)(C^*(\mathscr{A}))$. Note that $\mathscr{H}'$ is separable. Take ρ to be $S \mapsto f(S)|_{\mathscr{H}'}$ for $S \in p(C^*(\mathscr{A}))$. We obtain (iii).

REMARK. Since the assumption of Proposition 4.1 is slightly weaker than (ii) in Theorem 1.1, we have the following slight improvement of Theorem 1.1.

THEOREM 4.2. *Let $\mathscr{A}$ be a norm-separable norm closed algebra containing I, and $Q \in \mathscr{B}(\mathscr{H})$, $0 \leqslant Q \leqslant I$. Then the following statements are equivalent.*

(i) There exists a sequence $(P_n)_{n=1}^{\infty}$ of projections in $\mathscr{B}(\mathscr{H})$ such that $\lim_{n\to\infty} \|(I-P_n)TP_n\| = 0$ for all $T \in \mathscr{A}$ and $w \lim_{n\to\infty} P_n = Q$.

(ii) There exists a bounded sequence $(R_n)_{n=1}^{\infty}$ in $\mathscr{B}(\mathscr{H})$ such that $w \lim_{n\to\infty} (I-R_n^*)TR_n = 0$ for all $T \in \mathscr{A}$ and $w \lim_{n\to\infty} R_n = Q$.

(iii) There exists a representation ρ of $p(C^*(\mathscr{A}))$ on some separable Hilbert space $\mathscr{H}'$ and a subspace $L \subset \mathscr{H} \oplus \mathscr{H}'$ invariant under $(\mathrm{id} \oplus (\rho \circ p))(\mathscr{A})$ such that

$$P_{\mathscr{H}\oplus 0} P_L|_{\mathscr{H}\oplus 0} = Q.$$

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