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DENSITY OF A NORMAL SUBGROUP OF THE INVERTIBLES IN CERTAIN MULTIPLIER ALGEBRAS

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Abstract. Let $\mathscr{A}$ be a unital separable simple C*-algebra. Let $GL(\mathscr{M}(\mathscr{A}\otimes\mathscr{K}))$ be the group of invertible elements of the multiplier algebra of the stabilization of $\mathscr{A}$, and let $N \subseteq GL(\mathscr{M}(\mathscr{A}\otimes\mathscr{K}))$ be any (algebraic) normal subgroup that properly contains the scalar invertibles.

Then

$$\overline{N}^{\text{strict}} = \mathscr{M}(\mathscr{A}\otimes\mathscr{K}),$$

where $\overline{N}^{\text{strict}}$ is the closure of N in the strict topology.

1. Introduction

Let $\mathscr{C}$ be a unital C*-algebra and let $\mathscr{L} \subseteq \mathscr{C}$ be a (not necessarily closed) linear subspace. Recall that $\mathscr{L}$ is a *Lie ideal* of $\mathscr{C}$ if $[\mathscr{L},\mathscr{C}] \subseteq \mathscr{L}$.[1]

There is an extensive literature studying the Lie ideals of operator algebras, and this has had many consequences for operator theory and operator algebras. (See, for example, [7], [13], [28], [29], [30], [31], and the references therein.) Among other things, if $\mathscr{H}$ is a separable Hilbert space and $\mathbb{B}(\mathscr{H})$ is the algebra of bounded linear operators on $\mathscr{H}$, then a linear subspace $\mathscr{L} \subseteq \mathbb{B}(\mathscr{H})$ is a Lie ideal if and only if $\mathscr{L}$ is invariant under conjugation by unitaries if and only if $\mathscr{L}$ is invariant under conjugation by invertibles. (See [7, Theorem 1]; see also [12], [13], [14], [15], [29], [30], [31], [38].)

A basic question in the subject is to ask under what conditions is a Lie ideal "large" (e.g., contains all additive commutators or even the whole algebra). From fundamental results of the theory, one gets the following immediate consequence: If $\mathscr{H}$ is a separable Hilbert space and $\mathscr{L} \subseteq \mathbb{B}(\mathscr{H})$ is a Lie ideal which properly contains the scalars, then $\overline{\mathscr{L}}^{\text{strong}} = \mathbb{B}(\mathscr{H})$, where $\overline{\mathscr{L}}^{\text{strong}}$ is the closure of $\mathscr{L}$ in the strong operator topology. (See, for example, [7]; see also [12], [28], [29], [30].)

In this paper, we study a nonlinear, multiplicative version of the above result, where we also replace $\mathbb{B}(\mathscr{H})$ with more general multiplier algebras (and replace the

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[1]For subsets $E,E' \in \mathscr{C}$, $[E,E'] = \mathrm{Span}\{[e,e'] := ee' - e'e : e \in E, e' \in E'\}$, all finite sums of additive commutators.

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strong operator topology with the strict topology). Our focus will be on properly infinite multiplier algebras.

In the multiplicative analogue, linear subspaces will be replaced with subgroups of the invertible group (or general linear group) of a unital C*-algebra. Lie ideals will be replaced with normal subgroups.

The phenomenon that we are interested in has, to a certain extent, been studied in the form of topological groups associated with operator algebras. (We will not extensively discuss topological groups, though we may mention them.) We are mainly interested in the general linear group (or group of invertibles) of a unital C*-algebra. The most basic example of this is the full matrix algebra $\mathbb{M}_n(\mathbb{C})$. In this case, the projective general linear group $GL(\mathbb{M}_n(\mathbb{C}))/(\mathbb{C}-\{0\})$ is an algebraically simple group. Moreover, since $GL(\mathbb{M}_n)$ is norm-dense in $\mathbb{M}_n$, we immediately have the required multiplicative analogue: If $N \subseteq GL(\mathbb{M}_n)$ is an algebraic normal subgroup which properly contains the scalar invertibles then $N = GL(\mathbb{M}_n)$ and hence $\overline{N}^{\|\cdot\|} = \mathbb{M}_n$, where $\overline{N}^{\|\cdot\|}$ is the closure of N in the (operator) norm topology.

The first infinite dimensional generalizations were due to Kadison who studied the case of von Neumann factors ([18], [19], [20]), and these results have been extended to simple C*-algebras with implications to the study of automorphism groups and other areas (e.g., see [6], [33], [34], [35]). In the case of present interest for us, Kadison's result, stated in our language, implies the following: Let $\mathscr{M}$ be a properly infinite von Neumann factor[2], and let $N \subseteq GL(\mathscr{M})$ be a (algebraic) normal subgroup which properly contains the scalar invertibles, then $\overline{N}^{\text{strong}} = \mathscr{M}$. (We note that Kadison was interested in the structure of the topological group $GL(\mathscr{M})$ with the norm topology.)

In this paper, we will focus on the case of properly infinite multiplier algebras. The *multiplier algebra* $\mathscr{M}(\mathscr{B})$ of a C*-algebra $\mathscr{B}$ is the largest unital C*-algebra containing $\mathscr{B}$ as an essential ideal. The structure of various subgroups of $GL(\mathscr{M}(\mathscr{B}))$ have had important implications for extension theory and K-theory, and they are in themselves quite interesting (e.g., [17], [24], [25], [32], [36], [37], [41]).

In this paper, we prove the natural analogue of the above density results, when the relevant unital C*-algebra is the multiplier algebra of $\mathscr{A} \otimes \mathscr{K}$, where $\mathscr{A}$ is unital separable and simple, and where the relevant topology is the strict topology.

The proof techniques involve a combination of methods from algebra and operator theory, as well as the theory of absorbing extensions.

2. Preliminaries

In this section, we provide some preliminaries and notations, including some definitions and basic results about purely large extensions.

Recall that for a C*-algebra $\mathscr{B}$, the *multiplier algebra* $\mathscr{M}(\mathscr{B})$, of $\mathscr{B}$, is the largest unital C*-algebra containing $\mathscr{B}$ as an essential ideal. This is an object which encodes the extension theory of $\mathscr{B}$. The quotient $\mathscr{M}(\mathscr{B})/\mathscr{B}$ is called the *corona algebra* of $\mathscr{B}$. E.g., if $\mathscr{K}$ is the algebra of compact operators on a separable infinite dimensional

[2] In this paper, we assume that all our simple C*-algebras are separable and all our von Neumann algebras have separable preduals.

Hilbert space $\mathscr{H}$, then $\mathscr{M}(\mathscr{K}) = \mathbb{B}(\mathscr{H})$ and $\mathscr{M}(\mathscr{K})/\mathscr{K}$ is the Calkin algebra. Basic information about multiplier algebras, corona algebras, and their relationship with extension theory can be found in [2], [40] and the references therein.

For a C*-algebra $\mathscr{B}$, the *strict topology* on $\mathscr{M}(\mathscr{B})$ is the topology on $\mathscr{M}(\mathscr{B})$ that is generated by the family of seminorms $\{\|.\|_b\}_{b\in\mathscr{B}}$, where for all $T \in \mathscr{M}(\mathscr{B})$, for all $b \in \mathscr{B}$, $\|T\|_b := \|Tb\| + \|bT\|$. The strict topology on $\mathscr{M}(\mathscr{B})$ roughly plays a role analogous to the strong* topology on a von Neumann algebra. (See [2], [40].) We note that $\mathscr{M}(\mathscr{K})$, with the strict topology, is not a topological algebra, since multiplication is not strictly continuous.

For a C*-algebra $\mathscr{C}$ and for $S \subseteq \mathscr{C}$, we let $\overline{S}^{\|\cdot\|}$ denote the closure of S in the (C*-) norm topology on $\mathscr{C}$. Often, for simplicity, we write $\overline{S}$ in place of $\overline{S}^{\|\cdot\|}$.

Let $\mathscr{B}$ be a C*-algebra and let $R \subseteq \mathscr{M}(\mathscr{B})$. We let $\overline{R}^{\text{strict}}$ denote the closure of R in the strict topology on $\mathscr{M}(\mathscr{B})$.

For a C*-algebra $\mathscr{C}$ and an element $x \in \mathscr{C}$, $\sigma(x)$ denotes the spectrum of x. For a subset $S \subseteq \mathscr{C}$, $\text{Ideal}(S)$ denotes the C*-ideal of $\mathscr{C}$ generated by S; in particular, $\text{Ideal}(x)$ denotes the C*-ideal of $\mathscr{C}$ generated by the element x. If $a \in \mathscr{C}_+$, then $\text{Her}(a)$ denotes the the hereditary C*-subalgebra of $\mathscr{C}$ which is generated by a, i.e., $\text{Her}(a) := \overline{a\mathscr{C}a}$.

In the context of multiplier algebras, there will be special notation for certain hereditary C*-subalgebras. Let $\mathscr{B}$ be a nonunital C*-algebra and $A \in \mathscr{M}(\mathscr{B})_+$. $\text{Her}(A)$, as before, denotes the hereditary C*-subalgebra of $\mathscr{M}(\mathscr{B})$ generated by A, i.e., $\text{Her}(A) := \overline{A\mathscr{M}(\mathscr{B})A}$. On the other hand, $\text{her}(A)$ denotes the hereditary C*-subalgebra of $\mathscr{B}$ generated by A, i.e., $\text{her}(A) := \overline{A\mathscr{B}A}$.

Finally, let $\mathscr{A}, \mathscr{C}$ be C*-algebras with $\mathscr{C}$ unital. Let $\phi, \psi : \mathscr{A} \to \mathscr{C}$ be *-homomorphisms. Recall that ϕ and ψ are said to be *approximately unitarily equivalent* if there exists a sequence $\{u_n\}_{n=1}^{\infty}$ of unitaries in $\mathscr{C}$ such that for all $a \in \mathscr{A}$, $u_n\phi(a)u_n^* \to \psi(a)$ in the norm topology.

We will also need some techniques from extension theory, especially the theory of absorbing extensions which is a theory with connections to operator theory, extension theory, KK theory, the Elliott classification program and other subjects. We gave some thought on the best exposition of these results, given that the foundational proofs from the literature are phrased in the language of extension theory, but we nonetheless wish to give a short presentation which can be followed quickly by a nonexpert. We decided that to keep the exposition efficient, we will phrase all the relevant results in this paper in terms of *-homomorphisms into a multiplier algebra (so, in principle, the reader need not know extension theory). However, we will also make comments and give references connecting these concepts to extension theory and we will use the term "extension" and other terms without definition in these comments – we will single out almost all of these comments by placing them into statements labelled with "Remark". Good references for basic extension theory are [2], [16], [26] and [40]. (See also [3], [4] and [11].) In fact, a fast summary for the relevant basic extension theory can be found on page 386 of [5]. (A good starting point may be for a nonexpert to read this page while consulting the references cited above.) Good references for the theory of absorbing extensions are [5] and [27]. Section 15.12 of [2] gives a fast two page introduction. (See also [1], [2],

[4], and [26].)

DEFINITION 2.1. Let $\mathscr{A}$ and $\mathscr{B}$ be separable C*-algebras with $\mathscr{A}$ unital and $\mathscr{B}$ stable.

1. A unital injective *-homomorphism $\phi : \mathscr{A} \to \mathscr{M}(\mathscr{B})$ is *absorbing* if for every unital *-homomorphism $\psi : \mathscr{A} \to \mathscr{M}(\mathscr{B})$, for every $\varepsilon > 0$, for all isometries $S_1, S_2 \in \mathscr{M}(\mathscr{B})$ with $S_1S_1^* + S_2S_2^* = 1$ and for every finite subset $\mathscr{F} \subseteq \mathscr{A}$, there exists a unitary $U \in \mathscr{M}(\mathscr{B})$ such that
$$\phi(a) - U(S_1\phi(a)S_1^* + S_2\psi(a)S_2^*)U^* \in \mathscr{B}$$
for all $a \in \mathscr{A}$, and
$$\|\phi(a') - U(S_1\phi(a')S_1^* + S_2\psi(a')S_2^*)U^*\| < \varepsilon$$
for all $a' \in \mathscr{F}$.

2. A positive invertible element $a \in \mathscr{M}(\mathscr{B})$ is *absorbing* if the inclusion map $C^*(a) \hookrightarrow \mathscr{M}(\mathscr{B})$ is absorbing.

REMARK 2.1. In the terminology of extension theory, in Definition 2.1, we are working with the *Busby invariants* of trivial extensions, and the sum $S_1\phi(.)S_1^* + S_2\psi(.)S_2^*$ is a realization of the *Brown–Douglas–Fillmore sum* of the extensions ϕ and ψ. Also, ϕ is a *unital absorbing extension*. See, for example, [2], [5], [16], [26], [40].

The following useful approximate uniqueness result follows immediately from the definition of absorbing *-homomorphism (i.e., Definition 2.1).

LEMMA 2.2. *Let $\mathscr{A}$ and $\mathscr{B}$ be separable C*-algebras with $\mathscr{A}$ unital and $\mathscr{B}$ stable. Suppose that $\phi, \psi : \mathscr{A} \to \mathscr{M}(\mathscr{B})$ are two unital injective absorbing *-homomorphisms.*

Then for every $\varepsilon > 0$, for every finite subset $\mathscr{F} \subseteq \mathscr{A}$, there exists a unitary $U \in \mathscr{M}(\mathscr{B})$ such that
$$\phi(a) - U\psi(a)U^* \in \mathscr{B}$$
for all $a \in \mathscr{A}$, and
$$\|\phi(a') - U\psi(a')U^*\| < \varepsilon$$
for all $a' \in \mathscr{F}$.

Proof. Suppose that $\mathscr{A}, \mathscr{B}, \phi, \psi$ satisfy the hypotheses.

Let $\varepsilon > 0$ and a finite subset $\mathscr{F} \subset \mathscr{A}$ be given. Let $S_1, S_2 \in \mathscr{M}(\mathscr{B})$ be isometries with
$$S_1S_1^* + S_2S_2^* = 1.$$

Since ϕ is an absorbing homomorphism, let $U \in \mathscr{M}(\mathscr{B})$ be a unitary such that
$$\phi(a) - U(S_1\phi(a)S_1^* + S_2\psi(a)S_2^*)U^* \in \mathscr{B}$$

for all $a \in \mathscr{A}$, and

$$\|\phi(c) - U(S_1\phi(c)S_1^* + S_2\psi(c)S_2^*)U^*\| < \varepsilon/2$$

for all $c \in \mathscr{F}$.

Since ψ is an absorbing homomorphism, let $V \in \mathscr{M}(\mathscr{B})$ be a unitary such that

$$V\psi(a)V^* - (S_1\phi(a)S_1^* + S_2\psi(a)S_2^*) \in \mathscr{B}$$

for all $a \in \mathscr{A}$, and

$$\|V\psi(c)V^* - (S_1\phi(c)S_1^* + S_2\psi(c)S_2^*)\| < \varepsilon/2$$

for all $c \in \mathscr{F}$.

Then

$$\phi(a) - UV\psi(a)V^*U^* \in \mathscr{B}$$

for all $a \in \mathscr{A}$, and

$$\|\phi(c) - UV\psi(c)V^*U^*\| < \varepsilon$$

for all $c \in \mathscr{F}$. □

REMARK 2.3. There is an elegant and insightful short proof of Lemma 2.2 which is phrased in the language of extension theory (see, for example, [4, Corollary II.5.6]).

For the convenience of the reader, we presented this proof without mentioning extensions, and in a way which is slightly less short.

We will need two specific absorbing *-homomorphisms that were constructed by Lin and Kasparov. These examples actually fall under the Elliott–Kucerovsky result stated in Theorem 2.9 below (though Lin's and Kasparov's works came first). However, for the convenience of the reader, we state the results of Lin and Kasparov.

THEOREM 2.4. *Let $\mathscr{A}, \mathscr{C}$ be unital separable C*-algebras such that $\mathscr{C}$ is simple and $\mathscr{A}$ is a nuclear unital C*-subalgebra of $\mathscr{C}$.*

Let $d : \mathscr{A} \to \mathscr{M}(\mathscr{C} \otimes \mathscr{K})$ be defined by $d(a) := a \otimes 1_{\mathbb{B}(\mathscr{H})} = diag(a, a, a, a, \ldots\ldots)$. Then d is absorbing.

Proof. This follows from [27, Theorem 1.12]. □

REMARK 2.5. Lin introduced the above absorbing extensions result Theorem 2.4 (which we stated as a result about *-homomorphisms) in the course of proving an important stable uniqueness theorem, which is a foundational tool in the Elliott Program.

We also note that the statement and proof of [27, Theorem 1.12] is phrased in the language of extension theory.

Next, we have the following special case of the absorbing *-homomorphism due to Kasparov.

THEOREM 2.6. *Let $\mathscr{A}$ and $\mathscr{C}$ be separable C*-algebras with $\mathscr{A}$ unital and nuclear. Let $\mathscr{H}$ be a separable Hilbert space.*

*Let $\phi : \mathscr{A} \to \mathbb{B}(\mathscr{H})$ be a unital *-homomorphism which is full (i.e., for all $a \in \mathscr{A} - \{0\}$, $\phi(a)/\mathscr{K} \neq 0$).*

Let $\widetilde{\phi} : \mathscr{A} \to \mathscr{M}(\mathscr{C} \otimes \mathscr{K})$ be given by $\widetilde{\phi}(a) := 1_{\mathscr{C}} \otimes \phi(a) \in 1_{\mathscr{C}} \otimes \mathbb{B}(\mathscr{H}) \subseteq \mathscr{M}(\mathscr{C} \otimes \mathscr{K})$, for all $a \in \mathscr{A}$.

Then $\widetilde{\phi}$ is absorbing.

Proof. See [21, Theorem 6] and [22, 1.16]. See also [2, Theorem 15.12.3]. □

REMARK 2.7. The above is a special case of Kasparov's absorbing extension, which generalizes Voiculescu's absorbing extension. In both cases, the existence of these absorbing extensions gave a clean characterization of the relevant extension group (see, for example, [2, Theorem 15.12.2]). Voiculescu's absorbing extension is also the basis of his famous noncommutative Weyl–von Neumann theorem, with many applications and generalizations (see, for example, [1], [4], [5], [11], [26] and the references therein).

We note that the proofs referred to in the proof of Theorem 2.6 are phrased in the language of extension theory.

The notion of "pure largeness" is due to Elliott and Kucerovsky ([5]).

DEFINITION 2.2. Let $\mathscr{A}$ and $\mathscr{B}$ be separable C*-algebras with $\mathscr{B}$ stable.

1. An injective *-homomorphism $\phi : \mathscr{A} \to \mathscr{M}(\mathscr{B})$ is *purely large* if for all $c \in (\phi(A) + \mathscr{B})_+ - \mathscr{B}$, her$(c) := \overline{c\mathscr{B}c}$ contains a stable C*-subalgebra which is full in $\mathscr{B}$.

2. Let $a \in \mathscr{M}(\mathscr{B})_+$. Then a is *purely large* if the inclusion map $C^*(a) \hookrightarrow \mathscr{M}(\mathscr{B})$ is purely large.

REMARK 2.8. Elliott and Kucerovsky introduced the notion of a purely large extension in order to give a simple algebraic characterization of when an extension is absorbing. (See [5]; see also Theorem 2.9 below.)

This algebraic characterization answered a longstanding question (see, for example, [2, page 134, first line]), captures Kasparov's and Lin's extensions, and has many other applications.

We note that the proof referred to, in the proof of Theorem 2.9 below, is also phrased in the language of extension theory.

Elliott and Kucerovsky discovered a relationship between pure largeness and absorption. Here is a special case of their result, stated in terms of *-homomorphisms:

THEOREM 2.9. *Let $\mathscr{A}$ and $\mathscr{B}$ be separable C*-algebras with $\mathscr{A}$ unital and nuclear, and with $\mathscr{B}$ stable. Let $\phi : \mathscr{A} \to \mathscr{M}(\mathscr{B})$ be a unital injective *-homomorphism.*

Then ϕ is purely large if and only if ϕ is absorbing.

Proof. This is [5, Theorem 6]. □

We will also need the following characterization of pure largeness.

LEMMA 2.10. *Let $\mathscr{A}$ and $\mathscr{B}$ be separable C^*-algebras with $\mathscr{A}$ unital and nuclear, and with $\mathscr{B}$ stable. Let $\phi : \mathscr{A} \to \mathscr{M}(\mathscr{B})$ be a unital injective *-homomorphism. Then the following statements are equivalent:*

1. *ϕ is purely large.*
2. *For any $\varepsilon > 0$, for all $c \in (\phi(\mathscr{A}) + \mathscr{B})_+ - \mathscr{B}$, and for all $b \in \mathscr{B}_+$ such that $\|c/\mathscr{B}\| = \|b\| = 1$, there exists an $r \in \mathscr{B}$ with $\|r\| \leqslant 1$ such that $\|rcr^* - b\| < \varepsilon$.*

Proof. See [23, Theorem 3.1]. □

LEMMA 2.11. *Let $\mathscr{B}$ be a separable stable C*-algebra, let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive purely large invertible elements in $\mathscr{M}(\mathscr{B})$, and let $a \in \mathscr{M}(\mathscr{B})_+$ be an invertible element such that $a_n \to a$ in norm.*

Then a is also purely large.

Proof. We use the previous lemma. Let $c \in (C^*(a) + \mathscr{B})_+ - \mathscr{B}$ such that $\|c/\mathscr{B}\| = 1$. We can write c as $c = (f(a)+k)^*(f(a)+k)$ for some $f \in C(\sigma(a))$ and $k \in \mathscr{B}$. Then $(f(a_n)+k)^*(f(a_n)+k) \to c$ in norm. Let $c_n = (f(a_n)+k)^*(f(a_n)+k)$ for all n. Since $c \notin \mathscr{B}, c_n \notin \mathscr{B}$ for sufficiently large n. Since $\|c/\mathscr{B}\| = 1, \|c_n/\mathscr{B}\| \to 1$. We can find $\{\alpha_n\}_{n=1}^{\infty}$ in $[1/2, 3/2]$ such that $\alpha_n \to 1$ and $\alpha_n c_n \to c$ in norm and $\|\alpha_n c_n/\mathscr{B}\| = 1$ for all sufficiently large n. Throwing away finitely many terms if necessary, we may assume that $\|\alpha_n c_n/B\| = 1$ for all n.

Let $\varepsilon > 0$, let $b \in \mathscr{B}_+$ such that $\|b\| = 1$. Choose $N \geqslant 1$ such that $\alpha_N c_N \approx_{\varepsilon/2} c$ and $\alpha_N c_N \notin \mathscr{B}$. So, $\alpha_N c_N \in (C^*(a_N) + \mathscr{B})_+ - \mathscr{B}$ and since a_N is purely large, we can find $r \in \mathscr{B}$ with $\|r\| \leqslant 1$ such that $r(\alpha_N c_N)r^* \approx_{\varepsilon/2} b$. Also, $r(\alpha_N c_N)r^* \approx_{\varepsilon/2} rcr^*$. Hence, $rcr^* \approx_{\varepsilon} b$. Since c, ε, b were arbitrary, by Lemma 2.10, a is purely large. □

LEMMA 2.12. *Let $\mathscr{B}$ be a separable stable C*-algebra. Suppose that $a \in \mathscr{M}(\mathscr{B})$ is a purely large, positive, invertible element.*

Let $S_1, S_2 \in \mathscr{M}(\mathscr{B})$ be isometries such that $S_1S_1^* + S_2S_2^* = 1$.

Then $a' := S_1aS_1^* + S_2S_2^* \in \mathscr{M}(\mathscr{B})$ is a purely large, positive, invertible element.

Proof. It is clear that a' is positive and invertible.

Let $c \in (C^*(a') + \mathscr{B})_+ - \mathscr{B}$ with $\|c/\mathscr{B}\| = 1$. Let $\varepsilon > 0$ and $b \in \mathscr{B}_+$ with $\|b\| = 1$ be arbitrary.

Hence, let $f \in C(\sigma(a')) = C(\sigma(a) \cup \{1\})$ and $k \in \mathscr{B}$ be such that $c = f(a') + k = (f(S_1aS_1^*) + f(S_2S_2^*)) + k$. Hence, either $\|f(S_1aS_1^*)/\mathscr{B}\| = 1$ or $\|f(S_2S_2^*)/\mathscr{B}\| = 1$. Let us assume that $\|f(S_1aS_1^*)/\mathscr{B}\| = 1$ (the proof for the other case is easier; note that $S_2S_2^* \sim 1$).

Note that since a' is invertible, $0 \notin \sigma(a')$ and f can be uniformly approximated over $\sigma(a')$ arbitrarily close by polynomials with no constant term.

Hence, since $S_1^*S_1 = 1$, $f(S_1aS_1^*) = S_1f(a)S_1^*$. Similarly, $f(S_2S_2^*) = S_2f(1)S_2^*$. Hence, $S_1^*cS_1 = f(a) + S_1^*kS_1$ is a positive element.

Also, $\|f(a)/\mathscr{B}\| = \|S_1f(a)S_1^*/\mathscr{B}\| = \|f(S_1aS_1^*)/\mathscr{B}\| = 1$. Since a is purely large, by Lemma 2.10, let $r \in \mathscr{B}$ with $\|r\| \leqslant 1$ be such that $r(f(a) + S_1^*kS_1)r^* \approx_\varepsilon b$.

Now let $s \in \mathscr{B}$ be given by $s := rS_1^*$. Then $\|s\| \leqslant \|r\| \leqslant 1$. Also, $scs^* = rS_1^*(S_1f(a)S_1^* + S_2f(1)S_2^* + k)S_1r^* = r(f(a) + S_1^*kS_1)r^* \approx_\varepsilon b$.

Since c, ε, b were arbitrary, by Lemma 2.10, a' is purely large. □

3. Existence of a nonscalar positive invertible from $GL(\mathbb{C}1 + \mathscr{A} \otimes \mathscr{K})$

The first lemma is a result due to Kadison. We use this to attain nonnormal elements in a normal subgroup of the general linear group of a C*-algebra.

LEMMA 3.1. *If $\mathscr{A}$ is a unital C*-algebra and $a \in \mathscr{A}$, then a is in the center of $\mathscr{A}$ if and only if for all $x \in GL(\mathscr{A})$, $x^{-1}ax$ is a normal operator.*

Proof. See [20, Lemma 1]. □

For the next lemma we recall the definition of a prime and semi-prime C*-algebra. If $\mathscr{A}$ is a C*-algebra we say that a (C*-)ideal $\mathscr{P}$ of $\mathscr{A}$ is a *prime ideal* if $\mathscr{P} \neq \mathscr{A}$ and $\mathscr{I}\mathscr{J} \subseteq \mathscr{P}$ implies $\mathscr{I} \subseteq \mathscr{P}$ or $\mathscr{J} \subseteq \mathscr{P}$ for all (C*-)ideals $\mathscr{I}$ and $\mathscr{J}$ in $\mathscr{A}$. We say that $\mathscr{A}$ is a *prime C*-algebra* if the zero ideal is a prime ideal of $\mathscr{A}$. Recall that every C*-algebra $\mathscr{A}$ is *semi-prime*, i. e., the intersection of all prime ideals of $\mathscr{A}$ is $\{0\}$.

LEMMA 3.2. *Let $\mathscr{A}$ be a unital C*-algebra and let g be an invertible element of $\mathscr{A}$. If $(g^*, g) := (g^*)^{-1}g^{-1}g^*g$ is in the center of $\mathscr{A}$ then g is normal.*

Proof. Assume first that $\mathscr{A}$ is a prime C*-algebra. Then the center of $\mathscr{A}$ is $\mathbb{C}1_\mathscr{A}$. Hence, $(g^*, g) = \alpha 1_\mathscr{A}$ for some $\alpha \in \mathbb{C}$. Hence, $g^*g = \alpha gg^*$. Taking norms on each side we get that $|\alpha| = 1$. Since g^*g is positive, we have that $\alpha = 1$. Hence, g is normal. Now assume that $\mathscr{A}$ is a general C*- algebra. If $\mathscr{J}$ is a prime ideal of $\mathscr{A}$ then $\mathscr{A}/\mathscr{J}$ is a prime C*-algebra. Let π be the quotient map. Then $(\pi(g)^*, \pi(g))$ is in the center of $\mathscr{A}/\mathscr{J}$ and hence $(\pi(g)^*, \pi(g)) = 1_{\mathscr{A}/\mathscr{J}}$. Hence, $(g^*, g) - 1_\mathscr{A} \in \mathscr{J}$. Since $\mathscr{J}$ was an arbitrary prime ideal, $(g^*, g) - 1_\mathscr{A} \in \mathscr{J}$ for any prime ideal $\mathscr{J}$. Since C*-algebras are semi-prime, the zero ideal is the intersection of prime ideals, so $(g^*, g) - 1_\mathscr{A} = 0$. This implies $g^*g = gg^*$. Hence, g is normal. □

LEMMA 3.3. *Let $\mathscr{A}$ be a unital C*-algebra and let g be an invertible element of $\mathscr{A}$. If $x := |g|(gg^*)^{-1}|g|$ is in the center of $\mathscr{A}$ then g is normal.*

Proof. Since x is in the center of $\mathscr{A}$, x is in the center of $GL(\mathscr{A})$ and since the center of $GL(\mathscr{A})$ is a normal subgroup of $GL(\mathscr{A})$ we have that $|g|^{-1}x|g| = (gg^*)^{-1}(g^*g) = (g^*, g)$ is in the center of $GL(\mathscr{A})$. Therefore, (g^*, g) is in the center of $\mathscr{A}$. By Lemma 3.2, g is normal. □

LEMMA 3.4. *Let $\mathscr{A}$ be a unital C*-algebra. If a normal subgroup of $GL(\mathscr{A})$ is not contained in the center of $\mathscr{A}$ then it contains noncentral positive elements.*

Proof. Let G be a noncentral normal subgroup of $GL(\mathscr{A})$. Let f be a noncentral element of G. Then by Lemma 3.1, there exists $x \in GL(\mathscr{A})$ such that $g = x^{-1}fx$ is not normal and in G. Now $a = (g^*g)^{1/2}((g^*)^{-1}g^{-1}g^*g)(g^*g)^{-1/2} = |g|(gg^*)^{-1}|g|$ is positive and in G, and by Lemma 3.3, a is noncentral. □

LEMMA 3.5. *Let $\mathscr{B}$ be a nonunital C*-algebra and let G be a noncentral normal subgroup of $GL(\mathscr{M}(\mathscr{B}))$. Then G contains a nonscalar element of $GL(\mathscr{B}+\mathbb{C}1)$.*

Proof. By Lemma 3.4, let $x \in G$ be a positive noncentral and hence nonscalar element. Then for all $y \in GL(\mathscr{B}+\mathbb{C}1)$, $yxy^{-1}x^{-1} \in G \cap GL(\mathscr{B}+\mathbb{C}1)$. Suppose to the contrary that for all $y \in GL(\mathscr{B}+\mathbb{C}1), yxy^{-1}x^{-1} \in (\mathbb{C}-\{0\})1$. Hence, for all $y \in GL(\mathscr{B}+\mathbb{C}1)$, there exists $\alpha_y \in \mathbb{C}$ such that $yxy^{-1}x^{-1} = \alpha_y \Leftrightarrow yxy^{-1} = \alpha_y x$. If y is unitary then taking norms on each side we get that $|\alpha_y| = 1$ and since $x \geqslant 0, \alpha_y = 1$. Hence, for all $u \in U(\mathscr{B}+\mathbb{C}1), ux = xu$. Hence, by the Russo-Dye theorem, $zx = xz$ for all $z \in \mathscr{B}+\mathbb{C}1$. Let $T \in \mathscr{M}(\mathscr{B})$ and let $\{z_n\}$ be a sequence in $\mathscr{B}$ such that $z_n \to T$ strictly. Then since $z_n x = x z_n$ for all n, we have that $Tx = xT$. Hence, since T was arbitrary, x is central. This is a contradiction. Hence, there is a $y \in GL(\mathscr{B}+\mathbb{C}1)$ such that $yxy^{-1}x^{-1}$ is a nonscalar element of $G \cap GL(\mathscr{B}+\mathbb{C}1)$. □

LEMMA 3.6. *Let $\mathscr{B}$ be a nonunital simple C*-algebra. Let G be a nonscalar normal subgroup of $GL(\mathscr{M}(\mathscr{B}))$. Then G contains a nonscalar positive element of $GL(\mathscr{B}+\mathbb{C}1)$.*

Proof. By Lemma 3.5, let $g \in G \cap GL(\mathscr{B}+\mathbb{C}1)$ be a nonscalar element. So, g is a noncentral element of $\mathscr{B}+\mathbb{C}1$. By Lemma 3.1, there is $c \in GL(\mathscr{B}+\mathbb{C}1)$ such that $c^{-1}gc$ is not normal. Let $x = c^{-1}gc$, then $x \in G \cap GL(\mathscr{B}+\mathbb{C}1)$. Note that $G \cap GL(\mathscr{B}+\mathbb{C}1)$ is a normal subgroup of $GL(\mathscr{B}+\mathbb{C}1)$. Since x is not normal, by Lemma 3.4, $G \cap GL(\mathscr{B}+\mathbb{C}1)$ contains a nonscalar positive element. □

4. Constructing a nonscalar purely large positive element

Throughout this section, $\mathscr{A}$ is a unital separable simple C*-algebra and G is a (algebraic) normal subgroup of $GL(\mathscr{M}(\mathscr{A}\otimes\mathscr{K}))$ such that G properly contains the scalar invertibles. (So G itself is nonscalar.) Let $\{e_{j,k}\}_{1\leqslant j,k<\infty}$ be a system of matrix units for $\mathscr{K}$. For all $n \geqslant 1$, let $e_n := \sum_{j=1}^{n} 1_{\mathscr{A}} \otimes e_{j,j}$. Then $\{e_n\}_{n=1}^{\infty}$ is an approximate unit for $\mathscr{A}\otimes\mathscr{K}$ consisting of an increasing sequence of projections.

The goal of this section is to construct a nonscalar purely large positive element in the strict topology closure $\overline{G}^{\text{strict}}$. We firstly provide an inductive construction which will be referred to throughout this section.

By Lemma 3.6, G contains a nonscalar positive element $x \in \mathbb{C}1+\mathscr{A}\otimes\mathscr{K}$. Since G properly contains the scalar invertibles, we may assume that $x \in 1+\mathscr{A}\otimes\mathscr{K}$.

By the continuous functional calculus, x^2 is also nonscalar.

Since $x^2 = x^*x$ is nonscalar, choose an integer $n_1 \geqslant 1$ and an $\varepsilon > 0$ so that for every $a \in (\mathscr{A} \otimes \mathscr{K})_+$ and for all $n \geqslant n_1$, if $e_n x^* x e_n \approx_\varepsilon e_n a e_n$ then $e_n a e_n$ is not a scalar multiple of e_n. Let $\{\varepsilon_j\}_{j=1}^{\infty}$ be a strictly decreasing sequence in $(0,1)$ such that

$$\sum_{j=1}^{\infty} \varepsilon_j < \varepsilon. \tag{4.1}$$

We now construct a sequence $\{x_k\}_{k=1}^{\infty}$ in $G \cap (1 + \mathscr{A} \otimes \mathscr{K})$, a subsequence $\{N_k\}_{k=1}^{\infty}$ of the positive integers, a sequence $\{f_k\}_{k=1}^{\infty}$ of projections in $\mathscr{A} \otimes \mathscr{K}$, and a sequence $\{U_k\}_{k=1}^{\infty}$ of unitaries in $\mathbb{C}1 + \mathscr{A} \otimes \mathscr{K}$. The construction is by induction on k.

Basic step $k = 1$: Since $x \in 1 + \mathscr{A} \otimes \mathscr{K}$, we can find $N_1 \geqslant n_1$, a projection $f_1 \in 1_{\mathscr{A}} \otimes \mathscr{K}$, and a unitary $U_1 \in \mathbb{C}1 + \mathscr{A} \otimes \mathscr{K}$ such that the following statements hold:

1. $f_1 \perp e_{N_1}$, $f_1 \sim e_{N_1}$ and $f_1 \leqslant e_n$ for some n.

2. $U_1 e_{N_1} = f_1 U_1$, $e_{N_1} U_1 = U_1 f_1$ and $U_1(1 - e_{N_1} - f_1) = (1 - e_{N_1} - f_1)U_1 = 1 - e_{N_1} - f_1$.

   Hence, a matrix representation for U_1 is

   $$U_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   where the unit of the $(1,1)$ position is e_{N_1}, the $(2,2)$ position is f_1, and the $(3,3)$ position is $1 - e_{N_1} - f_1$ which is Murray–von Neumann equivalent to $1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})}$.

3. $x^s \approx_{\varepsilon_1} e_{N_1} x^s e_{N_1} + (1 - e_{N_1})$ and $\|x^s e_{N_1}\| \approx_{\varepsilon_1} \|x^s\| \approx_{\varepsilon_1} \|e_{N_1} x^s\|$ for $s = \pm 1$.

4. For $s = \pm 1$,

   $$\begin{aligned} x^s U_1 x^s U_1^* &\approx_{\varepsilon_1} e_{N_1} x^s e_{N_1} + U_1 e_{N_1} x^s e_{N_1} U_1^* + (1 - e_{N_1} - f_1) \\ &\approx_{\varepsilon_1} U_1 x^s U_1^* x^s \\ &\approx_{\varepsilon_1} x^s U_1 x^s U_1^*. \end{aligned}$$

   Note that a matrix representation for $e_{N_1} x^s e_{N_1} + U_1 e_{N_1} x^s e_{N_1} U_1^* + (1 - e_{N_1} - f_1)$ is

   $$\begin{bmatrix} e_{N_1} x^s e_{N_1} & 0 & 0 \\ 0 & e_{N_1} x^s e_{N_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   where the unit of the $(1,1)$ position is e_{N_1}, the $(2,2)$ position is f_1, and the $(3,3)$ position is $1 - e_{N_1} - f_1$ which is Murray–von Neumann equivalent to $1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})}$.

5. Let $x_1 := xU_1xU_1^*$. Then $x_1 \in G \cap (1 + \mathscr{A} \otimes \mathscr{K})$,
$$x_1^s \approx_{\varepsilon_1} e_{N_1}x^s e_{N_1} + U_1 e_{N_1} x^s e_{N_1} U_1^* + (1 - e_{N_1} - f_1)$$
for $s = \pm 1$, and
$$\begin{aligned} x_1^* x_1 &\approx_{\varepsilon_1} e_{N_1}x^2 e_{N_1} + U_1 e_{N_1} x^2 e_{N_1} U_1^* + (1 - e_{N_1} - f_1) \\ &= \begin{bmatrix} e_{N_1}x^2e_{N_1} & 0 & 0 \\ 0 & e_{N_1}x^2e_{N_1} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$
Note also that since x is positive, $x_1^* \in G$.

6. $\|(x_1^*x_1)^{-1}\| < \|x^{-2}\| + \varepsilon_1$.

7. $e_{N_1}x^2e_{N_1}$ is invertible in $e_{N_1}(\mathscr{A} \otimes \mathscr{K})e_{N_1}$ and $\|(e_{N_1}x^2e_{N_1})^{-1}\| < \|x^{-2}\| + \varepsilon_1$.

We collectively denote the above statements by "$(*)$".

Induction step: Suppose that $\{x_l\}_{l=1}^k$, $\{N_l\}_{l=1}^k$, $\{f_l\}_{l=1}^k$, and $\{U_l\}_{l=1}^k$ have been constructed. We now construct x_{k+1}, N_{k+1}, f_{k+1}, and U_{k+1}.

By the previous steps of the construction, we have the following statements:

For all $1 \leqslant l \leqslant k$, $e_{N_l} \perp f_l$, $e_{N_{l-1}} + f_{l-1} \leqslant e_{N_l}$ (define $e_0 =_{df} f_0 =_{df} 0$), $f_k \leqslant e_{k'}$ for some k', $e_{N_l} \sim f_l$, $U_l e_{N_l} = f_l U_l$, $e_{N_l} U_l = U_l f_l$, $U_l(1 - e_{N_l} - f_l) = (1 - e_{N_l} - f_l)U_l = 1 - e_{N_l} - f_l$, and a matrix representation for U_l is
$$U_l = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
where the unit of the $(1,1)$ position is e_{N_l}, the $(2,2)$ position is f_l, and the $(3,3)$ position is $1 - e_{N_l} - f_l$ which is Murray–von Neumann equivalent to $1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})}$.

For $s = \pm 1$,
$$\begin{aligned} x_k^s &\approx_{\varepsilon_k} e_{N_k}x_{k-1}^s e_{N_k} + U_k e_{N_k} x_{k-1}^s e_{N_k} U_k^* + (1 - e_{N_k} - f_k) \\ &= \begin{bmatrix} e_{N_k}x_{k-1}^s e_{N_k} & 0 & 0 \\ 0 & e_{N_k}x_{k-1}^s e_{N_k} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{4.2}$$
where, in the matrix representation, the unit of the $(1,1)$ position is e_{N_k}, the $(2,2)$ position is f_k, and the $(3,3)$ position is $1 - e_{N_k} - f_k$. (Here and in the rest of the section, we define $x_0 := x$.)

For all $0 \leqslant l \leqslant k-1$ and for $s = \pm 1$, define
$$\begin{aligned} x_{l,k,s} := {} & e_{N_{l+1}} x_l^s e_{N_{l+1}} \\ & + \sum\{U_{k_m} \cdots U_{k_1}(e_{N_{l+1}} x_l^s e_{N_{l+1}}) U_{k_1}^* \cdots U_{k_m}^* : l+1 \leqslant k_1 < k_2 < \cdots < k_m \leqslant k\} \\ & + (1 - e_{N_{l+1}} - \sum\{U_{k_m} \cdots U_{k_1} e_{N_{l+1}} U_{k_1}^* \cdots U_{k_m}^* : l+1 \leqslant k_1 < k_2 < \cdots < k_m \leqslant k\}) \\ = {} & e_{N_{l+1}} x_l^s e_{N_{l+1}} \oplus e_{N_{l+1}} x_l^s e_{N_{l+1}} \oplus \ldots \oplus e_{N_{l+1}} x_l^s e_{N_{l+1}} \oplus 1. \end{aligned}$$

(Note that the terms in the first sum are pairwise orthogonal, and so $x_{l,k,s}$ is a direct sum of finitely many pairwise orthogonal copies of $e_{N_{l+1}} x_l^s e_{N_{l+1}}$ together with a projection which is Murray-von Neumann equivalent to $1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})}$.)

Then

$$x_k^s \approx_{\sum_{j=l+1}^k \varepsilon_j} x_{l,k,s} \tag{4.3}$$

for $s = \pm 1$.

For all $l \leqslant k-1$, define

$$\begin{aligned} a_{l,k} &:= e_{N_{l+1}} x_l^* x_l e_{N_{l+1}} \\ &\quad + \sum \{ U_{k_m} \dots U_{k_1} (e_{N_{l+1}} x_l^* x_l e_{N_{l+1}}) U_{k_1}^* \dots U_{k_m}^* : l+1 \leqslant k_1 < k_2 < \dots < k_m \leqslant k \} \\ &\quad + (1 - e_{N_{l+1}} - \sum \{ U_{k_m} \dots U_{k_1} e_{N_{l+1}} U_{k_1}^* \dots U_{k_m}^* : l+1 \leqslant k_1 < k_2 < \dots < k_m \leqslant k \}) \\ &= e_{N_{l+1}} x_l^* x_l e_{N_{l+1}} \oplus e_{N_{l+1}} x_l^* x_l e_{N_{l+1}} \oplus \dots \oplus e_{N_{l+1}} x_l^* x_l e_{N_{l+1}} \oplus 1. \end{aligned}$$

(So $a_{l,k}$ is the same as $x_{l,k}$ except every occurrence of x_l is replaced with $x_l^* x_l$. In other words, $a_{l,k}$ is a direct sum of finitely many pairwise orthogonal copies of $e_{N_{l+1}} x_l^* x_l e_{N_{l+1}}$ together with a projection which is Murray-von Neumann equivalent to $1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})}$.)

Then

$$x_k^* x_k \approx_{\sum_{j=l+1}^k \varepsilon_j} a_{l,k}. \tag{4.4}$$

Now, by the previous steps in the construction, $x_k, x_k^* \in G \cap (1 + \mathscr{A} \otimes \mathscr{K})$. Hence, we can choose $N_{k+1} \geqslant N_k + 1$, a projection $f_{k+1} \in 1_{\mathscr{A}} \otimes \mathscr{K}$, and a unitary $U_{k+1} \in \mathbb{C}1 + \mathscr{A} \otimes \mathscr{K}$ such that the following statements are true:

1. $f_k \leqslant e_{N_{k+1}}$, $f_{k+1} \perp e_{N_{k+1}}$, $f_{k+1} \sim e_{N_{k+1}}$, $f_{k+1} \leqslant e_n$ for some n, and $e_1 \preceq e_{N_{k+1}} - e_{N_k} - f_k$.

2. $U_{k+1} e_{N_{k+1}} = f_{k+1} U_{k+1}$ and $e_{N_{k+1}} U_{k+1} = U_{k+1} f_{k+1}$ and $U_{k+1}(1 - e_{N_{k+1}} - f_{k+1}) = (1 - e_{N_{k+1}} - f_{k+1}) U_{k+1} = 1 - e_{N_{k+1}} - f_{k+1}$.

   Hence, a matrix representation for U_{k+1} is

   $$U_{k+1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   where the unit of the $(1,1)$ position is $e_{N_{k+1}}$, the $(2,2)$ position is f_{k+1}, and the $(3,3)$ position is $1 - e_{N_{k+1}} - f_{k+1}$ which is Murray–von Neumann equivalent to $1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})}$.

3. $x_k^s \approx_{\varepsilon_{k+1}} e_{N_{k+1}} x_k^s e_{N_{k+1}} + (1 - e_{N_{k+1}})$, and $\|e_{N_{k+1}} x_k^s\| \approx_{\varepsilon_{k+1}} \|x_k^s\| \approx_{\varepsilon_{k+1}} \|x_k^s e_{N_{k+1}}\|$, for $s = \pm 1$.

4. For $s = \pm 1$,

   $$\begin{aligned} x_k^s U_{k+1} x_k^s U_{k+1}^* &\approx_{\varepsilon_{k+1}} e_{N_{k+1}} x_k^s e_{N_{k+1}} + U_{k+1} e_{N_{k+1}} x_k^s e_{N_{k+1}} U_{k+1}^* + (1 - e_{N_{k+1}} - f_{k+1}) \\ &\approx_{\varepsilon_{k+1}} U_{k+1} x_k^s U_{k+1}^* x_k^s \\ &\approx_{\varepsilon_{k+1}} x_k^s U_{k+1} x_k^s U_{k+1}^*. \end{aligned}$$

Note that a matrix representation for $e_{N_{k+1}}x_k^s e_{N_{k+1}} + U_{k+1}e_{N_{k+1}}x_k^s e_{N_{k+1}}U_{k+1}^* + (1 - e_{N_{k+1}} - f_{k+1})$ is

$$\begin{bmatrix} e_{N_{k+1}}x_k^s e_{N_{k+1}} & 0 & 0 \\ 0 & e_{N_{k+1}}x_k^s e_{N_{k+1}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where the unit of the $(1,1)$ position is $e_{N_{k+1}}$, the $(2,2)$ position is f_{k+1}, and the $(3,3)$ position is $1 - e_{N_{k+1}} - f_{k+1}$ which is Murray–von Neumann equivalent to $1_{\mathscr{M}(\mathscr{A}\otimes\mathscr{K})}$.

5. Let $x_{k+1} := x_k U_{k+1} x_k U_{k+1}^* \in G \cap (1 + \mathscr{A}\otimes\mathscr{K})$.

   For all $0 \leqslant l \leqslant k$ and for $s = \pm 1$, let

$$\begin{aligned} x_{l,k+1,s} :=\ & e_{N_{l+1}}x_l^s e_{N_{l+1}} \\ & + \sum\{U_{k_m}\ldots U_{k_1} e_{N_{l+1}}x_l^s e_{N_{l+1}} U_{k_1}^*\ldots U_{k_m}^* : l+1 \leqslant k_1 < k_2 < \ldots < k_m \leqslant k+1\} \\ & + (1 - e_{N_{1+1}} - \sum\{U_{k_m}\ldots U_{k_1} e_{N_{l+1}} U_{k_1}^*\ldots U_{k_m}^* : l+1 \leqslant k_1 < k_2 < \ldots < k_m \leqslant k+1\}) \\ =\ & e_{N_{l+1}}x_l^s e_{N_{l+1}} \oplus e_{N_{l+1}}x_l^s e_{N_{l+1}} \oplus \ldots \oplus e_{N_{l+1}}x_l^s e_{N_{l+1}} \oplus 1. \end{aligned}$$

   (Hence, $x_{l,k+1,s}$ is a direct sum of finitely many pairwise orthogonal copies of $e_{N_{l+1}}x_l^s e_{N_{l+1}}$ with a projection which is Murray–von Neumann equivalent to $1_{\mathscr{M}(\mathscr{A}\otimes\mathscr{K})}$.)

   Then for $s = \pm 1$, $x_{k+1}^s \approx_{\sum_{j=l+1}^{k+1}\varepsilon_j} x_{l,k+1,s}$.

   Note also that since $x_k^* \in G$, $x_{k+1}^* \in G$.

6. For $0 \leqslant l \leqslant k$, let

$$\begin{aligned} a_{l,k+1} :=\ & e_{N_{l+1}}x_l^* x_l e_{N_{l+1}} \\ & + \sum\{U_{k_m}\ldots U_{k_1} e_{N_{l+1}}x_l^* x_l e_{N_{l+1}} U_{k_1}^*\ldots U_{k_m}^* : l+1 \leqslant k_1 < k_2 < \ldots < k_m \leqslant k+1\} \\ & + (1 - e_{N_{l+1}} - \sum\{U_{k_m}\ldots U_{k_1} e_{N_{l+1}} U_{k_1}^*\ldots U_{k_m}^* : l+1 \leqslant k_1 < k_2 < \ldots < k_m \leqslant k+1\}) \\ =\ & e_{N_{l+1}}x_l^* x_l e_{N_{l+1}} \oplus e_{N_{l+1}}x_l^* x_l e_{N_{l+1}} \oplus \ldots \oplus e_{N_{l+1}}x_l^* x_l e_{N_{l+1}} \oplus 1. \end{aligned}$$

   (Hence, $a_{l,k+1}$ is a direct sum of finitely many pairwise orthogonal copies of $e_{N_{l+1}}x_l^* x_l e_{N_{l+1}}$ with a projection which is Murray–von Neumann equivalent to $1_{\mathscr{M}(\mathscr{A}\otimes\mathscr{K})}$.)

   Then $x_{k+1}^* x_{k+1} \approx_{\sum_{j=l+1}^{k+1}\varepsilon_j} a_{l,k+1}$.

7. For $0 \leqslant l \leqslant k, \|(x_l^* x_l)^{-1}\| < \|x^{-2}\| + \sum_{j=1}^{l+1}\varepsilon_j$.

8. For $0 \leqslant l \leqslant k$, $e_{N_{l+1}}x_l^* x_l e_{N_{l+1}}$ is invertible in $e_{N_{l+1}}(\mathscr{A}\otimes\mathscr{K})e_{N_{l+1}}$ and $\|(e_{N_{l+1}}x_l^* x_l e_{N_{l+1}})^{-1}\| < \|x^{-2}\| + \sum_{j=1}^{l+1}\varepsilon_j$.

We denote the above statements by “(+)”.

This completes the inductive construction.

In the rest of this section, we will repeatedly refer to the above construction.

LEMMA 4.1. *$\{x_k\}_{k=1}^{\infty}$ and $\{x_k^{-1}\}_{k=1}^{\infty}$ are both (norm) bounded sequences.*

Proof. By (+) statement (5), we have that for all k

$$\|x_{k+1}\| \leqslant \|x\| + \sum_{j=1}^{k+1} \varepsilon_j < \|x\| + \sum_{j=1}^{\infty} \varepsilon_j < \infty.$$

(Recall that, by our convention, $x_0 := x$.) Hence, $\{x_k\}_{k=1}^{\infty}$ is bounded. By a similar argument, $\{x_k^{-1}\}_{k=1}^{\infty}$ is bounded. □

LEMMA 4.2. *$\{x_k\}$ and $\{x_k^{-1}\}$ both converge strictly in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$.*

Proof. Let $n \geqslant 1$ be given. Choose K_0 so that $N_{K_0} \geqslant n$. By (+) statements (3) and (5), we have that for all $k \geqslant K_0$,

$$x_{k+1}e_n \approx_{2\varepsilon_{k+1}} x_k e_n.$$

So for all $k > l \geqslant K_0$,

$$x_{k+1}e_n \approx_{\sum_{j=l+1}^{k+1} 2\varepsilon_j} x_l e_n.$$

Since $\sum_{j=1}^{\infty} \varepsilon_j < \infty$, $\{x_k e_n\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathcal{A} \otimes \mathcal{K}$. By a similar argument, $\{e_n x_k\}_{k=1}^{\infty}$ is a Cauchy sequence in $\mathcal{A} \otimes \mathcal{K}$. Since n was arbitrary and since, by Lemma 4.1, $\{x_k\}$ is bounded, we have that $\{x_k\}$ converges strictly in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$.

By a similar argument, $\{x_k^{-1}\}$ converges strictly in $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$. □

By Lemma 4.2, let $y, y_1 \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ be the strict limits of $\{x_k\}$, $\{x_k^{-1}\}$ respectively. It follows that $yy_1 = y_1 y = 1$. Hence, y is invertible with inverse y_1.

LEMMA 4.3. *$x_k \to y$ strictly, $x_k^{-1} \to y_1$ strictly, $\limsup \|x_k\| \leqslant \|y\|$, and $\limsup \|x_k^{-1}\| \leqslant \|y_1\|$.*

Proof. We already have that $x_k \to y$ and $x_k^{-1} \to y_1$ strictly.

Let $l \geqslant 1$ be arbitrary. By (+) statements (3) and (5), we have that for all $k \geqslant l$,

$$x_{k+1}e_{N_{l+1}} \approx_{\varepsilon_{l+1} + \sum_{j=l+1}^{k+1} \varepsilon_j} x_l e_{N_{l+1}}.$$

But by (+) statement (3),

$$\|x_l\| \approx_{\varepsilon_{l+1}} \|x_l e_{N_{l+1}}\|.$$

Hence, for all $k \geqslant l$,

$$\|x_{k+1}e_{N_{l+1}}\| \approx_{2\varepsilon_{l+1} + \sum_{j=l+1}^{k+1} \varepsilon_j} \|x_l\|.$$

Therefore, since $x_{k+1} \to y$ in the strict topology,

$$\|x_l\| \leqslant \|ye_{N_{l+1}}\| + 2\varepsilon_{l+1} + \sum_{j=l+1}^{\infty} \varepsilon_j \leqslant \|y\| + 2\varepsilon_{l+1} + \sum_{j=l+1}^{\infty} \varepsilon_j.$$

Since $\lim_{l\to\infty}\left(2\varepsilon_{l+1}+\sum_{j=l+1}^{\infty}\varepsilon_j\right)=0$,

$$\limsup_{l\to\infty}\|x_l\|\leqslant\|y\|.$$

By a similar argument,

$$\limsup_{l\to\infty}\|x_l^{-1}\|\leqslant\|y_1\|.\quad\square$$

LEMMA 4.4. *For all $z\in\{y,y_1,y^*,y_1^*,y^*y,(y^*y)^{-1}=y_1y_1^*\}$, z is the strict limit of elements of G with norm at most $\|z\|$.*

In particular, there exists a sequence $\{z_k\}$ in G such that $z_k\to y^*y$ strictly, $z_k^{-1}\to(y^*y)^{-1}$ strictly, $\limsup\|z_k\|\leqslant\|y^*y\|$ and $\limsup\|z_k^{-1}\|\leqslant\|(y^*y)^{-1}\|$.

Proof. By Lemma 4.3, y and $y^{-1}=y_1$ are the strict limits of elements of G with norm at most $\|y\|$ and $\|y^{-1}\|=\|y_1\|$ respectively.

Note that by $(*)$ item (5) and (+) item (5), $x_k^*\in G$ for all k. Hence, since the *-operation is continuous with respect to the strict topology, it follows from Lemma 4.3 that $y^*,(y^*)^{-1}=y_1^*$ are strict limits of elements of G with norm at most $\|y^*\|$ and $\|(y^*)^{-1}\|=\|y_1^*\|$ respectively.

Finally, since the strict topology on bounded subsets respects multiplication, $x_k^*x_k\to y^*y$ strictly and $x_k^{-1}(x_k^*)^{-1}\to y^{-1}(y^*)^{-1}=(y^*y)^{-1}$. Note also that by Lemma 4.3, $\limsup\|x_k^*x_k\|=\limsup\|x_k\|^2\leqslant\|y\|^2=\|y^*y\|$. Similarly, $\limsup\|x_k^{-1}(x_k^*)^{-1}\|\leqslant\|y^{-1}(y^*)^{-1}\|$. Take $z_k:=x_k^*x_k$ for all k. $\square$

LEMMA 4.5. *y^*y is nonscalar, i.e., $y^*y\notin\mathbb{C}1$.*

Proof. From (+) statement (6), we have that for all k,

$$e_{N_1}x_{k+1}^*x_{k+1}e_{N_1}\approx_{\sum_{j=1}^{k+1}\varepsilon_j}e_{N_1}x^2e_{N_1}.$$

(Recall that $x_0:=x$.)

Hence, since $x_{k+1}\to y$ strictly,

$$e_{N_1}y^*ye_{N_1}\approx_{\sum_{j=1}^{\infty}\varepsilon_j}e_{N_1}x^2e_{N_1}.$$

By our choice of $\{\varepsilon_j\}$ (and ε), it follows that $e_{N_1}y^*ye_{N_1}$ is not a scalar multiple of e_{N_1}. Hence, y^*y is nonscalar. $\square$

To continue, we fix some notation. For $l+1\leqslant k_1<k_2<\ldots<k_m$, let

$$b_{l,k_1,k_2,\ldots,k_m}:=U_{k_m}\ldots U_{k_1}e_{N_{l+1}}x_l^*x_le_{N_{l+1}}U_{k_1}^*\ldots U_{k_m}^*\in(\mathscr{A}\otimes\mathscr{K})_+$$

and let

$$e_{l,k_1,k_2,\ldots,k_m}:=U_{k_m}\ldots U_{k_1}e_{N_{l+1}}U_{k_1}^*\ldots U_{k_m}^*\in Proj(\mathscr{A}\otimes\mathscr{K}).$$

Note that, by the definitions of U_j and e_{N_j} for all j, we have that for $l+1\leqslant k_1<\ldots<k_m$ and $l+1\leqslant l_1<\ldots<l_r$, if $(k_1,k_2,\ldots,k_m)\neq(l_1,l_2,\ldots,l_r)$, then $b_{l,k_1,\ldots,k_m}\perp b_{l,l_1,\ldots,l_r}$ and $e_{l,k_1,\ldots,k_m}\perp e_{l,l_1,\ldots,l_r}$.

LEMMA 4.6. *For all l, $\{a_{l,k+1}\}_{k=1}^{\infty}$ converges strictly to a_l, where*

$$a_l := e_{N_{l+1}} x_l^* x_l e_{N_{l+1}} \\ + \sum\{b_{l,k_1,\ldots,k_m} : l+1 \leqslant k_1 < k_2 < \ldots < k_m\} \\ +(1 - \sum\{e_{l,k_1,\ldots,k_m} : l+1 \leqslant k_1 < k_2 < \ldots < k_m\})$$

(and where the sums, in the definition of a_l, converge strictly).

Sketch of proof. By the definitions of U_j and e_{N_j} for all j, for all $l+1 \leqslant k_1 < k_2 < \ldots < k_m$, for all $n \leqslant N_{k_m - 1}$, $e_n b_{l,k_1,\ldots,k_m} = b_{l,k_1,\ldots,k_m} e_n = 0$.

Hence, for all n_2, the set $\{e_{n_2} b_{l,l_1,\ldots,l_r} : l+1 \leqslant l_1 < l_2 < \ldots < l_r\} \cup \{b_{l,l_1,\ldots,l_r} e_{n_2} : l+1 \leqslant l_1 < l_2 < \ldots < l_r\}$ is a finite set (most expressions will be zero).

A similar argument can be made for sets of the form $\{e_{n_2} e_{l,l_1,\ldots,l_r} : l+1 \leqslant l_1 < l_2 < \ldots < l_r\} \cup \{e_{l,l_1,\ldots,l_r} e_{n_2} : l+1 \leqslant l_1 < l_2 < \ldots < l_r\}$. □

REMARK 4.7. Note that in Lemma 4.6, a_l is an infinite repeat of $e_{N_{l+1}} x_l^* x_l e_{N_{l+1}}$ direct summed with a projection which is Murray–von Neumann equivalent to $1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})}$. I.e., a_l has the form

$$a_l = 1 \oplus e_{N_{l+1}} x_l^* x_l e_{N_{l+1}} \oplus e_{N_{l+1}} x_l^* x_l e_{N_{l+1}} \oplus e_{N_{l+1}} x_l^* x_l e_{N_{l+1}} \oplus \ldots$$

where 1 (in the first position) is Murray–von Neumann equivalent to $1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})}$ and where there are infinitely many copies of $e_{N_{l+1}} x_l^* x_l e_{N_{l+1}}$. Also, the above infinite sum converges strictly.

LEMMA 4.8. *For all l, a_l is a purely large positive invertible.*

Proof. Clearly, $a_l \geqslant 0$ for all l.

By (+) item (1), $e_1 \preceq e_{N_{k+1}} - e_{N_k} - f_k$ for all k. Hence, for all l,

$$1_{\mathscr{M}(\mathscr{B})} \sim (1_{\mathscr{M}(\mathscr{B})} - \sum\{e_{l,k_1,\ldots,k_m : l+1 \leqslant k_1 < k_2 < \ldots < k_m}\}).$$

Hence, for all l, by the definition of a_l, by Theorem 2.4, and by Lemma 2.12, we have that a_l is purely large. □

LEMMA 4.9. *$\{a_l\}$ converges in norm to $y^* y$.*

Proof. Let $\delta > 0$ be given.

Choose $L \geqslant 1$ such that for all $l \geqslant L$, $\sum_{j=l+1}^{\infty} \varepsilon_j < \delta$.

Let $n \geqslant 1$ be given. Choose $K \geqslant L$ such that for all $l \geqslant L$, for all $k \geqslant \max\{l, K\}$, $a_l e_n = a_{l,k+1} e_n$ and $e_n a_l = e_n a_{l,k+1}$.

Hence, by (+) statement (6) and our choice of L, for all $l \geqslant L$, for all $k \geqslant \max\{l, K\}$, $a_l e_n = a_{l,k+1} e_n \approx_{\delta} x_{k+1}^* x_{k+1} e_n$.

Hence, since $x_{k+1}^* x_{k+1} \to y^* y$ strictly as $k \to \infty$, for all $l \geqslant L$, $a_l e_n \approx_{2\delta} y^* y e_n$. By the same argument, for all $l \geqslant L$, $e_n a_l \approx_{2\delta} e_n y^* y$.

Since n was arbitrary, for all $l \geqslant L$, $a_l \approx_{2\delta} y^* y$.

Since δ was arbitrary, $a_l \to y^* y$ in norm as $l \to \infty$. □

LEMMA 4.10. *Let $\mathscr{A}$ be a unital separable simple C*-algebra and $G \subseteq GL(\mathscr{M}(\mathscr{A} \otimes \mathscr{K}))$ a (algebraic) normal subgroup which properly contains the scalar invertibles.*

Then $\overline{G}^{\text{strict}}$ contains a nonscalar purely large positive invertible element a.

Moreover, we can require that for every $\varepsilon > 0$, we can find a sequence $\{z_k\}$ in G such that $z_k \to a$ strictly, $z_k^{-1} \to a^{-1}$ strictly, $\|z_k\| \leqslant \|a\|$ for all k and $\|z_k^{-1}\| \leqslant \|a^{-1}\| + \varepsilon$ for all k.

Proof. This follows immediately from Lemma 4.4, Lemma 4.9, Lemma 2.11, and Lemma 4.8. □

5. Main Theorem

LEMMA 5.1. *Let $\mathscr{H}$ be a separable infinite dimensional Hilbert space.*

Suppose that $G \subseteq GL(\mathbb{B}(\mathscr{H}))$ is a norm closed normal subgroup which properly contains the scalar invertibles such that G contains a normal operator in $GL(\mathbb{B}(\mathscr{H})) - GL(\mathbb{C}1 + \mathscr{K})$.

Then $G = GL(\mathbb{B}(\mathscr{H}))$.

Proof. This follows from [19, Lemma 6]. □

THEOREM 5.2. *Let $\mathscr{A}$ be a unital, separable, simple C*-algebra. Let $G \subseteq GL(\mathscr{M}(\mathscr{A} \otimes \mathscr{K}))$ be a (algebraic) normal subgroup. Suppose that G properly contains the scalar invertibles.*

Then $GL(\mathscr{M}(\mathscr{A} \otimes \mathscr{K})) \subseteq \overline{G}^{\text{strict}}$.

Proof. Note that since G is a normal subgroup of $GL(\mathscr{M}(\mathscr{A} \otimes \mathscr{K}))$, $\overline{G}^{\text{strict}}$ is invariant under conjugation by an invertible, i.e., for all $x \in GL(\mathscr{M}(\mathscr{A} \otimes \mathscr{K}))$, $x\overline{G}^{\text{strict}}x^{-1} \subseteq \overline{G}^{\text{strict}}$.

Also note that $\overline{G}^{\text{strict}}$ is closed under products. (Say that $x, y \in \overline{G}^{\text{strict}}$. Let $\{x_\lambda\}$ and $\{y_\mu\}$ be nets in G such that $x_\lambda \to x$ strictly and $y_\lambda \to y$ strictly. Hence, for all λ, $x_\lambda y_\mu \to x_\lambda y$ strictly. Hence, for all λ, $x_\lambda y \in \overline{G}^{\text{strict}}$. But since $x_\lambda y \to xy$ strictly, $xy \in \overline{G}^{\text{strict}}$.)

Let $\{e_{i,j}\}_{1 \leqslant i,j < \infty}$ be a system of matrix units for $\mathscr{K}$. For each $k \geqslant 1$ let $e_k = \sum_{i=1}^{k} (1_{\mathscr{A}} \otimes e_{i,i})$. Then $\{e_k\}_{k=1}^{\infty}$ is an approximate unit for $\mathscr{A} \otimes \mathscr{K}$ consisting of a properly increasing sequence of projections. By Lemma 4.10, let $a \in \overline{G}^{\text{strict}}$ be a purely large nonscalar positive invertible such that for every $\delta > 0$, there exists a net $\{x_\lambda\}$ in G such that $x_\lambda \to a$ strictly, $x_\lambda^{-1} \to a^{-1}$ strictly, and $\|x_\lambda\| \leqslant \|a\|$ and $\|x_\lambda^{-1}\| \leqslant \|a^{-1}\| + \delta$ for all λ. In particular, $a^{-1} \in \overline{G}^{\text{strict}}$.

Let $X = \sigma(a)$. Since a is nonscalar and invertible, X contains at least two distinct points and $0 \notin X$.

Let $\{t_n\}_{n=1}^{\infty}$ be a dense sequence in X such that each term repeats infinitely many times. Let $\phi : \mathscr{C}(X) \to 1_{\mathscr{A}} \otimes \mathbb{B}(\mathscr{H}) \subseteq \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ be the unital injective *-homomorphism given by $\phi(f) = f(t_1)e_1 + \sum_{i=1}^{\infty} f(t_{i+1})(e_{i+1} - e_i)$.

Since each term in the sequence $\{t_n\}_{n=1}^{\infty}$ repeats infinitely many times, ϕ is a full *-homomorphism. So, by Theorem 2.6, ϕ is absorbing. Let $i : C(X) \cong C^*(a) \to \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ be the natural inclusion map. Since a is purely large, i is purely large. Hence, by Theorem 2.9, i is absorbing. Hence, by Lemma 2.2, i and ϕ are approximately unitarily equivalent. Let $\{w_n\}_{n=1}^{\infty}$ be a sequence of unitaries that witnesses this, i.e., $w_n f(a) w_n^* = w_n i(f) w_n^* \to \phi(f)$ in norm for all $f \in C(X)$.

The function $g(s) = s$ in $C(X)$ corresponds to the positive element $a \in C^*(a) \cong C(X)$. Hence,

$$w_n a w_n^* = w_n i(g) w_n^* \to \phi(g)$$

in norm, and also,

$$w_n a^{-1} w_n^* = w_n i(g)^{-1} w_n^* \to \phi(g)^{-1}.$$

Since $w_n a w_n^*, w_n a^{-1} w_n^* \in \overline{G}^{\text{strict}} \subseteq GL(\mathscr{M}(\mathscr{A} \otimes \mathscr{K}))$, we must have that $\phi(g)$, $\phi(g)^{-1} \in \overline{G}^{\text{strict}}$.

Let $H \subseteq GL(\mathscr{M}(\mathscr{A} \otimes \mathscr{K}))$ be the (algebraic) normal subgroup generated by G and $\phi(g)$. Since $\overline{G}^{\text{strict}}$ is closed under products and conjugation by invertibles and since $\phi(g), \phi(g)^{-1} \in \overline{G}^{\text{strict}}$, $H \subseteq \overline{G}^{\text{strict}}$. Hence, the norm-closure $\overline{H}^{\|\cdot\|} \subseteq \overline{G}^{\text{strict}}$.

$\overline{H}^{\|\cdot\|} \cap GL(\mathscr{M}(\mathscr{A} \otimes \mathscr{K}))$ is a (relative) norm-closed normal subgroup of $GL(\mathscr{M}(\mathscr{A} \otimes \mathscr{K}))$, and hence, $\overline{H}^{\|\cdot\|} \cap GL(1_{\mathscr{A}} \otimes \mathbb{B}(\mathscr{H}))$ is a (relative) norm-closed normal subgroup of $GL(1_{\mathscr{A}} \otimes \mathbb{B}(\mathscr{H}))$. But $\overline{H}^{\|\cdot\|} \cap GL(1_{\mathscr{A}} \otimes \mathbb{B}(\mathscr{H}))$ contains all scalar invertibles and $\phi(g) \in \overline{H}^{\|\cdot\|} \cap GL(1_{\mathscr{A}} \otimes \mathbb{B}(\mathscr{H}))$ is a positive invertible which is not contained in $GL(\mathbb{C}1 + 1_{\mathscr{A}} \otimes \mathscr{K})$. Hence, by Lemma 5.1, $\overline{H}^{\|\cdot\|} \cap GL(1_{\mathscr{A}} \otimes \mathbb{B}(\mathscr{H})) = GL(1_{\mathscr{A}} \otimes \mathbb{B}(\mathscr{H}))$. So $GL(1_{\mathscr{A}} \otimes \mathbb{B}(\mathscr{H})) \subseteq \overline{H}^{\|\cdot\|} \subseteq \overline{G}^{\text{strict}}$.

Let N be the group of unitaries in $\overline{H}^{\|\cdot\|}$. Hence, $\overline{N}^{\text{strict}}$ is a strictly closed normal subgroup of $U(\mathscr{M}(\mathscr{A} \otimes \mathscr{K}))$ which properly contains the scalar unitaries. By [37], the unitary group $U(\mathscr{M}(\mathscr{A} \otimes \mathscr{K})) = \overline{N}^{\text{strict}} \subseteq \overline{G}^{\text{strict}}$.

Since $\overline{G}^{\text{strict}}$ is closed under multiplication, by the Polar Decomposition Theorem, to complete the proof, it suffices to prove that every positive invertible element is contained in $\overline{G}^{\text{strict}}$.

Let $c \in GL(\mathscr{M}(\mathscr{A} \otimes \mathscr{K}))$ be an arbitrary nonscalar positive element. We want to show that c is in $\overline{G}^{\text{strict}}$. Since $\overline{G}^{\text{strict}}$ is closed under products and contains all scalar invertibles, we may assume that $\|c\| \leqslant 1$. Since $\overline{G}^{\text{strict}}$ is strictly closed, it suffices to prove the following:

Let $\varepsilon > 0$ be given. Let $b_1, b_2, \ldots, b_m$ be a finite set of elements in $\mathscr{A} \otimes \mathscr{K}$. Then there exists $x \in \overline{G}^{\text{strict}}$ such that

$$\|x - c\|_{b_i} < \varepsilon$$

for $1 \leqslant i \leqslant m$. We denote the above statement by $(*)$.

Contracting ε if necessary, we may assume that $b_i \geqslant 0$ and $\|b_i\| \leqslant 1$ for $1 \leqslant i \leqslant m$. Choose a positive integer $N \geqslant 1$ such that for $1 \leqslant i \leqslant m$, $e_N b_i$, $b_i e_N$, $e_N b_i e_N$, and b_i are all norm within $\varepsilon/100$ of each other. Since $\mathscr{A} \otimes \mathscr{K}$ is stable, let S, T be isometries in $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ such that the following hold:

(a) $1_{\mathscr{M}(\mathscr{A} \otimes \mathscr{K})} = SS^* + TT^*$
(b) $e_N \leqslant SS^*$
(c) $Se_N = e_N S = e_N$
(d) $\|Sce_N - ce_N\| < \varepsilon/100$
(e) $\|e_N cS^* - e_N c\| < \varepsilon/100$.

Let $Y = \sigma(c)$. Since c is a nonscalar, Y contains at least two distinct points. Let $\{s_l\}_{l=1}^{\infty}$ be a dense sequence in Y such that each term repeats infinitely many times. Let $\psi : \mathscr{C}(Y) \to 1_{\mathscr{A}} \otimes B(\mathscr{H}) \subseteq \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ be the unital injective *-homomorphism given by $\psi(f) = f(s_1)e_1 + \sum_{i=1}^{\infty} f(s_{i+1})(e_{i+1} - e_i)$ for all $f \in \mathscr{C}(Y)$. Since each term in the sequence $\{s_l\}_{l=1}^{\infty}$ repeats infinitely many times, by Theorem 2.6, ψ is absorbing.

Let $i' : C(Y) \cong C^*(c) \to \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ be the natural inclusion map. Let $\psi' : C(Y) \cong C^*(c) \to \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ be the *-homomorphism that is given by $\psi'(f) := Si'(f)S^* + T\psi(f)T^*$. Since ψ is absorbing, ψ' is also absorbing. Hence, by Lemma 2.2, ψ and ψ' are approximately unitarily equivalent. Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of unitaries in $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ that witnesses this; in particular, $u_n \psi(f) u_n^* \to \psi'(f)$ for all $f \in C(Y)$. The function $h(s) = s$ in $C(Y)$ corresponds to the element $c \in C^*(c)$. Hence,

$$u_n \psi(h)(u_n)^* \to \psi'(c) = Si'(c)S^* + T\psi(h)T^* = ScS^* + T\psi(h)T^*$$

in the norm topology as $n \to \infty$. So choose $M \geqslant 1$ such that for all $n \geqslant M$,

$$\|u_n \psi(h)(u_n)^* - (ScS^* + T\psi(h)T^*)\| < \varepsilon/100.$$

Hence, since $\|b_i\| \leqslant 1$ for $1 \leqslant i \leqslant m$, we have that for $n \geqslant M$, for $1 \leqslant i \leqslant m$,

$$\|(u_n \psi(h)(u_n)^* - (ScS^* + T\psi(h)T^*))b_i\| < \varepsilon/100$$

$$\|b_i(u_n \psi(h)(u_n)^* - (ScS^* + T\psi(h)T^*))\| < \varepsilon/100.$$

Note also that from our choices of ψ and h,

$$\|\psi(h)\| = \|c\| \leqslant 1.$$

Now by our choice of N, we have that for $1 \leqslant i \leqslant m$,

$$\begin{aligned}
&\|b_i ScS^* + b_i T\psi(h)T^* - b_i c\| \\
&\leqslant \|(b_i - b_i e_N)ScS^*\| + \|b_i e_N ScS^* - b_i c\| + \|(b_i - b_i e_N)T\psi(h)T^*\| + \|b_i e_N T\psi(h)T^*\| \\
&\leqslant \varepsilon/100 + \|b_i e_N ScS^* - b_i c\| + \varepsilon/100 + 0 \\
&\leqslant \varepsilon/50 + \|b_i(e_N cS^* - e_N c)\| + \|(b_i e_N - b_i)c\| \\
&< \varepsilon/50 + \varepsilon/100 + \varepsilon/100 \\
&= \varepsilon/25.
\end{aligned}$$

From this we have that for $1 \leqslant i \leqslant m$,

$$\begin{aligned} &\|b_i(u_M\psi(h)(u_M)^* - c)\| \\ &\leqslant \|b_i(u_M\psi(h)(u_M)^* - (ScS^* + T\psi(h)T^*))\| + \|b_i((ScS^* + T\psi(h)T^*) - c)\| \\ &< \varepsilon/100 + \varepsilon/25 \\ &= 5\varepsilon/100. \end{aligned}$$

By similar arguments, we have that for $1 \leqslant i \leqslant m$,

$$\|(u_M\psi(h)(u_M)^* - c)b_i\| < 5\varepsilon/100.$$

Hence, for $1 \leqslant i \leqslant m$,

$$\|(u_M\psi(h)(u_M)^* - c)\|_{b_i} < 10\varepsilon/100 = \varepsilon/10.$$

Now since $GL(1_{\mathscr{A}} \otimes \mathbb{B}(\mathscr{H})) \subseteq \overline{G}^{\text{strict}}$ and since $\overline{G}^{\text{strict}}$ is closed under conjugating by invertibles from $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$, we have $u_M GL(1_{\mathscr{A}} \otimes \mathbb{B}(\mathscr{H}))(u_M)^* \subseteq \overline{G}^{\text{strict}}$. Hence, since $\psi(h) \in GL(1_{\mathscr{A}} \otimes \mathbb{B}(\mathscr{H})), u_M\psi(h)(u_M)^* \in \overline{G}^{\text{strict}}$. Since ε was arbitrary, this proves statement $(*)$. □

THEOREM 5.3. *Let $\mathscr{A}$ be a unital separable simple C*-algebra and let $G \subseteq GL(\mathscr{M}(\mathscr{A} \otimes \mathscr{K}))$ be a (algebraic) normal subgroup which properly contains the scalar invertibles.*

Then $\overline{G}^{\text{strict}} = \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$.

Proof. By Theorem 5.2, $GL(\mathscr{M}(\mathscr{A} \otimes \mathscr{K})) \subseteq \overline{G}^{\text{strict}}$. Hence, $GL(\mathbb{C}1 + \mathscr{A} \otimes \mathscr{K}) \subseteq \overline{G}^{\text{strict}}$. Since $\mathscr{A} \otimes \mathscr{K}$ is stable, $\mathscr{A} \otimes \mathscr{K} \subseteq \overline{GL(\mathbb{C}1 + \mathscr{A} \otimes \mathscr{K})}^{\|\cdot\|}$. Hence, $\mathscr{A} \otimes \mathscr{K} \subseteq \overline{G}^{\text{strict}}$. Since $\mathscr{A} \otimes \mathscr{K}$ is strictly dense in $\mathscr{M}(\mathscr{A} \otimes \mathscr{K})$, $\overline{G}^{\text{strict}} = \mathscr{M}(\mathscr{A} \otimes \mathscr{K})$ as required. □

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