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A NOTE ON OPERATOR TUPLES WHICH ARE (m,p)–ISOMETRIC AS WELL AS (μ,∞)–ISOMETRIC

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Abstract. We show that if a tuple of commuting, bounded linear operators $(T_1,\dots,T_d)\in B(X)^d$ is both an (m,p)-isometry and a (μ,∞)-isometry, then the tuple $(T_1^m,\dots,T_d^m)$ is a $(1,p)$-isometry. We further prove some additional properties of the operators $T_1,\dots,T_d$ and show a stronger result in the case of a commuting pair (T_1,T_2).

1. Introduction

Let in the following X be a normed vector space over $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$ and let the symbol $\mathbb{N}$ denote the natural numbers including 0.

A tuple of commuting linear operators $T:=(T_1,\dots,T_d)$ with $T_j:X\to X$ is called an (m,p)-*isometry* (or an (m,p)-*isometric tuple*) if, and only if, for given $m\in\mathbb{N}$ and $p\in(0,\infty)$,

$$\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\|T^\alpha x\|^p=0,\ \ \forall x\in X. \tag{1.1}$$

Here, $\alpha:=(\alpha_1,\dots,\alpha_d)\in\mathbb{N}^d$ is a multi-index, $|\alpha|:=\alpha_1+\dots+\alpha_d$ the sum of its entries, $\frac{k!}{\alpha!}:=\frac{k!}{\alpha_1!\cdots\alpha_d!}$ a multinomial coefficient and $T^\alpha:=T_1^{\alpha_1}\cdots T_d^{\alpha_d}$, where $T_j^0:=I$ is the identity operator.

Tuples of this kind have been introduced by Gleason and Richter [9] on Hilbert spaces (for $p=2$) and have been further studied on general normed spaces in [7]. The tuple case generalises the single operator case, originating in the works of Richter [10] and Agler [1] in the 1980s and being comprehensively studied in the Hilbert space case by Agler and Stankus [2]; the single operator case on Banach spaces has been introduced in the case $p=2$ in [6] and [11] and in its general form by Bayart in [3]. We remark that boundedness, although usually assumed, is not essential for the definition of (m,p)-isometries, as shown by Bermúdez, Martinón and Müller in [4]. Boundedness does, however, play an important role in the theory of objects of the following kind:

Let $B(X)$ denote the algebra of bounded (i.e. continuous) linear operators on X. Equating sums over even and odd k and then considering $p\to\infty$ in (1.1), leads to

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the definition of (m,∞)-*isometries* (or (m,∞)-*isometric tuples*). That is, a tuple of commuting, bounded linear operators $T \in B(X)^d$ is referred to as an (m,∞)-isometry if, and only if, for given $m \in \mathbb{N}$ with $m \geqslant 1$,

$$\max_{\substack{|\alpha|=0,\dots,m \\ |\alpha| \text{ even}}} \|T^\alpha x\| = \max_{\substack{|\alpha|=0,\dots,m \\ |\alpha| \text{ odd}}} \|T^\alpha x\|, \quad \forall x \in X. \tag{1.2}$$

These tupes have been introduced in [7], with the definition of the single operator case appearing in [8]. Although, tuples containing unbounded operators satisfying (1.2) exist, several important statements on (m,∞)-isometries require boundedness. Therefore, from now on, we will always assume the operators $T_1,\dots,T_d$ to be bounded.

In [7], the question is asked what necessary properties a commuting tuple $T \in B(X)^d$ has to satisfy if it is both an (m,p)-isometry and a (μ,∞)-isometry, where possibly $m \neq \mu$. In the single operator case this question is trivial and answered in [8]: If $T = T_1$ is a single operator, then the condition that T_1 is an (m,p)-isometry is equivalent to the mappings $n \mapsto \|T_1^n x\|^p$ being polynomial of degree $\leqslant m-1$ for all $x \in X$. This has been already been observed for operators on Hilbert spaces in [9] and shown in the Banach space/normed space case in [8]; the necessity of the mappings $n \mapsto \|T_1^n x\|^p$ being polynomial has also been proven in [3] and [5]. On the other hand, in [8] it is shown that if a bounded operator $T = T_1 \in B(X)$ is a (μ,∞)-isometry, then the mappings $n \mapsto \|T_1^n x\|$ are bounded for all $x \in X$. The conclusion is obvious: if $T = T_1 \in B(X)$ is both (m,p) and (μ,∞)-isometric, then for all $x \in X$ the $n \mapsto \|T_1^n x\|^p$ are always constant and T_1 has to be an isometry (and, since every isometry is (m,p) and (μ,∞)-isometric, we have equivalence).

The situation is, however, far more difficult in the multivariate, that is, in the operator tuple case. Again, we have equivalence between $T = (T_1,\dots,T_d)$ being an (m,p)-isometry and the mappings $n \mapsto \sum_{|\alpha|=n} \frac{n!}{\alpha!} \|T^\alpha x\|^p$ being polynomial of degree $\leqslant m-1$ for all $x \in X$. The necessity part of this statement has been proven in the Hilbert space case in [9] and equivalence in the general case has been shown in [7]. On the other hand, one can show that if $T \in B(X)^d$ is a (μ,∞)-isometry, then the families $(\|T^\alpha x\|)_{\alpha \in \mathbb{N}^d}$ are bounded for all $x \in X$, which has been proven in [7]. But this fact only implies that the polynomial growth of the $n \mapsto \sum_{|\alpha|=n} \frac{n!}{\alpha!} \|T^\alpha x\|^p$ has to caused by the factors $\frac{n!}{\alpha!}$ and does not immediately give us any further information about the tuple T.

There are several results in special cases proved in [7]. For instance, if a commuting tuple $T = (T_1,\dots,T_d) \in B(X)^d$ is an (m,p)-isometry as well as a (μ,∞)-isometry and we have $m = 1$ or $\mu = 1$ or $m = \mu = d = 2$, then there exists one operator $T_{j_0} \in \{T_1,\dots,T_d\}$ which is an isometry and the remaining operators T_k for $k \neq j_0$ are in particular nilpotent of order m. Although, we are not able to obtain such a results for general $m \in \mathbb{N}$ and $\mu, d \in \mathbb{N} \setminus \{0\}$, yet, we can prove a weaker property: In all proofs of the cases discussed in [7], the fact that the tuple $(T_1^m,\dots,T_d^m)$ is a $(1,p)$-isometry is of critical importance (see the proofs of Theorem 7.1 and Proposition 7.3 in [7]). We will show in this paper that this fact holds in general for any tuple which is both (m,p)-isometric and (μ,∞)-isometric, for general m, μ and d.

The notation we will be using is basically standard, with one possible exception: We will denote the tuple of $d-1$ operators obtained by removing one operator T_{j_0}

from $(T_1,\ldots,T_d)$ by T'_{j_0}, that is $T'_{j_0} := (T_1,\ldots,T_{j_0-1},T_{j_0+1},\ldots,T_d) \in B(X)^{d-1}$ (not to be confused with the dual of the operator T_{j_0}, which will not appear in this paper). Analogously, we denote by α'_{j_0} the multi-index obtained by removing α_{j_0} from $(\alpha_1,\ldots,\alpha_d)$. We will further use the notation $N(T_j)$ for the kernel (or nullspace) of an operator T_j.

2. Preliminaries

In this section, we introduce two needed definitions/notations and compile a number of propositions and theorems from [7], which are necessary for our considerations.

In the following, for $T \in B(X)^d$ and given $p \in (0,\infty)$, define for all $x \in X$ the sequences $(Q^{n,p}(T,x))_{n\in\mathbb{N}}$ by

$$Q^{n,p}(T,x) := \sum_{|\alpha|=n} \frac{n!}{\alpha!}\|T^\alpha x\|^p. \tag{2.1}$$

Define further for all $\ell \in \mathbb{N}$ the mappings $P_\ell^{(p)}(T,\cdot) : X \to \mathbb{R}$, by

$$\begin{aligned} P_\ell^{(p)}(T,x) &:= \sum_{k=0}^{\ell} (-1)^{\ell-k}\binom{\ell}{k} Q^{k,p}(T,x) \\ &= \sum_{k=0}^{\ell} (-1)^{\ell-k}\binom{\ell}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!}\|T^\alpha x\|^p. \end{aligned} \tag{2.2}$$

It is clear that $T \in B(X)^d$ is an (m,p)-isometry if, and only if, $P_m^{(p)}(T,\cdot) \equiv 0$.

If the context is clear, we will simply write $P_\ell(x)$ and $Q^n(x)$ instead of $P_\ell^{(p)}(T,x)$ and $Q^{n,p}(T,x)$.

Further, for $n,k \in \mathbb{N}$, define the (descending) Pochhammer symbol $n^{(k)}$ as follows:

$$n^{(k)} := \begin{cases} 0, & \text{if } k > n, \\ \binom{n}{k}k!\,, & \text{else.} \end{cases}$$

Then $n^{(0)} = 0^{(0)} = 1$ and, if $n,k > 0$ and $k \leqslant n$, we have

$$n^{(k)} = n(n-1)\cdots(n-k+1).$$

As mentioned above, a fundamental property of (m,p)-isometries is that their defining property can be expressed in terms of polynomial sequences.

THEOREM 2.1. ([7, Theorem 3.1]) *$T \in B(X)^d$ is an (m,p)-isometry if, and only if, there exists a family of polynomials $f_x : \mathbb{R} \to \mathbb{R}$, $x \in X$, of degree $\leqslant m-1$ with $f_x|_{\mathbb{N}} = (Q^n(x))_{n\in\mathbb{N}}$.* [1]

[1] Set $\deg 0 := -\infty$ to account for the case $m = 0$.

The following statement describes the Newton-form of the Lagrange-polynomial f_x interpolating $(Q^n(x))_{n\in\mathbb{N}}$.

COROLLARY 2.2. ([7, Proposition 3.2.(i)]) *Let $m \geqslant 1$ and $T \in B(X)^d$ be an (m,p)-isometry. Then we have for all $n \in \mathbb{N}$*

$$Q^n(x) = \sum_{k=0}^{m-1} n^{(k)} \left(\frac{1}{k!}P_k(x)\right), \quad \forall x \in X.$$

Regarding (m,∞)-isometries, we will need the following two statements. Theorem 2.4 is a combination of several fundamental properties of (m,∞)-isometric tuples.

THEOREM 2.3. ([7, Corollary 5.1]) *Let $T = (T_1,\ldots,T_d) \in B(X)^d$ be an (m,∞)-isometry. Then $(\|T^\alpha x\|)_{\alpha\in\mathbb{N}^d}$ is bounded, for all $x \in X$, and*

$$\max_{\alpha\in\mathbb{N}^d} \|T^\alpha x\| = \max_{|\alpha|=0,\ldots,m-1} \|T^\alpha x\|,$$

for all $x \in X$.

THEOREM 2.4. ([7, Proposition 5.5, Theorem 5.1 and Remark 5.2])
Let $T = (T_1,\ldots,T_d) \in B(X)^d$ be an (m,∞)-isometric tuple. Define the norm $|.|_\infty : X \to [0,\infty)$ via $|x|_\infty := \max_{\alpha\in\mathbb{N}^d} \|T^\alpha x\|$, for all $x \in X$, and denote

$$X_{j,|.|_\infty} := \{x \in X \mid |x|_\infty = |T_j^n x|_\infty \text{ for all } n \in \mathbb{N}\}.$$

Then

$$X = \bigcup_{j=1,\ldots,d} X_{j,|.|_\infty}.$$

(Note that, by Theorem 2.3, $|.|_\infty = \|.\|$ if $m = 1$.)

We will also require a fundamental fact on tuples which are both (m,p) and (μ,∞)-isometric and an (almost) immediate corollary.

LEMMA 2.5. ([7, Lemma 7.2]) *Let $T = (T_1,\ldots,T_d) \in B(X)^d$ be an (m,p)-isometry as well as a (μ,∞)-isometry. Let $\gamma = (\gamma_1,\ldots,\gamma_d) \in \mathbb{N}^d$ be a multi-index with the property that $|\gamma_j'| \geqslant m$ for every $j \in \{1,\ldots,d\}$. Then $T^\gamma = 0$.*

Conversely, this implies that if an operator T^α is not the zero-operator, the multi-index α has to be of a specific form. The proof in [7] of the following corollary appears to be overly complicated, the statement is just the negation of the previous lemma.

COROLLARY 2.6. ([7, Corollary 7.1]) *Let $T = (T_1,\ldots,T_d) \in B(X)^d$ be an (m,p)-isometry for some $m \geqslant 1$ as well as a (μ,∞)-isometry. If $\alpha \in \mathbb{N}^d$ is a multi-index with $T^\alpha \neq 0$ and $|\alpha| = n$, then there exists some $j_0 \in \{1,\ldots,d\}$ with $T^\alpha = T_{j_0}^{n-|\alpha_{j_0}'|}(T_{j_0}')^{\alpha_{j_0}'}$ and $|\alpha_{j_0}'| \leqslant m-1$.*

This fact has consequences for the appearance of elements of the sequences $(Q^n(x))_{n\in\mathbb{N}}$, since several summands become zero for large enough n. That is, we have trivially by definition (2.1) of $(Q^n(x))_{n\in\mathbb{N}}$:

COROLLARY 2.7. (see [7, proof of Theorem 7.1]) *Let $T=(T_1,\dots,T_d)\in B(X)^d$ be an (m,p)-isometry for some $m\geqslant 1$ as well as a (μ,∞)-isometry. Then, for all $n\in\mathbb{N}$ with $n\geqslant 2m-1$, we have*

$$Q^n(x)=\sum_{\substack{\beta\in\mathbb{N}^{d-1}\\ |\beta|=0,\dots,m-1}}\sum_{j=1}^{d}\frac{n!}{(n-|\beta|)!\beta!}\|T_j^{n-|\beta|}(T_j')^{\beta}x\|^p,\ \ \forall x\in X,$$

where $\frac{n!}{(n-|\beta|)!\beta!}=\frac{n^{(|\beta|)}}{\beta!}$. (We set $n\geqslant 2m-1$ to ensure that every multi-index only appears once.)

3. The main result

We first present the main result of this article, which is a generalisation of [7, Proposition 7.3], before stating a preliminary lemma needed for its proof.

THEOREM 3.1. *Let $T=(T_1,\dots,T_d)\in B(X)^d$ be an (m,p)-isometric as well a (μ,∞)-isometric tuple. Then*

(i) the sequences $n\mapsto\|T_j^n x\|$ become constant for $n\geqslant m$, for all $j\in\{1,\dots,d\}$, for all $x\in X$.

(ii) the tuple $(T_1^m,\dots,T_d^m)$ is a $(1,p)$-isometry, that is

$$\sum_{j=1}^{d}\|T_j^m x\|^p=\|x\|^p,\quad\forall x\in X.$$

(iii) for any $(n_1,\dots,n_d)\in\mathbb{N}^d$ with $n_j\geqslant m$ for all j, the operators $\sum_{j=1}^{d}T_j^{n_j}$ are isometries, that is

$$\left\|\sum_{j=1}^{d}T_j^{n_j}x\right\|=\|x\|,\quad\forall x\in X.$$

Of course, (i) and (ii) imply that, for any $(n_1,\dots,n_d)\in\mathbb{N}^d$ with $n_j\geqslant m$ for all j,

$$\sum_{j=1}^{d}\|T_j^{n_j}x\|^p=\|x\|^p,\quad\forall x\in X,$$

Theorem 3.1 is a consequence of the following lemma, which is a weaker version of 3.1.(i).

LEMMA 3.2. *Let $T=(T_1,\dots,T_d)\in B(X)^d$ be an (m,p)-isometric as well as a (μ,∞)-isometric tuple. Let further $\kappa\in\mathbb{N}^{d-1}$ be a multi-index with $|\kappa|\geqslant 1$. Then the mappings*

$$n\mapsto \|T_j^n\left(T_j'\right)^{\kappa}x\|$$

become constant for $n\geqslant m$, for all $j\in\{1,\dots,d\}$, for all $x\in X$.

Proof. If $m=0$, then $X=\{0\}$ and if $m=1$, the statement holds trivially, since $T_jT_i=0$ for all $i\neq j$ by Lemma 2.5. So assume $m\geqslant 2$. Further, it clearly suffices to consider $|\kappa|=1$, since the statement then holds for all $x\in X$. The proof, however, works by proving the theorem for $|\kappa|\in\{1,\dots,m-1\}$ in descending order. (Note that the case $|\kappa|\geqslant m$ is also trivial, again by Lemma 2.5.)

Now fix an arbitrary $j_0\in\{1,\dots,d\}$, let $\kappa\in\mathbb{N}^{d-1}$ with $|\kappa|\in\{1,\dots,m-1\}$ and set $\ell:=m-|\kappa|$. Then $\ell\in\{1,\dots,m-1\}$ and $|\kappa|=m-\ell$. We apply Lemma 2.5 to $Q^k(T_{j_0}^m\left(T_{j_0}'\right)^{\kappa}x)$.

By definition (2.1),

$$\begin{aligned}
Q^k(T_{j_0}^m\left(T_{j_0}'\right)^{\kappa}x) &= \sum_{|\alpha|=k}\frac{k!}{\alpha!}\|T^{\alpha}\left(T_{j_0}^m\left(T_{j_0}'\right)^{\kappa}x\right)\|^p\\
&= \|T_{j_0}^k\left(T_{j_0}^m\left(T_{j_0}'\right)^{\kappa}x\right)\|^p+\sum_{j=1}^{k}\sum_{\substack{\beta\in\mathbb{N}^{d-1}\\|\beta|=j}}\frac{k!}{(k-j)!\beta!}\|T_{j_0}^{k-j}\left(T_{j_0}'\right)^{\beta}\left(T_{j_0}^m\left(T_{j_0}'\right)^{\kappa}x\right)\|^p\\
&\overset{2.5}{=} \|T_{j_0}^{m+k}\left(T_{j_0}'\right)^{\kappa}x\|^p+\sum_{j=1}^{\min\{k,\ell-1\}}\sum_{\substack{\beta\in\mathbb{N}^{d-1}\\|\beta|=j}}\frac{k!}{(k-j)!\beta!}\|T_{j_0}^{m+k-j}\left(T_{j_0}'\right)^{\kappa+\beta}x\|^p\\
&= \|T_{j_0}^{m+k}\left(T_{j_0}'\right)^{\kappa}x\|^p+\sum_{j=1}^{\ell-1}k^{(j)}\sum_{\substack{\beta\in\mathbb{N}^{d-1}\\|\beta|=j}}\frac{1}{\beta!}\|T_{j_0}^{m+k-j}\left(T_{j_0}'\right)^{\kappa+\beta}x\|^p,
\end{aligned}\tag{3.1}$$

for all $k\in\mathbb{N}$, for all $x\in X$. Here, in the last line, we utilise the fact that $k^{(j)}=0$ if $j>k$.

We now prove our statement by (finite) induction on ℓ.

$\underline{\ell=1:}$
For $\ell=1$ and $|\kappa|=m-1$, we have, by (3.1),

$$Q^k\left(T_{j_0}^m\left(T_{j_0}'\right)^{\kappa}x\right)=\|T_{j_0}^{m+k}\left(T_{j_0}'\right)^{\kappa}x\|^p,\ \ \forall k\in\mathbb{N},\ \forall x\in X.$$

Since we know by Theorem 2.1 that the sequences $k\mapsto Q^k\left(T_{j_0}^m\left(T_{j_0}'\right)^{\kappa}x\right)$ are polynomial for all $x\in X$, and by Theorem 2.3 that the $k\mapsto\|T_{j_0}^{m+k}\left(T_{j_0}'\right)^{\kappa}x\|^p$ are bounded for all $x \in X$, it follows that

$$n \mapsto \|T_{j_0}^n \left(T'_{j_0}\right)^{\kappa} x\|$$

become constant for $n \geqslant m$, for all $x \in X$.

Since $\ell \in \{1,\dots,m-1\}$, if we have $m=2$, we are already done. So assume in the following that $m \geqslant 3$.

$\underline{\ell \to \ell+1:}$

Assume that the statement holds for some $\ell \in \{1,\dots,m-2\}$. That is, for all $\kappa \in \mathbb{N}^{d-1}$ with $|\kappa| = m-\ell$ the sequences

$$n \mapsto \|T_{j_0}^n \left(T'_{j_0}\right)^{\kappa} x\|$$

become constant for $n \geqslant m$, for all $x \in X$.

Now take a multi-index $\tilde{\kappa} \in \mathbb{N}^{d-1}$ with $|\tilde{\kappa}| = m-(\ell+1)$ and consider

$$Q^k(T_{j_0}^m \left(T'_{j_0}\right)^{\tilde{\kappa}} x) = \|T_{j_0}^{m+k} \left(T'_{j_0}\right)^{\tilde{\kappa}} x\|^p + \sum_{j=1}^{\ell} k^{(j)} \sum_{\substack{\beta\in\mathbb{N}^{d-1} \\ |\beta|=j}} \frac{1}{\beta!} \|T_{j_0}^{m+k-j} \left(T'_{j_0}\right)^{\tilde{\kappa}+\beta} x\|^p.$$

(Where we are now summing over all j running from 1 to $(\ell+1)-1=\ell$.)

Since $|\beta| \geqslant 1$, we have $|\tilde{\kappa}+\beta| \geqslant m-\ell$. Hence, if $k \geqslant j$, by our induction assumption,

$$\|T_{j_0}^{m+k-j} \left(T'_{j_0}\right)^{\tilde{\kappa}+\beta} x\|^p = \|T_{j_0}^{m} \left(T'_{j_0}\right)^{\tilde{\kappa}+\beta} x\|^p, \ \forall x \in X,$$

since $n \mapsto \|T_{j_0}^n \left(T'_{j_0}\right)^{\tilde{\kappa}+\beta} x\|$ become constant for $n \geqslant m$.

Hence, we have, for all $x \in X$,

$$Q^k(T_{j_0}^m \left(T'_{j_0}\right)^{\tilde{\kappa}} x) = \|T_{j_0}^{m+k} \left(T'_{j_0}\right)^{\tilde{\kappa}} x\|^p + \sum_{j=1}^{\ell} k^{(j)} \sum_{\substack{\beta\in\mathbb{N}^{d-1} \\ |\beta|=j}} \frac{1}{\beta!} \|T_{j_0}^{m} \left(T'_{j_0}\right)^{\tilde{\kappa}+\beta} x\|^p. \tag{3.2}$$

That is, for all $x \in X$, the sequences $k \mapsto Q^k(T_{j_0}^m \left(T'_{j_0}\right)^{\tilde{\kappa}} x)$ become almost polynomial (of degree $\leqslant \ell$), with the term $\|T_{j_0}^{m+k} \left(T'_{j_0}\right)^{\tilde{\kappa}} x\|^p$ instead of a (constant) trailing coefficient.

But, as before, by Theorem 2.1, we know that for any $x \in X$, the sequences $k \mapsto Q^k(T_{j_0}^m \left(T'_{j_0}\right)^{\tilde{\kappa}} x)$ are indeed polynomial. Through Corollary 2.2 we know that their trailing coefficients are $\|T_{j_0}^m \left(T'_{j_0}\right)^{\tilde{\kappa}} x\|^p$. Since, by Theorem 2.3, for each $x \in X$, the sequences $k \mapsto \|T_{j_0}^{m+k} \left(T'_{j_0}\right)^{\tilde{\kappa}} x\|^p$ are bounded, we can successively compare and remove coefficients of the formulae for $Q_k(T_{j_0}^m \left(T_{j_0}'\right)^{\tilde{\kappa}} x)$ as given through Corollary 2.2 and (3.2), until we eventually obtain that

$$\|T_{j_0}^{m+k} \left(T_{j_0}'\right)^{\tilde{\kappa}} x\|^p = \|T_{j_0}^m \left(T_{j_0}'\right)^{\tilde{\kappa}} x\|^p,$$

for all $k \in \mathbb{N}$, for all $x \in X$. That is, the sequences

$$n \mapsto \|T_{j_0}^n \left(T_{j_0}'\right)^{\tilde{\kappa}} x\|$$

become constant for $n \geqslant m$, for all $x \in X$. This concludes the induction step and the proof. $\square$

We can now prove the main result.

Proof of Theorem 3.1. By Corollary 2.7 and the lemma above, we have for $n \geqslant 2m-1$,

$$\begin{aligned} Q^n(x) &= \sum_{\substack{\beta \in \mathbb{N}^{d-1} \\ |\beta|=0,\ldots,m-1}} n^{(|\beta|)} \sum_{j=1}^d \frac{1}{\beta!} \|T_j^{n-|\beta|}(T_j')^\beta x\|^p \\ &= \sum_{\substack{\beta \in \mathbb{N}^{d-1} \\ |\beta|=1,\ldots,m-1}} n^{(|\beta|)} \sum_{j=1}^d \frac{1}{\beta!} \|T_j^m (T_j')^\beta x\|^p + \sum_{j=1}^d \|T_j^n x\|^p, \quad \forall x \in X. \end{aligned} \tag{3.3}$$

That is, for all $x \in X$, for $n \geqslant 2m-1$, the sequences $n \mapsto Q^n(x)$ become almost polynomial (of degree $\leqslant m-1$), with the term $\sum_{j=1}^d \|T_j^n x\|^p$ instead of a (constant) trailing coefficient.

Again, by Theorem 2.1, we know that for any $x \in X$, the sequences $n \mapsto Q^n(x)$ are indeed polynomial. And since, by Theorem 2.3, for each $x \in X$, the sequences $n \mapsto \sum_{j=1}^d \|T_j^n x\|^p$ are bounded, we can again successively compare and remove coefficients of the formulae for $Q_n(x)$ as given in Corollary 2.2 and (3.3), until we eventually obtain that

$$\sum_{j=1}^d \|T_j^n x\|^p = \|x\|^p, \quad \forall x \in X, \ \forall\, n \geqslant 2m-1\,. \tag{3.4}$$

Since $T_i^m T_j^m = 0$ for all $i \neq j$, by Lemma 2.5, replacing x by $T_j^\nu x$ with $\nu \geqslant m$ in this last equation, gives $\|T_j^\nu x\| = \|T_j^{n+\nu} x\|$ for all $n \geqslant 2m-1$, for all $x \in X$. Hence, the sequences $n \mapsto \|T_j^n x\|$ become constant for $n \geqslant m$, for all $j \in \{1,\ldots,d\}$, for all $x \in X$. This is 3.1.(i).

But then, (3.4) becomes

$$\sum_{j=1}^d \|T_j^m x\|^p = \|x\|^p, \quad \forall x \in X\,.$$

This is 3.1.(ii).

Now take any $(n_1,\dots,n_d)\in\mathbb{N}^d$ with $n_j\geqslant m$ for all j and replace x in the equation above by $\sum_{j=1}^d T_j^{n_j}$. Then, again, since $T_i^mT_j^m=0$ for $i\neq j$, and since $n\mapsto \|T_j^n x\|$ become constant for $n\geqslant m$,

$$\sum_{j=1}^d \|T_j^{m+n_j}x\|^p=\sum_{j=1}^d \|T_j^m x\|^p=\Big\|\sum_{j=1}^d T_j^{n_j}x\Big\|^p,\ \ \forall x\in X.$$

Together with 3.1.(i), this implies 3.1.(iii). $\square$

It is not hard to see that we actually have a stronger result in some special cases.

COROLLARY 3.3. *Let $T=(T_1,\dots,T_d)\in B(X)^d$ be an (m,p)-isometric as well a (μ,∞)-isometric tuple.*

(i) If one of the operators $T_{j_0}\in\{T_1,\dots,T_d\}$ is surjective, then T_{j_0} is actually an isometric isomorphism and the remaining operators are nilpotent of order m.

(ii) If one of the operators $T_{j_0}\in\{T_1,\dots,T_d\}$ is injective, then $T_{j_0}^m$ is an isometry and the remaining operators are nilpotent of order m.

Proof. (i): If one T_{j_0} is surjective, then $T_{j_0}^m$ is surjective and, since $\|T_{j_0}^m x\|=\|T_{j_0}^{m+1}x\|$ by Theorem 3.1.(i), T_{j_0} is isometric. The nilpotency of the remaining operators follows from 3.1.(ii).

(ii): If one T_{j_0} is injective, then $T_{j_0}^m$ is injective. Since Lemma 2.5 states that, in particular, $T_{j_0}^mT_i^m=0$ for all $i\neq j_0$, we must have $T_i^m=0$ for all $i\neq j_0$. Then $T_{j_0}^m$ is an isometry by 3.1.(ii). $\square$

With respect to part (ii) note that while, by definition of an (m,p)-isometry, we must have $\bigcap_{j=1}^d N(T_j)=\{0\}$, it is not clear that the kernel of a single operator has to be trivial.

4. Some further remarks and the case $d=2$

We finish this note with a stronger result for the case of a commuting pair $(T_1,T_2)\in B(X)^2$. We first state the following two easy corollaries of Theorem 3.1 which hold for general d.

COROLLARY 4.1. *Let $T=(T_1,\dots,T_d)\in B(X)^d$ be an (m,p)-isometry as well as a (μ,∞)-isometry. Then $T_j^m=0$ or $\|T_j^m\|=1$ for any $j\in\{1,\dots,d\}$.*

Proof. By Theorem 3.1.(ii) we have $\|T_j^m\|\leqslant 1$ for any j. On the other hand, by 3.1.(i) we have

$$\|T_j^m x\|=\|T_j^{m+1}x\|\leqslant \|T_j^m\|\cdot\|T_j^m x\|,\ \ \forall x\in X,$$

for any j. That is, $T_j^m=0$ or $\|T_j^m\|\geqslant 1$. $\square$

LEMMA 4.2. *Let* $T = (T_1, \dots, T_d) \in B(X)^d$ *be an* (m,p)*-isometry as well as a* (μ,∞)*-isometry. Define* $|.|_\infty : X \to [0,\infty)$ *and* $X_{j,|.|_\infty}$ *as in Theorem 2.4. Then*

$$X_{j,|.|_\infty} = \{x \in X \mid \exists \alpha(x) \in \mathbb{N}^d,\ s.th.\ |\alpha(x)| \leqslant \mu - 1 \text{ and}$$
$$|x|_\infty = \|T_j^n \left(T_j'\right)^{\alpha_j'(x)} x\|,\ \forall n \in \mathbb{N}\}.$$

Proof. By Theorem 2.3 we know that for every $x \in X$, there exists an $\alpha(x) \in \mathbb{N}^d$ with $\max_{\alpha \in \mathbb{N}^d} \|T^\alpha x\| = \|T^{\alpha(x)} x\|$ and $|\alpha(x)| \leqslant \mu - 1$.

Then $x \in X_{j,|.|_\infty}$ if, and only if, for all $n \in \mathbb{N}$, there exists an $\alpha(x,n) \in \mathbb{N}^d$ with $|\alpha(x,n)| \leqslant \mu - 1$ s.th. $|x|_\infty = \|T_j^n T^{\alpha(x,n)} x\|$. Hence, the inclusion "$\supset$" is clear.

To show "$\subset$" let $0 \neq x \in X_{j,|.|_\infty}$. Then $T_j^m \neq 0$ and, hence, $\|T_j^m\| = 1$.

Since $|\alpha(x,n)| \leqslant \mu - 1$ for all $n \in \mathbb{N}$, there are only finitely many choices for each $\alpha(x,n)$. Thus, there exists an $\alpha(x) \in \mathbb{N}^d$ and an infinite set $M(x) \subset \mathbb{N}$ s.th.

$$|x|_\infty = \|T_j^n T^{\alpha(x)} x\|,\ \forall n \in M(x).$$

By Theorem 3.1.(i), $M(x)$ contains all $n \geqslant m$ and further,

$$\|T_j^n T^{\alpha(x)} x\| = \|T_j^n \left(T_j'\right)^{\alpha_j'(x)} x\|, \text{ for all } n \geqslant m.$$

Since $\|T_j^m\| = 1$, the statement holds for all $n \in \mathbb{N}$. □

PROPOSITION 4.3. *Let* $T = (T_1, T_2) \in B(X)^2$ *be both an* (m,p)*-isometric and a* (μ,∞)*-isometric pair. Then* T_1^m *is an isometry and* $T_2^m = 0$ *or vice versa.*

Proof. By Theorem 2.4, we have $X = X_{1,|.|_\infty} \cup X_{2,|.|_\infty}$.

Let $x_1 \in X_{1,|.|_\infty}$. Then, by the previous lemma, there exists an $\alpha_2(x_1) \in \mathbb{N}$ with $\alpha_2(x_1) \leqslant \mu - 1$ s.th. $|x_1|_\infty = \|T_1^n T_2^{\alpha_2(x_1)} x_1\|$ for all $n \in \mathbb{N}$.

Furthermore, we have $\|x\|^p = \|T_1^m x\|^p + \|T_2^m x\|^p$, for all $x \in X$, by Theorem 3.1.(ii). Replacing x by $T_2^{\alpha_2(x_1)} x_1$ gives

$$\|T_2^{\alpha_2(x_1)} x_1\|^p = \|T_1^m T_2^{\alpha_2(x_1)} x_1\|^p + \|T_2^{m+\alpha_2(x_1)} x_1\|^p$$
$$\Leftrightarrow\ \|T_2^{\alpha_2(x_1)} x_1\|^p = |x_1|_\infty^p + \|T_2^m x_1\|^p.$$

This implies $\|T_2^{\alpha_2(x_1)} x_1\| = |x_1|_\infty$ and, moreover, $\|T_2^m x_1\| = 0$.

An analogous argument shows that $X_{2,|.|_\infty} \subset N(T_1^m)$. Hence,

$$X = N(T_1^m) \cup N(T_2^m),$$

which forces $T_1^m = 0$ or $T_2^m = 0$. The statement follows from $\|x\|^p = \|T_1^m x\|^p + \|T_2^m x\|^p$, for all $x \in X$. □

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