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KSGNS CONSTRUCTION FOR τ–MAPS ON S–MODULES AND $\mathfrak{K}$–FAMILIES

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Abstract. We introduce S-modules, which generalizes the notion of Krein C^*-modules and where a fixed unitary replaces the symmetry of Krein C^*-modules. The representation theory on S-modules is explored and for a given $*$-automorphism α on a C^*-algebra the KSGNS construction for α-completely positive maps is illustrated. An extention of this construction for τ-maps is also achieved, when τ is an α-completely positive map. We prove decomposition theorems for α-CPD-kernels and $\mathfrak{K}$-families.

1. Introduction

A *symmetry* on a Hilbert space is a bounded operator J such that $J = J^* = J^{-1}$. A Hilbert space along with a symmetry, forms a *Krein space* where the symmetry induces an indefinite inner-product on the space. Dirac and Pauli were among the pioneers to explore the quantum field theory using Krein spaces.

In quantum field theory one encounters Wightman functionals which are positive linear functionals on a Borchers algebra (cf. [8]). In the massless or the guage quantum field theory, Strocchi showed that, both the locality and the positivity cannot be assumed together in a model. The axiomatic field theory motivates theoretical physicists to keep the locality assumption and sacrifice the positivity by considering indefinite inner products (cf. [7]), and more specifically Krein spaces, in the gauge field theory. In this context Jakobczyk defined the α-positivity, where α is a $*$-automorphism of a Borchers algebra, in [12] and derived a reconstruction theorem for Strocchi-Wightman states.

DEFINITION 1. Let $\mathscr{B}$ be a $*$-subalgebra of a unital $*$-algebra $\mathscr{A}$ containing the unit. Assume $P:\mathscr{A}\to\mathscr{B}$ to be a conditional expectation (i.e. P is a linear map preserving the unit and the involution, such that $P(bab') = bP(a)b'$ for each $a\in\mathscr{A}$; $b,b'\in\mathscr{B}$). A Hermitian linear functional τ defined on $\mathscr{A}$ is called a *P-functional* (cf. [2]) if the following holds:

(i) $\tau\circ P = \tau$,

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(ii) $2\tau(P(a)^*P(a)) \geqslant \tau(a^*a)$ for all $a \in \mathscr{A}$.

If we define a linear mapping $\alpha : \mathscr{A} \to \mathscr{A}$ by $\alpha(a) = 2P(a) - a$ for each $a \in \mathscr{A}$, then the P-functional τ satisfies

$$\tau(\alpha(a)\alpha(a')) = \tau(aa') \text{ and } \tau(\alpha(a)^*a) \geqslant 0 \text{ for all } a, a' \in \mathscr{A}.$$

Thus P-functionals generalize α-positivity. The Gelfand-Naimark-Segal (GNS) construction for states, is a fundamental result in operator theory, which illustrates how using a state on a C^*-algebra we can obtain a cyclic representation of that C^*-algebra on a Hilbert space. Antoine and Ota [2] proved that using P-functionals one can obtain unbounded GNS representations of a $*$-algebra on a Krein space.

The completely positive maps are crucial to the study of the classification of C^*-algebras, the classification of E_0-semigroups, etc. The Stinespring theorem characterizes completely positive maps, and if we consider the special case where the completely positive maps are states, then for them the Stinespring theorem gives the GNS construction. Motivated by α-positivity and P-functional, J. Heo et al. introduced the concept of α-completely positive maps in [10], where α is a bounded Hermitian map from a C^*-algebra to itself such that $\alpha^2 = id$ (i.e., order of α is 2), and did a KSGNS type construction on certain module called the Krein C^*-module for any α-completely positive map. U. C. Ji et al. did a KSGNS construction in [13] for α-completely positive maps where α is a $*$-automorphism on a C^*-algebra such that $\alpha^2 = id$. We extend this study of α-completely positive maps for any $*$-automorphism α, not necessarily of order 2, and obtain a KSGNS type construction on a bigger class of modules called S-modules. To illustrate KSGNS construction we first need to recall some notions:

DEFINITION 2. Let E and F be Hilbert $\mathscr{A}$-modules over a C^*-algebra $\mathscr{A}$. For a given map $S : E \to F$ if there exists a map $S' : F \to E$ such that

$$\langle S(x), y\rangle = \langle x, S'(y)\rangle \text{ for all } x \in E, y \in F,$$

then S' is unique for S, and we say S is *adjointable* and denote S' by S^*. Every adjointable map $S : E \to F$ is *right* $\mathscr{A}$*-linear*, i.e., $S(xa) = S(x)a$ for all $x \in E$, $a \in \mathscr{A}$. The symbol $\mathscr{B}^a(E,F)$ denotes the collection of all adjointable maps from E to F. We use $\mathscr{B}^a(E)$ for $\mathscr{B}^a(E,E)$. The *strict topology* on $\mathscr{B}^a(E)$ is the topology induced by the seminorms $a \mapsto \|ax\|$, $a \mapsto \|a^*y\|$ for each $x, y \in E$.

Kasparov obtained the following theorem, called *Kasparov-Stinespring-Gelfand-Naimark-Segal (KSGNS) construction* (cf. [14]), which is a dilation theorem for strictly continuous completely positive maps:

THEOREM 1. *Let $\mathscr{B}$ and $\mathscr{C}$ be C^*-algebras. Assume E to be a Hilbert $\mathscr{C}$-module and $\tau : \mathscr{B} \to \mathscr{B}^a(E)$ to be a strictly continuous completely positive map. Then there is a Hilbert $\mathscr{C}$-module F with a nondegenerate $*$-homomorphism $\pi : \mathscr{B} \to \mathscr{B}^a(F)$ and $V \in \mathscr{B}^a(E,F)$ such that $\overline{\operatorname{span}}\, \pi(\mathscr{B})VE = F$ and*

$$\tau(b) = V^*\pi(b)V \text{ for all } b \in \mathscr{B}.$$

Szafraniec [25] obtained a dilation theorem, which extends the Sz-Nagy's principal theorem [19], for certain C^*-algebra valued positive definite functions defined on $*$-semigroups. The KSGNS construction is a special case of Szafraniec's dilation theorem.

Let $(E,\langle\cdot,\cdot\rangle)$ be a Hilbert C^*-module over a C^*-algebra $\mathscr{A}$ and let J be a fundamental symmetry on E, i.e., J is an invertible adjointable map on E such that $J=J^*=J^{-1}$. Define an $\mathscr{A}$-valued indefinite inner product on E by

$$[x,y] := \langle Jx,y\rangle \text{ for all } x,y\in E. \tag{1}$$

In this case we say $(E,\mathscr{A},J)$ is a *Krein $\mathscr{A}$-module* or *Krein C^*-module over $\mathscr{A}$*. If $\mathscr{A}=\mathbb{C}$, then $(E,\mathbb{C},J)$ is a *Krein space* and in addition if J is the identity operator, then it becomes a Hilbert space. In the definition of Krein spaces if we replace the symmetry J by a unitary, then we get *S-spaces*. The two sided shift is unitary and therefore it is normal. Szafraniec introduced the notion of S-spaces in [24] and proved that the closure of the two-sided weighted shift is S-normal. Phillipp, Szafraniec and Trunk [18] investigated invariant subspaces of selfadjoint operators in Krein spaces by using results obtained through a detailed analysis of S-spaces. We introduce the notion of S-modules below:

DEFINITION 3. Let $(E,\langle\cdot,\cdot\rangle)$ be a Hilbert C^*-module over a C^*-algebra $\mathscr{A}$ and let U be a unitary on E, i.e., U is an invertible adjointable map from E to E such that $U^*=U^{-1}$. Then we can define an $\mathscr{A}$-valued sesquilinear form by

$$[x,y] := \langle x,Uy\rangle \text{ for all } x,y\in E. \tag{2}$$

In this case we say $(E,\mathscr{A},U)$ is an *S-module*.

If $U=I$, then $\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]$ coincide for the S-module $(E,\mathscr{A},U)$. In the case when $U=U^*$, the S-module $(E,\mathscr{A},U)$ forms a Krein $\mathscr{A}$-module. The following is the definition of an α-completely positive map which will play an important role in this article:

DEFINITION 4. Let $\mathscr{A}$ be a unital C^*-algebra and $\alpha:\mathscr{A}\to\mathscr{A}$ be a $*$-automorphism, i.e., α is a unital bijective $*$-homomorphism. Let $\mathscr{B}$ be a C^*-algebra and E be a Hilbert $\mathscr{B}$-module. If $(E,\mathscr{B},U)$ is an S-module, then a map $\tau:\mathscr{A}\to\mathscr{B}^a(E)$ is called *α-completely positive (or α-CP)* if it is a $*$-preserving map such that

(i) $\tau(\alpha(a))=U^*\tau(a)U=\tau(a)$ for all $a\in\mathscr{A}$;

(ii) $\sum\limits_{i,j=1}^{n}\left\langle x_i,\tau(\alpha(a_i^*)a_j)x_j\right\rangle\geqslant 0$ for all $n\geqslant 1$; $a_1,\dots,a_n\in\mathscr{A}$ and $x_1,\dots,x_n\in E$;

(iii) for any $a\in\mathscr{A}$, there is $M(a)>0$ such that

$$\sum_{i,j=1}^{n}\left\langle x_i,\tau(\alpha(a_i^*a^*)aa_j)x_j\right\rangle\leqslant M(a)\sum_{i,j=1}^{n}\left\langle x_i,\tau(\alpha(a_i^*)a_j)x_j\right\rangle$$

for all $n\geqslant 1$; $x_1,\dots,x_n\in E$ and $a_1,\dots,a_n\in\mathscr{A}$.

For an α-CP map we define below certain maps associated to them:

DEFINITION 5. Let $\mathscr{A}$ and $\mathscr{B}$ be unital C^*-algebras, E be a Hilbert $\mathscr{A}$-module and $\alpha : \mathscr{A} \to \mathscr{A}$ be a $*$-automorphism. Let $(E_1, \mathscr{B}, U_1)$ and $(E_2, \mathscr{B}, U_2)$ be S-modules and $\tau : \mathscr{A} \to \mathscr{B}^a(E_1)$ be an α-CP map. A map $T : E \to \mathscr{B}^a(E_1, E_2)$ is called a (α completely positive) τ*-map* if

$$T(x)^* T(y) = \tau(\langle x, y \rangle) \text{ for all } x, y \in E.$$

The dilation theory of τ-maps, where τ is a CP map, has been explored in [5], [22], [23], [13], etc. In [13], for an order two $*$-automorphism α on a C^*-algebra $\mathscr{A}$, the authors did a KSGNS type construction for τ-maps where τ is an α-CP map. This and the study done in [11] are motivations for the approach taken by us to study the representation theory of τ-maps on S-modules in Section 2.

For any Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, let $\mathscr{B}(\mathscr{H}, \mathscr{K})$ denote the space of all bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$. Assume E to be a Hilbert $\mathscr{B}$-module where $\mathscr{B}$ is a von Neumann algebra such that there exist a non-degenarate representation of $\mathscr{B}$ on a Hilbert space $\mathscr{H}$. The interior tensor product $E \bigotimes \mathscr{H}$ is a Hilbert space. For a fixed $x \in E$ we define a bounded linear operator $L_x : \mathscr{H} \to E \bigotimes \mathscr{H}$ by

$$L_x(h) := x \otimes h \text{ for } h \in \mathscr{H}.$$

We have $L_{x_1}^* L_{x_2} = \langle x_1, x_2 \rangle$ for all $x_1, x_2 \in E$. This allows us to identify each $x \in E$ with L_x and thus E is identified with a concrete submodule of $\mathscr{B}(\mathscr{H}, E \bigotimes \mathscr{H})$. We say that E is a *von Neumann* $\mathscr{B}$*-module* or a *von Neumann module over* $\mathscr{B}$ if E is strongly closed in $\mathscr{B}(\mathscr{H}, E \bigotimes \mathscr{H}) \subset \mathscr{B}(\mathscr{H} \oplus (E \otimes \mathscr{H}))$. This notion of von Neumann modules is due to Skeide (cf. [21]). In fact, $a \mapsto a \otimes id_{\mathscr{H}}$ is a representation of $\mathscr{B}^a(E)$ on $E \bigotimes \mathscr{H}$, and therefore it is an isometry. Thus we are allowed to consider $\mathscr{B}^a(E) \subset \mathscr{B}(E \bigotimes \mathscr{H})$ and so $\mathscr{B}^a(E)$ is a von Neumann algebra acting non-degenerately on $E \bigotimes \mathscr{H}$. In [26], we proved a Stinespring type theorem for τ-maps, when $\mathscr{B}$ is any von Neumann algebra and F is any von Neumann $\mathscr{B}$-module. As in [26], in this article too at certain places we work with von Neumann modules instead of Hilbert C^*-modules because all von Neumann modules are self-dual (cf. [21]), and hence they are complemented in all Hilbert C^*-modules which contain them as submodules.

C^*-algebra valued positive definite kernels were defined by Murphy in [16]. In Section 3 we obtain a decomposition theorem for an α-CPD-kernel (see Section 3 for definition), for any $*$-automorphism α on a C^*-algebra, with the help of reproducing kernel S-correspondences. An α-CPD-kernel is a CPD kernel if $\alpha = id$. We obtain a new proof for the factorization theorem for $\mathfrak{K}$-families where $\mathfrak{K}$ is a CPD-kernel. Accardi and Kozyrev, in [1], considered semigroups of CPD-kernels over the set $\Omega = \{0, 1\}$. Barreto, Bhat, Liebscher and Skeide [4] studied several results regarding structure of type I product-systems of Hilbert C^*-modules based on the dilation theory of CPD-kernels over any set Ω. Their approach was based on the Kolmogorov decomposition of a CPD-kernel. Ball, Biswas, Fang and ter Horst [3] introduced the notion of reproducing kernel C^*-correspondences and identified Hardy spaces studied by Muhly-Solel [15] with a reproducing kernel C^*-correspondence for a CPD-kernel which is an analogue of the classical Szegö kernel.

2. KSGNS type construction for τ-maps

Assume $(E_1,\mathscr{B},U_1)$ and $(E_2,\mathscr{B},U_2)$ to be S-modules. For each $T\in\mathscr{B}^a(E_1,E_2)$, there exists an operator $T^\natural\in\mathscr{B}^a(E_2,E_1)$ such that

$$\langle T(x),U_2y\rangle=\langle x,U_1T^\natural(y)\rangle \text{ for all } x\in E_1,\ y\in E_2.$$

In fact, $T^\natural=U_1^*T^*U_2$. Suppose $\mathscr{A}$ is a C^*-algebra and $(E,\mathscr{B},U)$ be an S-module. An algebra homomorphism $\pi:\mathscr{A}\to\mathscr{B}^a(E)$ is called an *U-representation of $\mathscr{A}$ on* $(E,\mathscr{B},U)$ if $\pi(a^*)=U^*\pi(a)^*U=\pi(a)^\natural$, i.e.,

$$[\pi(a)x,y]=[x,\pi(a^*)y] \text{ for all } x,y\in E.$$

The theorems in this section are analogous to Theorem 3.2 of [11] and Theorem 4.4 of [10], and Theorem 2.6 of [13].

THEOREM 2. *Let $\mathscr{A}$ and $\mathscr{B}$ be unital C^*-algebras and $\alpha:\mathscr{A}\to\mathscr{A}$ be a $*$-automorphism. Suppose $(E_1,\mathscr{B},U_1)$ is an S-module. If $\tau:\mathscr{A}\to\mathscr{B}^a(E_1)$ is an α-CP map, then there exist*

(i) a Hilbert $\mathscr{B}$-module E_0 and a unitary U_0 such that $(E_0,\mathscr{B},U_0)$ is an S-module,

(ii) a map $V\in\mathscr{B}^a(E_1,E_0)$ such that $V^\natural=V^*$ and an U_0-representation π_0 of $\mathscr{A}$ on $(E_0,\mathscr{B},U_0)$ satisfying

$$V^*\pi_0(a)^*\pi_0(b)V=V^*\pi_0(\alpha(a)^*b)V \text{ for each } a,b\in\mathscr{A},$$

and

$$\tau(a)=V^*\pi_0(a)V \text{ for all } a\in\mathscr{A}.$$

Proof. Let $\mathscr{A}\bigotimes_{alg}E_1$ be the algebraic tensor product of $\mathscr{A}$ and E_1. Define a map $\langle\cdot,\cdot\rangle:(\mathscr{A}\bigotimes_{alg}E_1)\times(\mathscr{A}\bigotimes_{alg}E_1)\to\mathscr{B}$ by

$$\left\langle\sum_{i=1}^n a_i\otimes x_i,\sum_{j=1}^m a'_j\otimes y_j\right\rangle=\sum_{i=1}^n\sum_{j=1}^m\langle x_i,\tau\left(\alpha\left(a_i^*\right)a'_j\right)y_j\rangle$$

for all $a_1,\dots,a_n;\ a'_1,\dots,a'_m\in\mathscr{A}$ and $x_1,\dots,x_n;\ y_1,\dots,y_m\in E_1$. The condition (ii) of Definition 4 implies that $\langle\cdot,\cdot\rangle$ is a positive definite semi-inner product. Using the Cauchy-Schwarz inequality for positive-definite sesquilinear forms we observe that K is a submodule of $\mathscr{A}\bigotimes_{alg}E_1$ where

$$K:=\left\{\sum_{i=1}^n a_i\otimes x_i\in\mathscr{A}\bigotimes_{alg}E_1\ :\ \sum_{i,j=1}^n\langle x_i,\tau(\alpha(a_i^*)a_j)x_j\rangle=0\right\}.$$

Therefore $\langle\cdot,\cdot\rangle$ induces naturally on the quotient module $\left(\mathscr{A}\bigotimes_{alg} E_1\right)/K$, a $\mathscr{B}$-valued inner product. We denote this induced inner-product also by $\langle\cdot,\cdot\rangle$. Assume that E_0 denote the Hilbert $\mathscr{B}$-module obtained by the completion of $\left(\mathscr{A}\bigotimes_{alg} E_1\right)/K$.

It is easy to check that $(E_0,\mathscr{B},U_0)$ is an S-module, where the unitary U_0 is defined by

$$U_0\left(\sum_{i=1}^n a_i\otimes x_i+K\right)=\sum_{i=1}^n \alpha(a_i)\otimes U_1x_i+K \text{ where } a\in\mathscr{A}, x\in E_1.$$

Indeed, U_0 is a unitary, because for all $a,a'\in\mathscr{A}$ and $x,y\in E_1$ we get

$$\begin{aligned}&\left\langle U_0\left(\sum_{i=1}^n a_i\otimes x_i+K\right),U_0\left(\sum_{j=1}^n a_j\otimes x_j+K\right)\right\rangle\\
&=\sum_{i,j=1}^n\langle\alpha(a_i)\otimes U_1x_i+K,\alpha(a_j)\otimes U_1x_j+K\rangle=\sum_{i,j=1}^n\langle U_1x_i,\tau(\alpha(\alpha(a_i)^*)\alpha(a_j))U_1x_j\rangle\\
&=\sum_{i,j=1}^n\langle x_i,\tau(\alpha(a_i^*)a_j)x_j\rangle=\left\langle\sum_{i=1}^n a_i\otimes x_i+K,\sum_{j=1}^n a_j\otimes x_j+K\right\rangle,\end{aligned}$$

and because U_0 is surjective. Since

$$\begin{aligned}&\left\langle U_0\left(\sum_{i=1}^n a_i\otimes x_i+K\right),\sum_{j=1}^m a'_j\otimes y_j+K\right\rangle\\
&=\sum_{i=1}^n\sum_{j=1}^m\langle\alpha(a_i)\otimes U_1x_i+K,a'_j\otimes y_j+K\rangle=\sum_{i=1}^n\sum_{j=1}^m\langle U_1x_i,\tau(\alpha(\alpha(a_i)^*)a'_j)y_j\rangle\\
&=\sum_{i=1}^n\sum_{j=1}^m\langle x_i,\tau(\alpha(a_i^*)\alpha^{-1}(a'_j))U_1^*y_j\rangle=\left\langle\sum_{i=1}^n a_i\otimes x_i+K,\sum_{j=1}^m\alpha^{-1}(a'_j)\otimes U_1^*y_j+K\right\rangle,\end{aligned}$$

we obtain $U_0^*\left(\sum\limits_{j=1}^m a'_j\otimes y_j+K\right)=\sum\limits_{j=1}^m\alpha^{-1}(a'_j)\otimes U_1^*y_j+K$. Define a map $V:E_1\to E_0$ by

$$Vx:=1\otimes U_1x+K \text{ where } x\in E_1.$$

For each $x\in E_1$ we have

$$\begin{aligned}\|Vx\|^2=\|\langle Vx,Vx\rangle\|&=\|\langle 1\otimes U_1x+K,1\otimes U_1x+K\rangle\|=\|\langle U_1x,\tau(1)U_1x\rangle\|\\
&\leqslant\|\tau(1)\|\|x\|^2.\end{aligned}$$

This implies that V is bounded. For each $a_1,a_2,\ldots,a_n\in\mathscr{A}$ and $x,y_1,y_2,\ldots,y_n\in E_1$ we have

$$\begin{aligned}\left\langle Vx,\sum_{i=1}^n a_i\otimes y_i+K\right\rangle&=\left\langle 1\otimes U_1x+K,\sum_{i=1}^n a_i\otimes y_i+K\right\rangle=\left\langle U_1x,\sum_{i=1}^n\tau(\alpha(1)a_i)y_i\right\rangle\\
&=\left\langle x,\sum_{i=1}^n U_1^*\tau(a_i)y_i\right\rangle=\left\langle x,\sum_{i=1}^n\tau(a_i)U_1^*y_i\right\rangle.\end{aligned}\tag{3}$$

From Lemma 2.8 of [10] there exists $M > 0$ such that

$$(\tau(a_i^*)\tau(a_j)) \leqslant M(\tau(\alpha(a_i^*)a_j)).$$

Thus, for each $a_1, a_2, \dots a_n \in \mathscr{A}$ and $y_1, y_2, \dots y_n \in E_1$ we get

$$\begin{aligned}
\left\| \sum_{i=1}^n \tau(a_i) U_1^* y_i \right\|^2 &= \left\| \left\langle \sum_{i=1}^n \tau(a_i) U_1^* y_i, \sum_{j=1}^n \tau(a_j) U_1^* y_j \right\rangle \right\| \\
&= \left\| \sum_{i=1}^n \sum_{j=1}^n \langle U_1^* y_i, \tau(a_i^*)\tau(a_j) U_1^* y_j \rangle \right\| \\
&\leqslant M \left\| \sum_{i=1}^n \sum_{j=1}^n \langle U_1^* y_i, \tau(\alpha(a_i^*)a_j) U_1^* y_j \rangle \right\| \\
&= M \left\| \sum_{i=1}^n a_i \otimes y_i + K \right\|^2 . \qquad (4)
\end{aligned}$$

Therefore using Equations 3 and 4, we conclude that V is an adjointable map with adjoint

$$V^* \left(\sum_{i=1}^n a_i \otimes x_i + K \right) := \sum_{i=1}^n U_1^* \tau(a_i) x_i \quad \text{where } a_i \in \mathscr{A};\ x_i \in E_1 \text{ for } 1 \leqslant i \leqslant n.$$

For each $a_1, a_2, \dots, a_n \in \mathscr{A}$; $x_1, x_2, \dots, x_n \in E_1$ we obtain

$$\begin{aligned}
V^\natural \left(\sum_{i=1}^n a_i \otimes x_i + K \right) &= U_1^* V^* U_0 \left(\sum_{i=1}^n a_i \otimes x_i + K \right) = U_1^* V^* \left(\sum_{i=1}^n \alpha(a_i) \otimes U_1 x_i + K \right) \\
&= U_1^* \sum_{i=1}^n \tau(\alpha(a_i)) U_1^* U_1 x_i = U_1^* \sum_{i=1}^n \tau(\alpha(a_i)) x_i \\
&= V^* \left(\sum_{i=1}^n a_i \otimes x_i + K \right)
\end{aligned}$$

which implies that $V^\natural = V^*$. Define the map $\pi_0' : \mathscr{A} \to \mathscr{B}^a(E_0)$ by

$$\pi_0'(a) \left(\sum_{i=1}^n b_i \otimes x_i + K \right) = \sum_{i=1}^n a b_i \otimes x_i + K \qquad (5)$$

for all $a, b_1, b_2, \ldots, b_n \in \mathscr{A}; x_1, x_2, \ldots, x_n \in E_1$. We have

$$\begin{aligned}
&\left\| \pi_0'(a)\left(\sum_{i=1}^n a_i \otimes x_i + K\right)\right\|^2 = \left\|\sum_{i=1}^n aa_i \otimes x_i + K\right\|^2 \\
&= \left\| \left\langle \sum_{i=1}^n aa_i \otimes x_i + K, \sum_{j=1}^n aa_j \otimes x_j + K \right\rangle \right\| = \left\| \sum_{i,j=1}^n \left\langle x_i, \tau(\alpha(a_i^* a^*) aa_j) x_j \right\rangle \right\| \\
&\leqslant M(a) \left\| \sum_{i,j=1}^n \left\langle x_i, \tau(\alpha(a_i^*) a_j) x_j \right\rangle \right\| = M(a) \left\| \left(\sum_{i=1}^n a_i \otimes x_i + K \right) \right\|^2
\end{aligned}$$

where $a, a_1, \ldots, a_n \in \mathscr{A}$ and $x_1, \ldots, x_n \in E_1$. Thus for each $a \in \mathscr{A}$, $\pi_0'(a)$ is a well-defined bounded linear operator from E_0 to E_0. Using

$$\begin{aligned}
&\left\langle \pi_0'(a)\left(\sum_{i=1}^n a_i \otimes x_i + K\right), \sum_{j=1}^m a_j' \otimes x_j' + K \right\rangle \\
&= \left\langle \sum_{i=1}^n aa_i \otimes x_i + K, \sum_{j=1}^m a_j' \otimes x_j' + K \right\rangle = \sum_{i=1}^n \sum_{j=1}^m \left\langle x_i, \tau(\alpha(a_i^* a^*) a_j') x_j' \right\rangle \\
&= \sum_{i=1}^n \sum_{j=1}^m \left\langle x_i, \tau(\alpha(a_i^*) \alpha(a^*) a_j') x_j' \right\rangle \\
&= \left\langle \sum_{i=1}^n a_i \otimes x_i + K, \sum_{j=1}^m \alpha(a^*) a_j' \otimes x_j' + K \right\rangle
\end{aligned}$$

and

$$\begin{aligned}
U_0 \pi_0'(a^*) U_0^* \left(\sum_{j=1}^m a_j' \otimes x_j' + K \right) &= U_0 \pi_0'(a^*) \left(\sum_{j=1}^m \alpha^{-1}(a_j') \otimes U_1^* x_j' + K \right) \\
&= U_0 \left(\sum_{j=1}^m (a^* \alpha^{-1}(a_j')) \otimes U_1^* x_j' + K \right) \\
&= \sum_{j=1}^m \alpha(a^*) a_j' \otimes x_j' + K
\end{aligned}$$

for all $a_1, \ldots, a_n, a_1', \ldots, a_m' \in \mathscr{A}$ and $x_1, \ldots, x_n, x_1', \ldots, x_m' \in E_1$, it follows that $\pi_0' : \mathscr{A} \to \mathscr{B}^a(E_0)$ is a well-defined map. Indeed, $\pi_0' : \mathscr{A} \to \mathscr{B}^a(E_0)$ is an U_0-representation. Define an U_0-representation $\pi_0 : \mathscr{A} \to \mathscr{B}^a(E_0)$ by $\pi_0(a) := \pi_0'(\alpha(a))$ for all $a \in \mathscr{A}$. Since $V^\natural = V^*$, for all $a \in \mathscr{A}$, $x \in E_1$ we obtain

$$V^\natural \pi_0'(a) V x = V^* (a \otimes U_1 x + K) = U_1^* \tau(a) U_1 x = \tau(a) x.$$

Therefore $\tau(a) = \tau(\alpha(a)) = V^\natural \pi_0(a) V$ for all $a \in \mathscr{A}$. Moreover, for each $x \in E_1$ and

$a,b \in \mathscr{A}$ we get

$$\begin{aligned}
V^*\pi_0'(a)^*\pi_0'(b)Vx &= V^*U_0\pi_0'(a^*)U_0^*\pi_0'(b)Vx = V^*U_0\pi_0'(a^*)U_0^*(b\otimes U_1x+K)\\
&= V^*U_0\pi_0'(a^*)(\alpha^{-1}(b)\otimes x+K)\\
&= V^*U_0(a^*\alpha^{-1}(b)\otimes x+K) = V^*(\alpha(a^*\alpha^{-1}(b))\otimes U_1x+K)\\
&= U_1^*\tau(\alpha(a^*\alpha^{-1}(b)))U_1x = \tau(\alpha(a)^*b)x = V^*\pi_0'(\alpha(a)^*b)Vx.
\end{aligned}$$

From this equality, it follows that

$$\begin{aligned}
V^*\pi_0(a)^*\pi_0(b)V &= V^*\pi_0'(\alpha(a))^*\pi_0'(\alpha(b))V = V^*\pi_0'(\alpha(\alpha(a))^*\alpha(b))V\\
&= V^*\pi_0'(\alpha(\alpha(a)^*b))V = V^*\pi_0(\alpha(a)^*b)V
\end{aligned}$$

for each $a,b \in \mathscr{A}$. □

In the following theorem we extend the KSGNS construction for τ-maps:

THEOREM 3. *Assume $\mathscr{A}$ to be a unital C^*-algebra and $\alpha : \mathscr{A} \to \mathscr{A}$ be a $*$-automorphism. Suppose $\mathscr{B} \subset \mathscr{B}(\mathscr{H})$ is a von Neumann algebra for some Hilbert space $\mathscr{H}$ and E is a Hilbert $\mathscr{A}$-module. Let E_1 be a Hilbert $\mathscr{B}$-module and E_2 be a von Neumann $\mathscr{B}$-module, and $(E_1,\mathscr{B},U_1)$ and $(E_2,\mathscr{B},U_2=\mathrm{id}_{E_2})$ be S-modules. If $\tau : \mathscr{A} \to \mathscr{B}^a(E_1)$ is an α-CP map and $T : E \to \mathscr{B}^a(E_1,E_2)$ is a τ-map, then there exist*

(i) (a) a von Neumann $\mathscr{B}$-module E_3 with a unitary U_3 such that $(E_3,\mathscr{B},U_3)$ is an S-module,

(b) an U_3-representation π of $\mathscr{A}$ on $(E_3,\mathscr{B},U_3)$ with a map $V \in \mathscr{B}^a(E_1,E_3)$ such that $V^\natural = V^*$, and

$$\tau(a) = V^*\pi(a)V \text{ for all } a \in \mathscr{A},$$

(ii) (a) a von Neumann $\mathscr{B}$-module E_4 such that $(E_4,\mathscr{B},U_4=id_{E_4})$ is an S-module and a map $\Psi : E \to \mathscr{B}^a(E_3,E_4)$ which is a π-map,

(b) a coisometry W from E_2 onto E_4 satisfying $W^\natural = W^*$,

$$T(x) = W^*\Psi(x)V \text{ for all } x \in E.$$

Proof. By Theorem 2 we obtain the triple (π_0,V,E_0) associated to τ where $(E_0,\mathscr{B},U_0)$ is an S-module. Here $V \in \mathscr{B}^a(E_1,E_0)$, the Hilbert $\mathscr{B}$-module E_0 satisfies $\overline{\operatorname{span}}\,\pi_0(\mathscr{A})VE_1 = E_0$, and π_0 is an U_0-representation of $\mathscr{A}$ to $\mathscr{B}^a(E_0)$ such that

$$\tau(a) = V^*\pi_0(a)V \text{ for all } a \in \mathscr{A}.$$

We obtain a von Neumann $\mathscr{B}$-module E_3 by taking the strong operator topology closure of E_0 in $\mathscr{B}(\mathscr{H},E_0 \bigotimes \mathscr{H})$. Consider the element of $\mathscr{B}^a(E_1,E_3)$ which gives the same value as V when evaluated on the elements of E_1, because E_0 is canonically embedded in E_3. We denote this element of $\mathscr{B}^a(E_1,E_3)$ by V. Fix $\lim_\alpha x_\alpha^0 \in E_3$ with $x_\alpha^0 \in E_0$. It is easy to check that sot-$\lim_\alpha \pi_0(a)x_\alpha^0$ exists for each $a \in \mathscr{A}$. The U_0-representation $\pi_0 : \mathscr{A} \to \mathscr{B}^a(E_0)$ extends to a representation of $\mathscr{A}$ on E_3 as follows: For each $a \in \mathscr{A}$ and x=sot-$\lim_\alpha x_\alpha^0 \in E_3$ with $x_\alpha^0 \in E_0$, define

$$\pi(a)(x) := \text{sot-}\lim_\alpha \pi_0(a)x_\alpha^0.$$

For each $a \in \mathscr{A}$, x=sot-$\lim_\alpha x_\alpha^0$ and y=sot-$\lim_\beta y_\beta^0 \in E_3$ with $x_\alpha^0, y_\beta^0 \in E_0$ we have

$$\begin{aligned}\langle \pi(a)x, y\rangle &= \text{sot-}\lim_\beta \langle \pi(a)x, y_\beta^0\rangle = \text{sot-}\lim_\beta(\text{sot-}\lim_\alpha \langle y_\beta^0, \pi_0(a)x_\alpha^0\rangle)^* \\ &= \text{sot-}\lim_\beta(\text{sot-}\lim_\alpha \langle \pi_0(a)^* y_\beta^0, x_\alpha^0\rangle)^* = \langle x, \pi(a)^* y\rangle,\end{aligned}$$

i.e., $\pi(a) \in \mathscr{B}^a(E_3)$ for each $a \in \mathscr{A}$. Let $U_3 : E_3 \to E_3$ be a map defined by

$$U_3(x) := \text{sot-}\lim_\alpha U_0(x_\alpha^0)) \text{ where } x\text{=sot-}\lim_\alpha x_\alpha^0 \in E_3 \text{ with } x_\alpha^0 \in E_0.$$

It is easy to observe that U_3 is a unitary, $(E_3, \mathscr{B}, U_3)$ is an S-module and the triple (π, V, E_3) satisfies all the conditions of the statement (i).

Let E_4' be the Hilbert $\mathscr{B}$-module $\overline{\text{span}}\, T(E)E_1$. For each $x \in E$, define a map $\Psi_0(x) : E_0 \to E_4'$ by

$$\Psi_0(x)\left(\sum_{i=1}^n \pi_0(a_i)Vx_i\right) = \sum_{i=1}^n T(xa_i)x_i \tag{6}$$

for all $a_1, a_2, \ldots, a_n \in \mathscr{A}$ and $x_1, x_2, \ldots, x_n \in E_1$. Each $\Psi_0(x)$ is a bounded right $\mathscr{B}$-linear map from E_0 to E_4'. Indeed, we have

$$\begin{aligned}&\left\langle \Psi_0(x)\left(\sum_{i=1}^n \pi_0(a_i)Vx_i\right), \Psi_0(y)\left(\sum_{j=1}^m \pi_0\left(a_j'\right)Vx_j'\right)\right\rangle \\ &= \left\langle \sum_{i=1}^n T(xa_i)x_i, \sum_{j=1}^m T\left(ya_j'\right)x_j'\right\rangle = \sum_{i=1}^n\sum_{j=1}^m \left\langle x_i, T(xa_i)^* T\left(ya_j'\right)x_j'\right\rangle \\ &= \sum_{i=1}^n\sum_{j=1}^m \left\langle x_i, \tau(\langle xa_i, ya_j'\rangle)x_j'\right\rangle = \sum_{i=1}^n\sum_{j=1}^m \left\langle x_i, V^*\pi_0\left(a_i^*\langle x,y\rangle a_j'\right)Vx_j'\right\rangle \\ &= \sum_{i=1}^n\sum_{j=1}^m \left\langle x_i, V^*\pi_0(a_i)^*\pi_0(\langle x,y\rangle a_j')Vx_j'\right\rangle \\ &= \left\langle \sum_{i=1}^n \pi_0(a_i)Vx_i, \pi_0(\langle x,y\rangle)\sum_{j=1}^m \pi_0\left(a_j'\right)Vx_j'\right\rangle \end{aligned}\tag{7}$$

for all $x,y \in E$; and $a_i, a'_j \in \mathscr{A}$ and $x_i, x'_j \in E_1$ for $1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m$. We denote by E_4 the strong operator topology closure of E'_4 in $\mathscr{B}(\mathscr{H}, E'_4 \otimes \mathscr{H})$. For each $x \in E$ and z=sot-$\lim\limits_{\alpha} z^0_\alpha \in E_3$ with $z^0_\alpha \in E_0$, define a mapping $\Psi(x) : E_3 \to E_4$ by

$$\Psi(x)(z) := \text{sot-}\lim_{\alpha} \Psi_0(x) z^0_\alpha.$$

Note that the limit sot-$\lim_\alpha \Psi_0(x) z^0_\alpha$ exists. For all z=sot-$\lim\limits_{\alpha} z^0_\alpha \in E_3$ with $z^0_\alpha \in E_0$ and $x,y \in E$ we have

$$\begin{aligned}\langle \Psi(x)z, \Psi(y)z\rangle &= \text{sot-}\lim_{\alpha}\{\text{sot-}\lim_{\beta}\langle \Psi_0(y)z^0_\alpha, \Psi_0(x)z^0_\beta\rangle\}^* \\ &= \text{sot-}\lim_{\alpha}\{\text{sot-}\lim_{\beta}\langle z^0_\alpha, \pi_0(\langle x,y\rangle)z^0_\beta\rangle\}^* = \langle z, \pi(\langle x,y\rangle)z\rangle.\end{aligned}$$

Since E_3 is a von Neumann $\mathscr{B}$-module, we conclude that $\Psi : E \to \mathscr{B}^a(E_3, E_4)$ is a π-map. Because E_4 is a von Neumann $\mathscr{B}$-submodule of E_2, we get an orthogonal projection from E_2 onto E_4 (cf. Theorem 5.2 of [20]) which we denote by W. Therefore W^* is the inclusion map from E_4 to E_2, and hence $WW^* = id_{E_4}$, i.e., W is a coisometry. Considering the S-module $(E_4, \mathscr{B}, U_4 = id_{E_4})$ it is evident that $W^\natural(x) = U_2^* W^* U_4(x) = W^*(x)$ for all $x \in E_2$. Eventually

$$W^*\Psi(x)V = \Psi(x)V = \Psi(x)(\pi(1)V) = T(x) \text{ for all } x \in E. \quad \square$$

3. Reproducing kernel S-correspondences

Suppose $\mathscr{B}$ and $\mathscr{C}$ are unital C^*-algebras. We denote the set of all bounded linear maps from $\mathscr{B}$ to $\mathscr{C}$ by $\mathscr{B}(\mathscr{B},\mathscr{C})$. Let α be a $*$-automorphism on $\mathscr{B}$. For a set Ω, a *kernel* $\mathfrak{K}$ over Ω from $\mathscr{B}$ to $\mathscr{C}$ is called *Hermitian* if $\mathfrak{K}^{\sigma,\sigma'}(b^*) = \mathfrak{K}^{\sigma',\sigma}(b)^*$ for all $\sigma,\sigma' \in \Omega$ and $b \in \mathscr{B}$. We say that a Hermitian kernel $\mathfrak{K}$ over Ω from $\mathscr{B}$ to $\mathscr{C}$ is an *α-completely positive definite kernel* or an *α-CPD-kernel* over Ω from $\mathscr{B}$ to $\mathscr{C}$ if for finite choices $\sigma_i \in \Omega$, $b_i \in \mathscr{B}$, $c_i \in \mathscr{C}$ we have

(i) $\sum_{i,j} c_i^* \mathfrak{K}^{\sigma_i,\sigma_j}(\alpha(b_i)^* b_j) c_j \geqslant 0$,

(ii) $\mathfrak{K}^{\sigma_i,\sigma_j}(\alpha(b)) = \mathfrak{K}^{\sigma_i,\sigma_j}(b)$ for all $b \in \mathscr{B}$,

(iii) for each $b \in \mathscr{B}$ there exists $M(b) > 0$ such that

$$\left\| \sum_{i,j=1}^{n} c_i^* \mathfrak{K}^{\sigma_i,\sigma_j}(\alpha(b_i^* b^*) b b_j) c_j \right\| \leqslant M(b) \left\| \sum_{i,j=1}^{n} c_i^* \mathfrak{K}^{\sigma_i,\sigma_j}(\alpha(b_i^*) b_j) c_j \right\|.$$

In this section we discuss the decomposition of $\mathfrak{K}$-families for an α-CPD-kernel in terms of a reproducing kernel S-correspondence which is defined as follows:

DEFINITION 6. Let $\mathscr{A}$ and $\mathscr{B}$ be unital C^*-algebras. An S-module $(\mathscr{F},\mathscr{B},U)$ is called an *S-correspondence over* Ω *from* $\mathscr{A}$ *to* $\mathscr{B}$ if there exists a U-representation π of $\mathscr{A}$ on $(\mathscr{F},\mathscr{B},U)$. We define

$$af := \pi(a)f \text{ for all } a \in \mathscr{A}, f \in \mathscr{F}.$$

Let Ω be a set. If $(\mathscr{F},\mathscr{B},U)$ is an S-correspondence from $\mathscr{A}$ to $\mathscr{B}$, consisting of functions from $\Omega \times \mathscr{A}$ to $\mathscr{B}$, which forms a vector space with point-wise vector space operations, and for each $\sigma \in \Omega$ there exists an element k_σ in $\mathscr{F}$ called the *kernel element* satisfying

$$f(\sigma,a) = \langle k_\sigma, af\rangle \text{ for all } a \in \mathscr{A},\ f \in \mathscr{F},$$

then this S-correspondence is called a *reproducing kernel S-correspondence* over Ω from $\mathscr{A}$ to $\mathscr{B}$. The mapping $\mathfrak{K} : \Omega \times \Omega \to \mathscr{B}(\mathscr{A},\mathscr{B})$ defined by

$$\mathfrak{K}^{\sigma,\sigma'}(a) = k_{\sigma'}(\sigma,a) \text{ for all } a \in \mathscr{A},\ \sigma' \in \Omega$$

is called the *reproducing kernel* for the reproducing kernel S-correspondence.

In Theorem 3.1 of [6], Bhattacharyya, Dritschel and Todd proved that a kernel $\mathfrak{K}$ is dominated by a CPD-kernel if and only if $\mathfrak{K}$ has a Kolmogorov decomposition in which the associated module forms a Krein C^*-correspondence. Skeide's factorization theorem for τ-maps [22] is based on the Paschke's GNS construction (cf. Theorem 5.2, [17]) for CP map τ. Using the Kolmogorov decomposition we proved a factorization theorem for $\mathfrak{K}$-families in Theorem 2.2 of [9] when $\mathfrak{K}$ is a CPD-kernel. In Theorem 3.5 of [3], a characterization of a CPD-kernel in terms of reproducing kernel C^*-correspondences was obtained.

THEOREM 4. *Let $\mathfrak{K}$ be a Hermitian kernel over a set Ω from a unital C^*-algebra $\mathscr{B}$ to a unital C^*-algebra $\mathscr{C}$. Assume α to be a $*$-automorphism on $\mathscr{B}$. Then the following statements are equivalent:*

(i) $\mathfrak{K}$ is an α-CPD-kernel.

(ii) $\mathfrak{K}$ is the reproducing kernel for an reproducing kernel S-correspondence $\mathscr{F} = \mathscr{F}(\mathfrak{K})$ over Ω from $\mathscr{B}$ to $\mathscr{C}$, i.e., there is an S-correspondence $\mathscr{F} = \mathscr{F}(\mathfrak{K})$ whose elements are $\mathscr{C}$-valued functions on $\Omega \times \mathscr{B}$ such that for any $\sigma' \in \Omega$ the function $k_{\sigma'}$ defined by

$$k_{\sigma'}(\sigma,b) := \mathfrak{K}^{\sigma,\sigma'}(b) \textit{ for all } \sigma \in \Omega;\ b \in \mathscr{B}$$

belongs to $\mathscr{F}(\mathfrak{K})$ and has the reproducing property

$$\langle k_\sigma, bf\rangle = \langle \alpha(b^*)k_\sigma, f\rangle = f(\sigma,b) \textit{ for all } \sigma \in \Omega,\ f \in \mathscr{F}(\mathfrak{K}),\ b \in \mathscr{B}$$

where $bk_\sigma \in \mathscr{F}$ is given by

$$(bk_\sigma)(\sigma',b') := \mathfrak{K}^{\sigma',\sigma}(b'b) \textit{ for all } b' \in \mathscr{B}.$$

Proof. Suppose (ii) holds. Thus from the reproducing property it follows that

$$\sum_{i,j} c_i^* \mathfrak{K}^{\sigma_i,\sigma_j}(\alpha(b_i^*)b_j)c_j = \sum_{i,j} c_i^* k_{\sigma_j}(\sigma_i, \alpha(b_i^*)b_j)c_j = \sum_{i,j} c_i^* \langle k_{\sigma_i}, \alpha(b_i^*)b_j k_{\sigma_j}\rangle c_j$$
$$= \left\langle \sum_i b_i k_{\sigma_i} c_i, \sum_j b_j k_{\sigma_j} c_j \right\rangle \geqslant 0$$

for all finite choices of $\sigma_i \in \Omega$, $b_i \in \mathscr{B}$, $c_i \in \mathscr{C}$. Further, for all $b \in \mathscr{B}$ and $\sigma, \sigma' \in \Omega$ we get

$$\mathfrak{K}^{\sigma,\sigma'}(\alpha(b)) = k_{\sigma'}(\sigma, \alpha(b)) = \langle k_\sigma, \alpha(b)k_{\sigma'}\rangle = \langle b^* k_\sigma, k_{\sigma'}\rangle$$
$$= (\langle k_{\sigma'}, b^* k_\sigma\rangle)^* = k_\sigma(\sigma', b^*)^* = \mathfrak{K}^{\sigma',\sigma}(b^*)^* = \mathfrak{K}^{\sigma,\sigma'}(b).$$

Finally, for a fixed $b \in \mathscr{B}$ and each finite choices $\sigma_i \in \Omega$, $b_i \in \mathscr{B}$, $c_i \in \mathscr{C}$ we obtain

$$\left\| \sum_{i,j=1}^n c_i^* \mathfrak{K}^{\sigma_i,\sigma_j}(\alpha(b_i^* b^*)bb_j)c_j \right\| = \left\| \sum_{i,j=1}^n c_i^* k_{\sigma_j}(\sigma_i, \alpha(b_i^* b^*)bb_j)c_j \right\|$$
$$= \left\| \sum_{i,j=1}^n c_i^* \langle k_{\sigma_i}, (\alpha(b_i^* b^*)bb_j)k_{\sigma_j}\rangle c_j \right\| = \left\| \left\langle \sum_{i=1}^n b_i k_{\sigma_i} c_i, \alpha(b)^* b \left(\sum_{j=1}^n b_j k_{\sigma_j} c_j \right) \right\rangle \right\|$$
$$\leqslant \|\alpha(b)^* b\| \left\| \sum_{i=1}^n b_i k_{\sigma_i} c_i \right\|^2 \leqslant \|b\|^2 \left\| \left\langle \sum_{i=1}^n b_i k_{\sigma_i} c_i, \sum_{j=1}^n b_j k_{\sigma_j} c_j \right\rangle \right\|$$
$$= \|b\|^2 \left\| \sum_{i,j=1}^n c_i^* \mathfrak{K}^{\sigma_i,\sigma_j}(\alpha(b_i^*)b_j)c_j \right\|.$$

Thus the function $\mathfrak{K}$ is an α-CPD-kernel, i.e., (i) holds.

Conversely, suppose (i) holds. For each $\sigma' \in \Omega$ let $k_{\sigma'} : \Omega \times \mathscr{B} \to \mathscr{C}$ be a map defined by $k_{\sigma'}(\sigma, b) := \mathfrak{K}^{\sigma,\sigma'}(b)$ where $\sigma \in \Omega$, $b \in \mathscr{B}$. Let us define the mapping $bk_{\sigma'}$ by $(\sigma, b') \mapsto \mathfrak{K}^{\sigma,\sigma'}(b'b) = k_{\sigma'}(\sigma, b'b)$ where $\sigma, \sigma' \in \Omega$ and $b, b' \in \mathscr{B}$. For fixed $c \in \mathscr{C}$ we define the function $k_{\sigma'}c$ by $(\sigma, b) \mapsto \mathfrak{K}^{\sigma,\sigma'}(b)c = k_{\sigma'}(\sigma, b)c$ for all $\sigma, \sigma' \in \Omega$ and $b \in \mathscr{B}$. In a canonical way define $(bk_\sigma)c$ and $b(k_\sigma c)$ for all $\sigma \in \Omega$, $b \in \mathscr{B}$, and $c \in \mathscr{C}$. Let $\mathscr{F}_0$ be the right $\mathscr{C}$-module generated by the set $\{bk_\sigma : b \in \mathscr{B},\ \sigma \in \Omega\}$ consisting of $\mathscr{C}$-valued functions on $\Omega \times \mathscr{B}$, i.e., $\mathscr{F}_0 = \{\sum_{j=1}^m (b_j k_{\sigma_j})c_j : b_1, \ldots, b_m \in \mathscr{B}; c_1, \ldots, c_m \in \mathscr{C}; \sigma_1, \ldots, \sigma_m \in \Omega; m \in \mathbb{N}\}$. Note that $(bk_\sigma)c = b(k_\sigma c)$ for all $\sigma \in \Omega$, $b \in \mathscr{B}$, and $c \in \mathscr{C}$ and hence we write $\mathscr{F}_0 = \{\sum_{j=1}^m b_j k_{\sigma_j} c_j : b_1, \ldots, b_m \in \mathscr{B}; c_1, \ldots, c_m \in \mathscr{C}; \sigma_1, \ldots, \sigma_m \in \Omega; m \in \mathbb{N}\}$. Define a map $\langle \cdot, \cdot \rangle : \mathscr{F}_0 \times \mathscr{F}_0 \to \mathscr{C}$ by

$$\langle f, g\rangle := \sum_{j=1}^m \sum_{i=1}^n c_j^* \mathfrak{K}^{\sigma_j,\sigma_i'}(\alpha(b_j)^* b_i')c_i' \tag{8}$$

where $f=\sum_{j=1}^{m} b_j k_{\sigma_j} c_j,\ g=\sum_{i=1}^{n} b'_i k_{\sigma'_i} c'_i \in \mathscr{F}_0$. With $f=\sum_{j=1}^{m} b_j k_{\sigma_j} c_j$ and $g=\sum_{i=1}^{n} b'_i k_{\sigma'_i} c'_i$ in $\mathscr{F}_0$, we obtain

$$\begin{aligned}
&\sum_{j=1}^{m} c_j^* g(\sigma_j, \alpha(b_j)^*)\\
&=\sum_{j=1}^{m}\sum_{i=1}^{n} c_j^* b'_i k_{\sigma'_i}(\sigma_j, \alpha(b_j)^*) c'_i = \sum_{j=1}^{m}\sum_{i=1}^{n} c_j^* k_{\sigma'_i}(\sigma_j, \alpha(b_j)^* b'_i) c'_i\\
&=\sum_{j=1}^{m}\sum_{i=1}^{n} c_j^* \mathfrak{K}^{\sigma_j,\sigma'_i}(\alpha(b_j)^* b'_i) c'_i = \sum_{j=1}^{m}\sum_{i=1}^{n} c_j^* \mathfrak{K}^{\sigma_j,\sigma'_i}(b_j^* \alpha^{-1}(b'_i)) c'_i\\
&=\sum_{j=1}^{m} c_j^* \sum_{i=1}^{n} (\mathfrak{K}^{\sigma'_i,\sigma_j}(\alpha^{-1}(b'_i)^* b_j))^* c'_i = \sum_{j=1}^{m} c_j^* \sum_{i=1}^{n} (k_{\sigma_j}(\sigma'_i, \alpha^{-1}(b'_i)^* b_j))^* c'_i\\
&=\sum_{j=1}^{m} c_j^* \sum_{i=1}^{n} (b_j k_{\sigma_j}(\sigma'_i, \alpha^{-1}(b'_i)^*))^* c'_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} b_j k_{\sigma_j}(\sigma'_i, \alpha^{-1}(b'_i)^*) c_j\right)^* c'_i\\
&=\sum_{i=1}^{n} (f(\sigma'_i, \alpha^{-1}(b'_i)^*))^* c'_i. \qquad (9)
\end{aligned}$$

Thus the function $\langle \cdot,\cdot \rangle$ defined above does not depend on the representations chosen for f and g. Since $\mathfrak{K}$ is an α-CPD-kernel,

$$\left\langle \sum_{j=1}^{m} b_j k_{\sigma_j} c_j, \sum_{i=1}^{m} b_i k_{\sigma_i} c_i \right\rangle = \sum_{=1}^{m}\sum_{i=1}^{m} c_j^* \mathfrak{K}^{\sigma_j,\sigma_i}(\alpha(b_j)^* b_i) c_i \geqslant 0.$$

Therefore the map $\langle \cdot,\cdot \rangle$ is positive definite. For $f := \sum_{j=1}^{m} b_j k_{\sigma_j} c_j \in \mathscr{F}_0,\ b \in \mathscr{B},\ c \in \mathscr{C}$ and $\sigma \in \Omega$, Equations 8 and 9, and the Cauchy-Schwarz inequality gives

$$\|f(\sigma,b)c\|^2 = \|\langle f, \alpha(b)^* k_\sigma c\rangle\|^2 \leqslant \|\langle \alpha(b)^* k_\sigma c, \alpha(b)^* k_\sigma c\rangle\| \|\langle f,f\rangle\|.$$

So $f \in \mathscr{F}_0$ vanishes pointwise if $\langle f,f\rangle = 0$. This implies that $\mathscr{F}_0$ is a right inner-product $\mathscr{C}$-module with respect to $\langle \cdot,\cdot \rangle$. Let $\mathscr{F}$ be the completion of $\mathscr{F}_0$. It is easy to observe that the linear map $f \mapsto ((\sigma,b) \mapsto \langle \alpha(b^*) k_\sigma, f\rangle)$, from $\mathscr{F}$ to the set of all functions from $\Omega \times \mathscr{B}$ to $\mathscr{C}$, is injective. Therefore we identify $\mathscr{F}$ as a subspace of the set of all functions from $\Omega \times \mathscr{B}$ to $\mathscr{C}$.

If $\sum_{j=1}^{m} b_j k_{\sigma_j} c_j$ and $\sum_{i=1}^{n} b'_i k_{\sigma'_i} c'_i$ are elements of $\mathscr{F}_0$, then we get

$$\begin{aligned}
\left\langle \sum_{j=1}^{m} b_j k_{\sigma_j} c_j, \sum_{i=1}^{n} b'_i k_{\sigma'_i} c'_i \right\rangle &= \sum_{j=1}^{m}\sum_{i=1}^{n} c_j^* \mathfrak{K}^{\sigma_j,\sigma'_i}(\alpha(b_j)^* b'_i) c'_i\\
&= \sum_{j=1}^{m}\sum_{i=1}^{n} c_j^* \mathfrak{K}^{\sigma_j,\sigma'_i}(\alpha(\alpha(b_j^*) b'_i)) c'_i\\
&= \left\langle \sum_{j=1}^{m} \alpha(b_j) k_{\sigma_j} c_j, \sum_{i=1}^{n} \alpha(b'_i) k_{\sigma'_i} c'_i \right\rangle. \qquad (10)
\end{aligned}$$

Therefore we get an isometry $U : \mathscr{F} \to \mathscr{F}$ by $\sum_{i=1}^{n} b_i k_{\sigma_i} c_i \mapsto \sum_{i=1}^{n} \alpha(b_i) k_{\sigma_i} c_i$. Moreover, from Equation 10, it is easy to check that U is a unitary with the adjoint $U^* : \mathscr{F} \to \mathscr{F}$ defined by $\sum_{i=1}^{n} b_i k_{\sigma_i} c_i \mapsto \sum_{i=1}^{n} \alpha^{-1}(b_i) k_{\sigma_i} c_i$. We define a sesquilinear form $[\cdot,\cdot] : \mathscr{F} \times \mathscr{F} \to \mathscr{C}$ as follows:

$$[f, f'] := \langle f, Uf' \rangle$$

where $f, f' \in \mathscr{F}$, i.e., for $\sum_{j=1}^{m} b_j k_{\sigma_j} c_j$, $\sum_{i=1}^{n} b'_i k_{\sigma'_i} c'_i \in \mathscr{F}$ we obtain

$$\left[\sum_{j=1}^{m} b_j k_{\sigma_j} c_j, \sum_{i=1}^{n} b'_i k_{\sigma'_i} c'_i \right] = \left\langle \sum_{j=1}^{m} b_j k_{\sigma_j} c_j, \sum_{i=1}^{n} \alpha(b'_i) k_{\sigma'_i} c'_i \right\rangle.$$

For each $b \in \mathscr{B}$ define $\pi(b) : \mathscr{F} \to \mathscr{F}$ by

$$\pi(b) \left(\sum_{j=1}^{m} b_j k_{\sigma_j} c_j \right) := \sum_{j=1}^{m} b b_j k_{\sigma_j} c_j \text{ for all } b' \in \mathscr{B}, \sigma \in \Omega, c \in \mathscr{C}.$$

Therefore for $b, b_1, \dots, b_n \in \mathscr{B}$; $c_1, \dots, c_n \in \mathscr{C}$ and $\sigma_1, \dots, \sigma_n \in \Omega$ we have

$$\begin{aligned}
\left\| \pi(b) \left(\sum_{i=1}^{n} b_i k_{\sigma_i} c_i \right) \right\|^2 &= \left\| \sum_{i=1}^{n} b b_i k_{\sigma_i} c_i \right\|^2 = \left\| \left\langle \sum_{i=1}^{n} b b_i k_{\sigma_i} c_i, \sum_{j=1}^{n} b b_j k_{\sigma_j} c_j \right\rangle \right\| \\
&= \left\| \sum_{i,j=1}^{n} c_i^* \mathfrak{K}^{\sigma_i,\sigma_j} (\alpha(b_i^* b^*) b b_j) c_j \right\| \\
&\leqslant M(b) \left\| \sum_{i,j=1}^{n} c_i^* \mathfrak{K}^{\sigma_i,\sigma_j} (\alpha(b_i^*) b_j) c_j \right\| = M(b) \left\| \sum_{i=1}^{n} b_i k_{\sigma_i} c_i \right\|^2.
\end{aligned}$$

This implies that for each $b \in \mathscr{B}$, $\pi(b)$ is a well defined bounded linear operator from $\mathscr{F}$ to $\mathscr{F}$. From

$$\begin{aligned}
&\left\langle \pi(b) \left(\sum_{i=1}^{n} b_i k_{\sigma_i} c_i \right), \sum_{j=1}^{m} b'_j k_{\sigma'_j} c'_j \right\rangle = \left\langle \sum_{i=1}^{n} b b_i k_{\sigma_i} c_i, \sum_{j=1}^{m} b'_j k_{\sigma'_j} c'_j \right\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} c_i^* \mathfrak{K}^{\sigma_i,\sigma'_j} (\alpha(b_i^* b^*) b'_j) c'_j = \sum_{i=1}^{n} \sum_{j=1}^{m} c_i^* \mathfrak{K}^{\sigma_i,\sigma'_j} (\alpha(b_i^*) \alpha(b^*) b'_j) c'_j \\
&= \left\langle \sum_{i=1}^{n} b_i k_{\sigma_i} c_i, \sum_{j=1}^{m} \alpha(b^*) b'_j k_{\sigma'_j} c'_j \right\rangle
\end{aligned}$$

and

$$\begin{aligned}
U\pi(b^*)U^* \left(\sum_{j=1}^{m} b'_j k_{\sigma'_j} c'_j \right) &= U\pi(b^*) \left(\sum_{j=1}^{m} \alpha^{-1}(b'_j) k_{\sigma'_j} c'_j \right) = U \left(\sum_{j=1}^{m} b^* \alpha^{-1}(b'_j) k_{\sigma'_j} c'_j \right) \\
&= \left(\sum_{j=1}^{m} \alpha(b^* \alpha^{-1}(b'_j)) k_{\sigma'_j} c'_j) \right) = \left(\sum_{j=1}^{m} \alpha(b^*) b'_j k_{\sigma'_j} c'_j \right)
\end{aligned}$$

for all $b, b'_1, \ldots, b'_n \in \mathscr{B}$; $c'_1, \ldots, c'_n \in \mathscr{C}$ and $\sigma'_1, \ldots, \sigma'_n \in \Omega$, it follows that π is an U-representation from $\mathscr{B}$ to the S-module $(\mathscr{F}, \mathscr{C}, U)$ and $\mathscr{F}$ becomes an S-correspondence with left action induced by π. Using Equations 8 and 9, we can realize elements g of $\mathscr{F}$ as $\mathscr{C}$-valued functions on $\Omega \times \mathscr{B}$ which satisfy the following reproducing property:

$$g(\sigma, b) = \langle k_\sigma, bg \rangle \text{ for all } \sigma \in \Omega,\ b \in \mathscr{B}. \quad \square$$

The "$(ii) \implies (i)$" part of the Theorem 4 gives a typical example of an α-CPD-kernel.

Motivated by the definition of τ-map, we introduced the following notion of $\mathfrak{K}$-family in [9] which we recall below: Let E and F be Hilbert C^*-modules over C^*-algebras $\mathscr{B}$ and $\mathscr{C}$ respectively. Assume Ω to be a set and $\mathfrak{K} : \Omega \times \Omega \to \mathscr{B}(\mathscr{B}, \mathscr{C})$ to be a kernel. Let $\mathscr{K}^\sigma$ be a map from E to F for each $\sigma \in \Omega$. The family $\{\mathscr{K}^\sigma\}_{\sigma \in \Omega}$ is called $\mathfrak{K}$*-family* if

$$\langle \mathscr{K}^\sigma(x), \mathscr{K}^{\sigma'}(x') \rangle = \mathfrak{K}^{\sigma,\sigma'}(\langle x, x' \rangle) \text{ for } x, x' \in E;\ \sigma, \sigma' \in \Omega.$$

REMARK 1. The U-representation π in Theorem 4 is not necessarily $*$-preserving, and $\pi(b^*)^* = \pi(\alpha(b))$ for all $b \in \mathscr{B}$.

Let us, in addition, assume $\alpha = id_{\mathscr{B}}$ in Theorem 4 (i.e., $\mathfrak{K}$ is a CPD-kernel). Then U is the identity map and π is a $*$-preserving representation, and hence $\mathscr{F}$ becomes a C^*-correspondence. This yields a new proof of our earlier result from Section 2 of [9] on a factorization for $\mathfrak{K}$-families where $\mathfrak{K}$ is a CPD-kernel:

COROLLARY 1. *Under the setting of Theorem 4, let E and F be Hilbert C^*-modules over $\mathscr{B}$ and $\mathscr{C}$ respectively, and let $\mathscr{K}^\sigma$ be a map from E to F, for each $\sigma \in \Omega$. Then $\{\mathscr{K}^\sigma\}_{\sigma \in \Omega}$ is a $\mathfrak{K}$-family where $\mathfrak{K}$ is a CPD-kernel if and only if $\mathfrak{K}$ is the reproducing kernel for an reproducing kernel S-correspondence $\mathscr{F} = \mathscr{F}(\mathfrak{K})$ over Ω from $\mathscr{B}$ to $\mathscr{C}$ with kernel elements $k_\sigma \in \mathscr{F}$ and there exists an isometry $v : E \bigotimes_{\mathscr{B}} \mathscr{F} \to F$ such that*

$$v(x \otimes bk_\sigma c) = \mathscr{K}^\sigma(xb)c \textit{ for all } x \in E,\ b \in \mathscr{B},\ c \in \mathscr{C},\ \sigma \in \Omega. \tag{11}$$

Proof. Suppose the family $\{\mathscr{K}^\sigma\}_{\sigma \in \Omega}$ is a $\mathfrak{K}$-family. For each $b, b' \in \mathscr{B}$; $c, c' \in \mathscr{C}$; $x, x' \in E$; $\sigma, \sigma' \in \Omega$ we get

$$\begin{aligned} \langle \mathscr{K}^\sigma(xb)c, \mathscr{K}^{\sigma'}(x'b')c' \rangle &= c^* \mathfrak{K}^{\sigma,\sigma'}(\langle xb, x'b' \rangle)c' \\ &= \langle k_\sigma c, b^* \langle x, x' \rangle b' k_{\sigma'} c' \rangle \\ &= \langle bk_\sigma c, \langle x, x' \rangle b' k_{\sigma'} c' \rangle. \end{aligned}$$

Define a linear map v from the interior tensor product $E \bigotimes_{\mathscr{B}} \mathscr{F}$ to F by

$$v(x \otimes bk_\sigma c) := \mathscr{K}^\sigma(xb)c \text{ for all } x \in E,\ b \in \mathscr{B},\ c \in \mathscr{C},\ \sigma \in \Omega.$$

We obtain

$$\begin{aligned}\langle \nu(x\otimes bk_\sigma c), \nu(x'\otimes b'k_{\sigma'}c')\rangle &= \langle \mathscr{K}^\sigma(xb)c, \mathscr{K}^{\sigma'}(x'b')c'\rangle = \langle bk_\sigma c, \langle x,x'\rangle b'k_{\sigma'}c'\rangle \\ &= \langle x\otimes bk_\sigma c, x'\otimes b'k_{\sigma'}c'\rangle\end{aligned}$$

for all $x,x'\in E;\ b,b'\in\mathscr{B};\ c,c'\in\mathscr{C};\ \sigma,\sigma'\in S$. Hence ν is an isometry.

Conversely, assume that there exist an isometry $\nu : E\bigotimes_{\mathscr{B}}\mathscr{F}\to F$ defined by Equation 11. For each $x,x'\in E;\ \sigma,\sigma'\in\Omega$ we obtain

$$\begin{aligned}\langle \mathscr{K}^\sigma(x), \mathscr{K}^{\sigma'}(x')\rangle &= \langle \nu(x\otimes k_\sigma), \nu(x'\otimes k_{\sigma'})\rangle \\ &= \langle x\otimes k_\sigma, x'\otimes k_{\sigma'}\rangle = \langle k_\sigma, (\langle x,x'\rangle)k_{\sigma'}\rangle \\ &= k_{\sigma'}(\sigma,\langle x,x'\rangle) = \mathfrak{K}^{\sigma,\sigma'}(\langle x,x'\rangle).\end{aligned}$$

So $\{\mathscr{K}^\sigma\}_{\sigma\in\Omega}$ is a $\mathfrak{K}$-family. $\square$

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