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ON THE DIFFERENCE OF A CONTRACTION AND AN INVERSE STRONGLY MONOTONE OPERATOR

DINU TEODORESCU

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Abstract. In this paper we prove a unique fixed point result in real Hilbert spaces for the difference operator $T-F$, where T is a contraction and F is an inverse strongly monotone operator.

1. Introduction

Let E be a Banach space with the norm $\|\cdot\|_E$. An operator $T: E \to E$ is said to be α-contraction if there exist a real number $\alpha \in (0,1)$ such that

$$||Tx-Ty||_E \leqslant \alpha||x-y||_E$$

for all $x,\ y \in E$.

The famous Banach fixed point theorem affirm that every α-contraction has a unique fixed point in E.

Applying the same Banach fixed point theorem it is easy to obtain

THEOREM 1.1. *Let $\alpha \in (0,1)$ and $T: E \to E$ be a α-contraction. If $\alpha+\beta<1$ and $V: E \to E$ is a β-contraction, then the operator $T-V$ has a unique fixed point in E.*

In this paper we prove that, in some particular case, the Theorem 1.1. holds even if the β-*contraction* V do not satisfies the condition $\alpha+\beta<1$.

To expose our particular case we need the notion of inverse strongly monotone operator.

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm denoted by $\|\cdot\|$. An operator $F: H \to H$ is said to be m-inverse strongly monotone $(m>0)$ if

$$\langle Fx-Fy, x-y\rangle \geqslant m\|Fx-Fy\|^2$$

for all $x,\ y \in H$.

Clearly, using the Schwartz inequality, we deduce that every m-inverse strongly monotone operator F is a $\frac{1}{m}$-Lipschitz operator (i.e. $\|Fx-Fy\| \leqslant \frac{1}{m}||x-y||$ for all

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$x,\ y \in H$). Particularly, if $m > 1$, then the m-inverse strongly monotone operator F is a $\frac{1}{m}$-contraction.

The definition of monotone operator was first given by Kachurovski [6] (see Browder and Petryshyn [2]). The notion of inverse strongly monotone operator appears firstly in 1967 (Browder and Petryshyn [2]).

As examples of inverse strongly monotone operators we give:

— the projection operator P_K, where K is a nonempty closed convex subset of H;

— the operator $I-T$, where T is a nonexpansive operator from H into itself and I is the identity of H;

— every η-strongly monotone and θ-Lipschitz operator A from H into itself is a $\frac{\eta}{\theta^2}$-inverse strongly monotone operator (see [5]).

Many recent papers involving inverse strongly monotone operators are dedicated to the study of iterative schemes for finding a common element of the set of fixed points of a nonexpansive operator and the set of solutions of the variational inequality for an inverse strongly monotone operator (see for example [1, 3, 5, 7]).

In the following we prove using a simple method, based on an application of the Banach fixed point theorem in real Hilbert spaces, that the difference operator of a contraction and an inverse strongly monotone operator has a unique fixed point.

2. The result

The result below can be regarded as a consequence of the following more general theorem, whose proof uses results involving differential operatorial equations in Hilbert spaces:

THEOREM 2.1. *Let X be a real Hilbert space and $F : X \to X$ be a mapping. Then*

a) If F is monotone, hemicontinuous and coercive, then it is a surjection (i.e.: the equation $Fx = h$ has a solution, for each $h \in X$).

b) If F is continuous and strongly monotone, then it is a homeomorphism (i.e.: $F^{-1} : X \to X$ exists and is continuous) (see Deimling [4], page 100).

Now we are in position to give the main result of this paper:

THEOREM 2.2. *Let H be a real Hilbert space and $T : H \to H$ be a α-contraction. If $F : H \to H$ is a m-inverse strongly monotone operator, then the operator $T-F$ has a unique fixed point in H.*

Proof. The operator $I-T$, where I is the identity of H, satisfies

$$\langle (I-T)x-(I-T)y, x-y\rangle \geqslant (1-\alpha)||x-y||^2 \quad \text{for all} \quad x,y \in H.$$

Indeed, using the Schwartz inequality, we have

$$\begin{aligned}\langle (I-T)x-(I-T)y, x-y\rangle &= ||x-y||^2 - \langle Tx-Ty, x-y\rangle \\ &\geqslant ||x-y||^2 - ||Tx-Ty|| \cdot ||x-y|| \geqslant (1-\alpha)||x-y||^2\end{aligned}$$

for all $x,y \in H$ $(1-\alpha > 0)$.

The operator F satisfies $\langle Fx-Fy, x-y\rangle \geqslant 0$ and $\|Fx-Fy\| \leqslant \frac{1}{m}||x-y||$ for all $x,\ y \in H$.

Let $A : H \to H$ be the operator defined by $Au = (I-T+F)u$. We obtain for all $x,\ y \in H$

$$||Ax-Ay|| \leqslant \left(1+\alpha+\frac{1}{m}\right)||x-y|| = \frac{1+m+m\alpha}{m}||x-y||$$

and

$$\begin{aligned}\langle Ax-Ay, x-y\rangle &= \langle (I-T)x-(I-T)y, x-y\rangle + \langle Fx-Fy, x-y\rangle \\ &\geqslant \langle (I-T)x-(I-T)y, x-y\rangle \geqslant (1-\alpha)||x-y||^2.\end{aligned}$$

Now let us define, for $\gamma > 0$, the operator

$$S_\gamma : H \to H$$

given by

$$S_\gamma u = (I-\gamma A)u.$$

We have

$$\begin{aligned}||S_\gamma x - S_\gamma y||^2 &= \langle x-\gamma Ax-(y-\gamma Ay), x-\gamma Ax-(y-\gamma Ay)\rangle \\ &= ||x-y||^2 - 2\gamma\langle Ax-Ay, x-y\rangle + \gamma^2||Ax-Ay||^2 \\ &\leqslant \left[1-2\gamma(1-\alpha)+\gamma^2\frac{(1+m+m\alpha)^2}{m^2}\right]\cdot||x-y||^2,\end{aligned}$$

so

$$||S_\gamma x - S_\gamma y|| \leqslant \sqrt{1-2\gamma(1-\alpha)+\gamma^2\frac{(1+m+m\alpha)^2}{m^2}}\cdot||x-y||,$$

for all $x,\ y \in H$. Further, remark that if

$$\gamma \in \left(0, \frac{2(1-\alpha)m^2}{(1+m+m\alpha)^2}\right),$$

then S_γ is a $\sqrt{1-2\gamma(1-\alpha)+\gamma^2\frac{(1+m+m\alpha)^2}{m^2}}$-contraction, because

$$\sqrt{1-2\gamma(1-\alpha)+\gamma^2\frac{(1+m+m\alpha)^2}{m^2}} < 1$$

and consequently, applying the Banach fixed point theorem, S_γ has a unique fixed point in H. In other words, there exists a unique element $u^* \in H$ such that

$$u^* = S_\gamma u^*,$$

which is succesive equivalent with

$$u^* = (I - \gamma A)u^* \Leftrightarrow u^* = u^* - \gamma A u^* \Leftrightarrow Au^* = 0.$$

Further,

$$Au^* = 0 \Leftrightarrow (I - T + F)u^* = 0 \Leftrightarrow u^* = (T - F)u^*,$$

thus u^* is the unique fixed point of $T - F$ and the proof of Theorem 2.2 is complete. □

Remark that, if $m > 1$, then the operators T and F are contractions, the operator $T - F$ has a unique fixed point, but it is not necessary that $\alpha + \frac{1}{m} < 1$.

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Dinu Teodorescu
Valahia University of Târgovişte
Department of Mathematics
Bd. Unirii 18, 130082 Târgovişte, Romania
e-mail: dinu.teodorescu@valahia.ro,
dteodorescu2003@yahoo.com