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HYPONORMAL TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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Abstract. A Hilbert space operator is hyponormal if $T^*T - TT^*$ is positive. We consider hyponormality of Toeplitz operators on the Bergman space with a symbol in the class of $f+\overline{g}$ where f is a monomial and g is a polynomial. We give sufficient conditions for hyponormality in this case.

1. Introduction

Let U denote the unit disk, dA the area measure on the plane. The Bergman space L_a^2 is the Hilbert space of analytic functions on U such that $\int_U |f|^2 dA < \infty$ and $L^\infty(U)$ is the space of bounded measurable functions on U. If P denotes the orthogonal projection of $L^2(U,dA)$ onto L_a^2, the Toelpitz operators on the Bergman space are defined by $T_f(k) = P(fk)$ for f bounded measurable and k in L_a^2. Hankel operators on the Bergman space are defined by $H_f(k) = (I-P)(fk)$ where f and k are as before. Basic properties of the Bergman space and their operators can be found in [16]. In this work we consider the hyponormality of Toeplitz operators on the Bergman space. More specifically we give sufficient conditions for hyponormality of Toeplitz operators with a symbol of the form $f+\overline{g}$ where f is a monomial and g is a polynomial. We begin by recalling some general properties relevant to our problem.

2. Some general properties

We list some well known properties of Topeplitz operators on the Bergman space (see [2], [3], [16]).

We assume f,g are in $L^\infty(U)$. Then we have

1) $T_{f+g} = T_f + T_g$
2) $T_f^* = T_{\overline{f}}$
3) $T_{\overline{f}}T_g = T_{\overline{f}g}$ if f or g analytic

Using these properties enables us to describe hyponormality in more than one form.

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PROPOSITION 1. *Let f, g be bounded and analytic on U. Then the following are equivalent*

i) $T_{f+\overline{g}}$ *is hyponormal*

ii) $H_{\overline{g}}^* H_{\overline{g}} \leqslant H_{\overline{f}}^* H_{\overline{f}}$

iii) $||(I-P)(\overline{g}k)|| \leqslant ||(I-P)(\overline{f}k)||$ *for any* k *in* L_a^2

iv) $||\overline{g}k||^2 - ||P(\overline{g}k)||^2 \leqslant ||\overline{f}k||^2 - ||P(\overline{f}k)||^2$ *for any* k *in* L_a^2

v) $H_{\overline{g}} = KH_{\overline{f}}$ *where* K *is of norm less than or equal to one.*

Proof. Only $ii) \Rightarrow v)$ needs to be proved and this is a well known lemma ([8]). □

The following lemma is needed. We will omit its proof ([15]).

LEMMA 2. *Let* $f = \sum_0^\infty a_n z^n$ *be bounded and analytic on* U. *The matrix of the operator* $H_{\overline{f}}^* H_{\overline{f}}$ *with respect to orthonormal basis* $\left\{\sqrt{n+1}z^n, n \geqslant 0\right\}$ *is given by:*

$$\lambda_{i,j} = \sum_{\substack{m \geqslant j-i \\ m \geqslant 0}} a_{m+i-j}\overline{a_m}\frac{\sqrt{i+1}\sqrt{j+1}}{i+m+1} - \sum_{\substack{i-j \leqslant m \leqslant i \\ 0 \leqslant m}} a_m \overline{a_{m+j-i}} \frac{i-m+1}{\sqrt{i+1}\sqrt{j+1}}$$

LEMMA 3. *The matrix of* $H_{\overline{z^q}}^* H_{\overline{z^q}}$, *where* q *is a positive integer, is a diagonal matrix where the diagonal term is given by:* $D_{i,i} = \begin{cases} \frac{i+1}{i+q+1} & \text{if } q > i \\ \frac{q^2}{(i+q+1)(i+1)} & \text{if } q \leqslant i \end{cases}$.

3. The results

In this section we give sufficient conditions for hyponormality. We set $E = H_{\overline{g}}^* H_{\overline{g}}$ and $C = H_{\overline{z^q}}^* H$, where g is bounded analytic on U and q a positive integer.

LEMMA 4. *If* $||C^{-1/2}EC^{-1/2}|| \leqslant 1$ *then* $T_{z^q+\overline{g}}$ *is hyponormal.*

Proof. Note that C is a positive operator and has a square root and E is a positive operator. So formally $C^{-1/2}EC^{-1/2}$ exists as a possibly unbounded positive operator. Moreover $||C^{-1/2}EC^{-1/2}|| \leqslant 1$ leads to $C^{-1/2}EC^{-1/2} \leqslant I$ which implies $E \leqslant C$ and thus $T_{z^q+\overline{g}}$ is hyponormal by ii) of proposition1. □

In what follows we set $g = \sum_q^r \alpha_n z^n$. We will show the following theorem:

THEOREM 5. *If* $|g'| \leqslant 1$ *on* ∂U *then* $T_{z^q+\overline{g}}$ *is hyponormal.*

The plan of the proof of the theorem is as follows: we set $A = C^{-1/2}EC^{-1/2}$. By the previous lemma it is enough to show $\|A\| \leqslant 1$. We define an operator G (a modification of A) the norm of which can be estimated. Under the assumption of positivity of G we show that $|g'| \leqslant 1$ on ∂U implies $\|G\| \leqslant 1/q^2$. Then we define an operator G' (a finite rank perturbation of G) which satisfies $A \leqslant G'$. From the definition of G' and the previous inequality we will deduce the positivity of G and the estimate $\|G'\| \leqslant 1$. This leads to hyponormality by the previous lemma.

The proofs rely on matrices. We begin by recalling that the matrix of A is given by $a_{i,j} = d_i d_j \lambda_{i,j}$ where $d_i = \frac{1}{\sqrt{D_{i,i}}}$ and $\lambda_{i,j}$ is as in Lemma2. From the expression of $\lambda_{i,j}$ we get

$$a_{i,i+p} = \left(\sum_{m \geqslant p+q}^{r} \alpha_{m-p} \overline{\alpha_m} \frac{\sqrt{i+1}\sqrt{i+p+1}}{i+m+1} - \sum_{\substack{m \geqslant q \\ i \geqslant m}}^{r-p} \alpha_m \overline{\alpha_{m+p}} \frac{i-m+1}{\sqrt{i+1}\sqrt{i+p+1}} \right) d_i d_{i+p} \tag{1}$$

where $p \leqslant r-q$ (a banded matrix of band width $2(r-q)+1$).

LEMMA 6. *For $i \geqslant r-p$ we have:*

$$a_{i,i+p} = \frac{1}{q^2} \sum_{l \geqslant p+q}^{r} \alpha_{l-p} \overline{\alpha_l} (l-p) l \frac{\sqrt{i+q+1}\sqrt{i+p+q+1}}{i+l+1}.$$

Proof. In this case $\sum_{\substack{m \geqslant q \\ i \geqslant m}}^{r-p} \alpha_m \overline{\alpha_{m+p}} \frac{i-m+1}{\sqrt{i+1}\sqrt{i+p+1}} = \sum_{l \geqslant p+q}^{r} \alpha_{l-p} \overline{\alpha_l} \frac{i-l+p+1}{\sqrt{i+1}\sqrt{i+p+1}}$ (set $m = l-p$). Now set $m = l$ in the first sum in (1) and compute. $\square$

Define G to be the operator with matrix

$$b_{i,i+p} = \frac{1}{q^2} \sum_{l \geqslant p+q}^{r} \alpha_{l-p} \overline{\alpha_l} (l-p) l \frac{\sqrt{i+q+1}\sqrt{i+p+q+1}}{i+l+1}, \quad i \geqslant 0$$

and $b_{i+p,i} = \overline{b_{i,i+p}}$ (a banded matrix with bandwidth $2(r-q)+1$).

Notice that for $i \geqslant r-p$ we have $a_{i,i+p} = b_{i,i+p}$. To obtain an estimate of $\|G\|$ the following partially defined matrices (assume all are $n \times n$ with $n \geqslant r$) will be needed.

Define M_1 as follows:

$$m^1_{i,i+p} = \frac{1}{q^2} \frac{\sqrt{i+q+1}\sqrt{i+p+q+1}}{i+p+q+1},$$
$$m^1_{i+p,i} = m^1_{i,i+p}$$

where $0 \leqslant p \leqslant r-q$ (a banded matrix of band width $2(r-q)+1$).

For $2 \leqslant s \leqslant r-q+1$ define M_s as follows:

$$m^s_{i,i+p} = \frac{1}{q^2} \left(\frac{\sqrt{i+q+1}\sqrt{i+p+q+1}}{i+p+q+s-1} - \frac{\sqrt{i+q+1}\sqrt{i+p+q+1}}{i+p+q+s} \right),$$
$$m^s_{i+p,i} = m^s_{i,i+p}$$

where $0 \leqslant p \leqslant r-q-s+1$ (a banded matrix of band width $2(r-q-s+1)+1$).

Note that M_{r-q+1} is a diagonal matrix. To find positive extensions of the matrices M_s $(1 \leqslant s \leqslant r-q+1)$ we need a slight generalization of a theorem of Dym and Gohberg ([11], Theorem 6.1) on positive definite extensions of partially defined matrices.

LEMMA 7. *Suppose the entries $b_{i,j}$ for $|i-j| \leqslant k$ are specified in an $n \times n$ matrix B $(n \geqslant k+2)$ and suppose that every principal $(k+1)\times(k+1)$ submatrix of B is positive then there is a positive matrix $\widetilde{B}$ such that $\widetilde{b_{i,j}} = b_{i,j}$ for $|i-j| \leqslant k$.*

Using this lemma we can find extensions of the matrices M_s.

LEMMA 8. *For $1 \leqslant s \leqslant r-q+1$, the matrix M_s has a positive extension $\widetilde{M_s}$.*

Proof. By the previous lemma it is enough to show that the principal submatices are positive. First we consider the case $s=1$.

For any $(r-q+1)\times(r-q+1)$ principal submatrix of M_1, denoted by M_1^c, we can write $M_1^c = L_1 \circ F_1$ where $\circ$ is the Hadamard product of matrices and L_1 and F_1 defined as follows:

$$L_1(i,i+p) = \frac{1}{q^2(i+p+q+1)}, \quad 0 \leqslant p \leqslant r-q$$

$$L_1(i+p,i) = L_1(i,i+p), \quad i_0 \leqslant i \leqslant i_0+r-q.$$

The $(r-q+1)\times(r-q+1)$ matrix F_1 is given by:

$$F_1(i,i+p) = \sqrt{i+q+1}.\ \sqrt{i+p+q+1}$$

$$F_1(i+p,i) = F_1(i,i+p), \quad p \leqslant r-q, \quad i_0 \leqslant i \leqslant i_0+r-q$$

where i_0 is the index of the first element on the diagonal of M_1^c. Clearly L_1 is an L-shaped matrix, so by [13, Lemma 4] L_1 is positive. F_1 is of rank one and in fact we have $F_1 = X_1X_1^*$ where X_1 is the $(r-q+1)\times 1$ matrix $X_1 = \begin{pmatrix} \sqrt{i_0+q+1} \\ \vdots \\ \sqrt{i_0+r+1} \end{pmatrix}$, so M_1^c as a Hadamard product of positive matrices, is positive [13, Theorem 7.5.3 p. 458]. For $1 < s \leqslant r-q+1$ one can write for any $(r-q-s+2)\times(r-q-s+2)$ principal submatrix of M_s an equality similar to the case $s=1$, $M_s^c = L_s \circ F_s$ where

$$L_s(i,i+p) = \frac{1}{q^2(i+p+q+s)(i+p+q+s-1)}$$

$$L_s(i+p,i) = L_s(i,i+p), \quad i_0 \leqslant i \leqslant i_0+r-q-s+1$$

where i_0 is as defined before. L_s is a Hadamard product of two L-shaped matrices so it is positive. $F_s = X_sX_s^*$ where X_s is the $(r-q-s+2)\times 1$ matrix given by: $X_s = \begin{pmatrix} \sqrt{i_0+q+1} \\ \vdots \\ \sqrt{i_0+r-s+2} \end{pmatrix}$ and the rest of the argument is similar to the case $s=1$. $\square$

The following computational lemma will be needed in the sequel.

LEMMA 9. *Given two sets of complex numbers* $\{A_l, u \leqslant l \leqslant v\}$ *and* $\{B_l, u \leqslant l \leqslant v\}$ *where* u, v *are fixed integers such that* $1 \leqslant u \leqslant v$ *the following equality holds:*

$$\sum_u^v A_l B_l = A_u \sum_u^v B_l - \sum_u^{v-1} (A_l - A_{l+1}) (\sum_{m=l+1}^{m=v} B_m).$$

The matrix of the Toeplitz operator on the Hardy space with symbol $|g'|^2$, where g is as in Theorem5 is given by

$$(T_{|g'|2})_{i,i+p} = \sum_{p+q}^{r} l(l-p)\overline{\alpha_l}\alpha_{l-p}, (T_{|g'|2})_{i,i+p} = (T_{|g'|2})_{i+p,i}$$

(a banded matrix of band width $2(r-q)+1$).

If $g_s = \sum_{q+s-1}^{r} \alpha_l z^l$ and $1 \leqslant s \leqslant r-q+1$ then the matrix of $T_{|g_s'|^2}$ is given by

$$(T_{|g_s'|^2})_{i,i+p} = \sum_{p+q+s-1}^{r} l(l-p)\overline{\alpha_l}\alpha_{l-p}, (T_{|g_s'|^2})_{i,i+p} = (T_{|g_s'|^2})_{i+p,i}$$

(a banded matrix of band width $2(r-q-s+1)+1$).

In the last equality p satisfies $p \leqslant r-q-s+1$. If $p > r-q-s+1$ then $(T_{|g_s'|^2})_{i,i+p} = 0$. Since the band width of the matrix of $T_{|g_s'|^2}$ is the same as the band width of M_s, we have $T_{|g_s'|^2} \circ M_s = T_{|g_s'|^2} \circ \widetilde{M_s}$ $(*)$. For a fixed integer i and a fixed integer p satisfying $p \leqslant r-q$, where r and q are as in theorem 5, choose $A_l = \frac{\sqrt{i+q+1}\sqrt{i+p+q+1}}{q^2(i+l+1)}$ and $B_l = l(l-p)\overline{\alpha_l}\alpha_{l-p}$.

Using the previous lemma, lemma7, the definitions of the matrices M_s, and $(*)$ we see that the following equality holds:

$$G = (T_{|g'|^2} \circ \widetilde{M_1} - (T_{|g_2'|^2} \circ \widetilde{M_2} + \ldots + T_{|g_{r-q+1}'|^2} \circ \widetilde{M_{r-q+1}}))$$

Denote by M^c the upper left corner of size n of a matrix M. Then we have

$$G^c = (T_{|g'|^2} \circ \widetilde{M_1} - (T_{|g_2'|^2} \circ \widetilde{M_2} + \ldots + T_{|g_{r-q+1}'|^2} \circ \widetilde{M_{r-q+1}}))^c \tag{2}$$

We are now ready to find an upper bound of the norm of G.

LEMMA 10. *If G is positive, then* $|g'| \leqslant 1$ *on* ∂U *implies* $\|G\| \leqslant \frac{1}{q^2}$

Proof. It is enough to show that an upper left corner of arbitrary size n satisfies the estimate. The Hadamard product of positive matrices is a positive matrix and the sum of positive matrices is a positive matrix. It follows from (2) that $0 \leqslant G^c \leqslant (T_{|g'|^2} \circ \widetilde{M_1})^c$.

Since the diagonal term of $\widetilde{M_1}$ is smaller than $\frac{1}{q^2}$, we see by a theorem on completely positive maps [14, Proposition 3.4] that $\|G^c\| \leqslant \sup\left|m^1_{i,i}\right| |g'|^2$. It follows that $|g'| \leqslant 1$ implies that $\|G^c\| \leqslant \frac{1}{q^2}$. This being true for any upper left corner of the matrix of G the lemma is proved. □

Now recall that the matrix of A is given by (from (1)):

$$a_{i,i+p} = \left(\sum_{l \geqslant p+q}^{r} \alpha_{l-p}\overline{\alpha_l} \frac{\sqrt{i+1}\sqrt{i+p+1}}{i+l+1} - \sum_{\substack{l \geqslant p+q \\ i \geqslant l-p}}^{r-p} \alpha_{l-p}\overline{\alpha_l} \frac{i-l+p+1}{\sqrt{i+1}\sqrt{i+p+1}} \right) d_i d_{i+p}$$

where d_i is as before, and notice that $b_{i,i+p} = a_{i,i+p}$ for $i \geqslant r-p$. Define an operator C' with a diagonal matrix given by:

$$c'_{i,i} = \begin{cases} \frac{q}{i+1} & \text{if } i \leqslant q-1 \\ 1 & \text{if } i \geqslant q \end{cases}.$$

It is obvious that $\|C'\| \leqslant q$. Set $G' = C'GC'$ we have

$$G' - A = C'GC' - A = C'(G - (C')^{-1}A(C')^{-1})C'$$

It is easy to see that the matrix of $(C')^{-1}A(C')^{-1}$ is given by

$$\left(\sum_{l \geqslant p+q}^{r} \alpha_{l-p}\overline{\alpha_l} \frac{\sqrt{i+1}\sqrt{i+p+1}}{i+l+1} - \sum_{\substack{l \geqslant p+q \\ i \geqslant l-p}}^{r-p} \alpha_{l-p}\overline{\alpha_l} \frac{i-l+p+1}{\sqrt{i+1}\sqrt{i+p+1}} \right) e_i e_{i+p}$$

where $e_i = \frac{1}{q}\sqrt{i+1}\sqrt{i+q+1}$. We note that $e_i = d_i$ if $i \geqslant q$.
Using this we obtain

LEMMA 11. *The following inequality holds* $0 \leqslant A \leqslant G'$

Proof. Writing

$$\begin{aligned} b_{i,i+p} &= \frac{1}{q^2} \sum_{l \geqslant p+q}^{r} \alpha_{l-p}\overline{\alpha_l}(l-p)l \frac{\sqrt{i+q+1}\sqrt{i+p+q+1}}{i+l+1} \\ &= \left(\sum_{l \geqslant p+q}^{r} \alpha_{l-p}\overline{\alpha_l} \frac{\sqrt{i+1}\sqrt{i+p+1}}{i+l+1} - \sum_{l \geqslant p+q}^{r} \alpha_{l-p}\overline{\alpha_l} \frac{i-l+p+1}{\sqrt{i+1}\sqrt{i+p+1}} \right) e_i e_{i+p} \end{aligned}$$

we see that $G - (C')^{-1}A(C')^{-1}$ has a matrix given by:

$$\begin{aligned} h_{i,i+p} &= \left(\sum_{\substack{l \geqslant p+q \\ l-p > i}}^{r} \alpha_{l-p}\overline{\alpha_l} \frac{l-p-i-1}{\sqrt{i+1}\sqrt{i+p+1}} \right) e_i e_{i+p} \\ &= \frac{1}{q^2} \sum_{\substack{l-p > i \\ l \geqslant p+q}}^{r} \alpha_{l-p}\overline{\alpha_l}(l-p-i-1)\sqrt{i+q+1}\sqrt{i+p+q+1} \end{aligned}$$

where $p \leqslant r-q$ and $h_{i,i+p} = \overline{h_{i+p,i}}$. Define an $(r+1)\times(r+1)$ matrix T by:

$$t_{i,j} = \frac{1}{q}\sqrt{j}\sqrt{i+q+1}\alpha_{i+j+1}$$

where $\alpha_s = 0$ if $s > r$. Then the matrix of $V = TT^*$ is given by

$$v_{i,j} = \sum_{k=0}^{r} t_{i,k}t^*_{k,j} = \sum_{k=0}^{r} t_{i,k}\overline{t_{j,k}}.$$

We set $j = i+p$ to get

$$v_{i,i+p} = \frac{1}{q^2}\sum_{k=0}^{r}\sqrt{k}\sqrt{i+q+1}\sqrt{i+p+q+1}\sqrt{k}\alpha_{i+k+1}\overline{\alpha_{i+p+k+1}}.$$

Put $l = i+p+k+1$ to get

$$v_{i,i+p} = \frac{1}{q^2}\sqrt{i+q+1}\sqrt{i+p+q+1}\sum_{\substack{l \geqslant p+q \\ l-p>i}}^{r}(l-p-i-1)\alpha_{l-p}\overline{\alpha_l}.$$

We see that $v_{i,i+p} = h_{i,i+p}$ and it follows that $G-(C')^{-1}A(C')^{-1} \geqslant 0$ and $0 \leqslant A \leqslant G'$. □

We can now prove Theorem 5.

Proof. By lemma 4 it is enough to show $\|A\| \leqslant 1$. From the definition of G' and the previous lemma we see that G is positive. Consequently we have $\|G\| \leqslant \frac{1}{q^2}$ by lemma 10. It follows that $\|G'\| \leqslant \|C'\|^2\|G\| \leqslant q^2\|G\| \leqslant 1$ and $\|A\| \leqslant 1$. □

In the particular case $q = 1$ we have a necessary and sufficient condition for hyponormality.

COROLLARY 12. *Let* $g = \sum_1^r \alpha_n z^n$. *Then* $T_{z+\overline{g}}$ *is hyponormal if and only if* $|g'| \leqslant 1$ *on* ∂U.

Proof. Only the necessary condition needs to be proved and this a particular case of a general theorem [15] (see also theorem 5 in [1]). □

REMARK 13. It easy to see that an operator R is hyponormal if and only if $R+\lambda I$ is hyponormal, where λ is any complex number.Thus in the case of $T_{z^q+\overline{g}}$ it is enough to consider the case where $g(0) = 0$.

We also have the following theorem.

THEOREM 14. *Let* $g=\sum_{1}^{q}\alpha_n z^n$. *Then* $|g'|\leqslant\sqrt{\frac{2}{q+1}}$ *on* ∂U *implies* $T_{z^q+\overline{g}}$ *is hyponormal.*

Proof. We give an outline of the proof since the method used is the same as the one used to prove theorem 5. We define an operator G_1 by its matrix

$$a_{i,i+p}=\frac{1}{q^2}\sum_{l=1}^{l=q-p}\alpha_l\overline{\alpha_{l+p}}l(l+p)\frac{\sqrt{i+q+1}\sqrt{i+p+q+1}}{i+l+p+1},\ \ a_{i+p,i}=\overline{a_{i,i+p}}$$

(a banded matrix of bandwidth $2(q-1)+1$). As in the proof of theorem 5 we define partially matrices and find positive extensions of these matrices. Using an identity similar to (2) we show $|g'|\leqslant\sqrt{\frac{2}{q+1}}$ on ∂U implies $\|G_1\|\leqslant\frac{1}{q^2}$. We also show, as in theorem 5, that $G_1-C'^{-1}A_1C'^{-1}\geqslant 0$, where $A_1=C^{-1/2}H_{\overline{g}}^{*}H_{\overline{g}}C^{-1/2}$ and C and C' are as before. This leads to $\|A_1\|\leqslant 1$. □

We conclude with a corollary.

COROLLARY 15. *Let* $g=\sum_{q}^{r}\alpha_n z^n$ *and* $f=\sum_{1}^{q}\alpha_n z^n$. *Assume that* $|g'|\leqslant 1$, $|f'|\leqslant\sqrt{\frac{2}{q+1}}$ *on* ∂U, *and* β,γ *are two complex numbers satisfying* $|\beta|+|\gamma|\leqslant 1$. *If* $h=\beta f+\gamma g$, *then* $T_{z^q+\overline{h}}$ *is hyponormal.*

Proof. Use theorem 5 and theorem 14 and the fact that $W_{z^q}=\{\varphi$ analytic and bounded on U such that $T_{z^q+\overline{\varphi}}$ is hyponormal is convex and balanced ([15]). □

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REFERENCES

[1] P. AHERN AND Z. CUCKOVIC, *A mean value inequality with applications to Bergman space operators*, Pac. J. Math, vol. **173**, no. 2 (1996), 295–305.

[2] S. AXLER, *Bergman spaces and their operators*, in Surveys of Some Recent results in Operator Theory, vol. **1**, Pitman Research Notes in Math, vol. **171**, Longman, Harlow, 1988, pp. 1–50.

[3] S. AXLER, *Bergman space, the Bloch space and commutators of multiplication operators*, Duke Math. J. **53** (1986), pp. 315–332.

[4] J. CONWAY, *Subnormal Operators*, Pitman, Boston, 1981.

[5] C. COWEN, *Hyponormal and subnormal Toeplitz operators*, in Surveys of Some Results in Operator Theory, vol. **1**, Pitman Research Notes in Math, vol. **171**, Longman, Harlow, 1988, pp. 155–167.

[6] C. COWEN, *Hyponormality of Toeplitz operatos*, Proc. Amer. Math. Soc. **103** (1988), pp. 809–812.

[7] C. COWEN, *An application of Hadamard multiplication to operators on weighted Hardy spaces*, Linear Alg. Appl. **133** (1990), pp. 21–32.

[8] R. DOUGLAS, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, 1972.

[9] R. DOUGLAS, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), pp. 21–32.

[10] P. DUREN, *Theory of* H^p *spaces*, Academic press, New York, 1970.

[11] H. DYM AND I. GOHBERG, *Extensions of band matrices with band inverses*, Linear. Alg. Appl. **36** (1981), pp. 1–24.

[12] U. GRENANDER AND G. SZEGO, *Toeplitz Forms and Their Applications*, University of California Press, Berkeley and Los Angeles, 1958.
[13] R. HORN AND C. JOHNSON, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.
[14] V. PAULSEN,, *Completely Bouded Maps and Dilations*, Pitman Research Notes in Math, vol. **146**, Wiley, New York, 1986.
[15] H. SADRAOUI, *Hyponormality of Toeplitz operators and composition operators*, Thesis, Purdue University, 1992.
[16] K. ZHU, *Operator Theory in Function Spaces*, Dekker, New York, 1990.

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