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NORMAL WEIGHTED COMPOSITION OPERATORS ON THE FOCK SPACE OF $\mathbb{C}^N$

LIANKUO ZHAO

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Abstract. This paper completely characterizes normal weighted composition operators on the Fock space of $\mathbb{C}^N$, the form of such operators are expressed explicitly. The characterization of self-adjoint weighted composition operators on the Fock space of $\mathbb{C}^N$ is obtained also.

1. Introduction

Let $\mathbb{C}^N$ be the $N-$dimensional complex Euclidean space with $N \geqslant 1$. dm_{2N} denotes usual Lebesgue measure on $\mathbb{C}^N$. The Fock space $\mathscr{F}^2$ is the space of analytic functions f on $\mathbb{C}^N$ with

$$\|f\|^2 = \frac{1}{(2\pi)^N} \int_{\mathbb{C}^N} |f(z)|^2 \exp\Big(-\frac{|z|^2}{2}\Big) dm_{2N}(z),$$

where $|z|$ denotes the norm for $z \in \mathbb{C}^N$. It is well-known that $\mathscr{F}^2$ is a reproducing kernel Hilbert space with inner product

$$\langle f, g\rangle = \frac{1}{(2\pi)^N} \int_{\mathbb{C}^N} f(z)\overline{g(z)} \exp\Big(-\frac{|z|^2}{2}\Big) dm_{2N}(z), \quad f, g \in \mathscr{F}^2$$

and reproducing kernel function

$$K_w(z) = \exp\Big(\frac{\langle z, w\rangle}{2}\Big), \quad w, z \in \mathbb{C}^N,$$

here $\langle z, w\rangle$ denotes the inner product for $z, w \in \mathbb{C}^N$ and $|z|^2 = \langle z, z\rangle$. It is unnecessary to distinguish the inner product symbol in $\mathscr{F}^2$ from that in $\mathbb{C}^N$.

In this paper, we study weighted composition operators on $\mathscr{F}^2$ and completely characterize normal weighted composition operators on $\mathscr{F}^2$. When $N = 1$, such operators have been characterized in [3]. As in other function spaces, weighted composition operators on the Fock space have been studied intensively and extensively. In [6, 7], bounded, compact and Hilbert-Schmidt weighted composition operators on the Fock

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space of complex plane were characterized, the results can be extended to the Fock space of $\mathbb{C}^N$ trivially. It is noteworthy that another simple characterizations of bounded and compact weighted composition operators on the Fock space of complex plane were presented in [3], which were extended to the case of $\mathbb{C}^N$ partially [9]. In [8], unitary weighted composition operators on the Fock space of $\mathbb{C}^N$ and their spectrum were studied. In [11], self-adjoint and complex symmetric weighted composition operators on the Fock space of complex plane were considered. For studies on normal weighted composition operators on other function spaces such as Hardy space and weighted Dirichlet space, see [1, 2, 4, 5]. A complete characterization for normal weighted composition operator on such function spaces is still unsolved.

Recall a weighted composition operator $C_{\psi,\varphi}$ on $\mathscr{F}^2$ with ψ an analytic function on $\mathbb{C}^N$ and φ an analytic self-map of $\mathbb{C}^N$ is defined as

$$C_{\psi,\varphi}f = \psi(f\circ\varphi), \quad f\in\mathscr{F}^2.$$

For an operator A on $\mathbb{C}^N$, $|A|$ means the norm of A. Our main results read as follows.

THEOREM 1.1. *Let ψ be a nonzero analytic function on $\mathbb{C}^N$ and φ be an analytic self-map of $\mathbb{C}^N$. Then $C_{\psi,\varphi}$ is a bounded normal operator on $\mathscr{F}^2$ if and only if*

$$\varphi(z) = Az+b, \quad \psi(z) = sK_c(z), \quad z\in\mathbb{C}^N,$$

where A is a normal operator on $\mathbb{C}^N$ with $|A|\leqslant 1$, $b,c\in\mathbb{C}^N$ with $(I-A)c = (I-A^*)b$, $|b| = |c|$ and s is a nonzero constant. Furthermore

$$\langle A\xi, b+Ac\rangle = 0$$

whenever $|A\xi| = |\xi|$ for $\xi\in\mathbb{C}^N$.

THEOREM 1.2. *Let ψ be a nonzero analytic function on $\mathbb{C}^N$ and φ be an analytic self-map of $\mathbb{C}^N$. Then $C_{\psi,\varphi}$ is a bounded self-adjoint operator on $\mathscr{F}^2$ if and only if*

$$\varphi(z) = Az+b, \quad \psi(z) = sK_b(z),\ z\in\mathbb{C}^N,$$

where A is a self-adjoint operator on $\mathbb{C}^N$ with $|A|\leqslant 1$, $b\in\mathbb{C}^N$ and s is a nonzero real constant. Furthermore,

$$\langle A\xi, (I+A)b\rangle = 0$$

whenever $|A\xi| = |\xi|$ for $\xi\in\mathbb{C}^N$.

2. Proof of main results

In this section, we give the proof of Theorem 1.1 and Theorem 1.2. Firstly, We cite some known results.

LEMMA 2.1. *Let $\psi_1,\cdots,\psi_n$ be analytic functions on $\mathbb{C}^N$ and $\varphi_1,\cdots,\varphi_n$ be analytic self-map of $\mathbb{C}^N$. If $C_{\psi_1,\varphi_1},\cdots,C_{\psi_n,\varphi_n}$ are bounded operators on $\mathscr{F}^2$, then*

$$C_{\psi_1,\varphi_1}C_{\psi_2,\varphi_2}\cdots C_{\psi_n,\varphi_n}=C_{\psi_1(\psi_2\circ\varphi_1)\cdots(\psi_n\circ\varphi_{n-1}\circ\cdots\circ\varphi_1),\ \varphi_n\circ\varphi_{n-1}\circ\cdots\circ\varphi_1}.$$

LEMMA 2.2. *Let ψ be an analytic function on $\mathbb{C}^N$ and φ be an analytic self-map of $\mathbb{C}^N$. If $C_{\psi,\varphi}$ is bounded on $\mathscr{F}^2$, then for $z\in\mathbb{C}^N$,*

$$C^*_{\psi,\varphi}K_z=\overline{\psi(z)}K_{\varphi(z)}.$$

For $p\in\mathbb{C}^N$, let k_p be the normalization of K_p, $\varphi_p(z)=z-p$, $z\in\mathbb{C}^N$ and $U_p=C_{k_p,\varphi_p}$.

LEMMA 2.3. [8, Proposition 2.3] *U_p is a unitary operator on $\mathscr{F}^2$ and $U_p^{-1}=U_{-p}$.*

The following lemma is a combination of Lemma 6 and Proposition 7 in [9].

LEMMA 2.4. *Let ψ be an analytic function on $\mathbb{C}^N$ with $\psi(0)\neq 0$ and φ be an analytic self-map of $\mathbb{C}^N$. If $C_{\psi,\varphi}$ is bounded on $\mathscr{F}^2$, then there exists an operator A on $\mathbb{C}^N$ with $|A|\leqslant 1$, $b\in\mathbb{C}^N$ and a positive constant M such that*

$$\varphi(z)=Az+b,$$

$$|\psi(z)|^2\exp\Big(\frac{|\varphi(z)|^2-|z|^2}{2}\Big)\leqslant M,\ \ z\in\mathbb{C}^N.$$

Moreover, there exists a constant $s\in\mathbb{C}$ such that

$$\psi_\xi(u):=\psi(\xi u)=s\ \exp\Big(-\frac{u\langle A\xi,b\rangle}{2}-\frac{|b|^2}{4}\Big),\ \ u\in\mathbb{C}$$

whenever $|A\xi|=|\xi|$ for $\xi\in\mathbb{C}^N$.

Now we begin the study of normal weighted composition operators on $\mathscr{F}^2$.

LEMMA 2.5. *Let ψ be a nonzero analytic function on $\mathbb{C}^N$ and φ be an analytic self-map of $\mathbb{C}^N$. If $C_{\psi,\varphi}$ is a bounded normal operator on $\mathscr{F}^2$, then ψ has no zeros.*

Proof. If $\psi(w)=0$ for some $w\in\mathbb{C}^N$, then $C^*_{\psi,\varphi}K_w=\overline{\psi(w)}K_{\varphi(w)}=0$. Since $C_{\psi,\varphi}$ is a bounded normal operator on $\mathscr{F}^2$, we have

$$\|C_{\psi,\varphi}K_w\|=\|C^*_{\psi,\varphi}K_w\|=0.$$

So $C_{\psi,\varphi}K_w = \psi\,(K_w \circ \varphi) = 0$, which implies that

$$\psi(z) = 0,\ z \in \mathbb{C}^N,$$

a contradiction. $\square$

The following propositions partially characterize normal weighted composition operators on $\mathscr{F}^2$ from different aspects. In fact, Theorem 1.1 states that any normal weighted composition operator on $\mathscr{F}^2$ has the form as in Proposition 2.6.

PROPOSITION 2.6. [10, Theorem 1.1] *Let $\varphi(z) = Az+b$, $\psi(z) = K_c(z)$, $z \in \mathbb{C}^N$ with A an operator on $\mathbb{C}^N$ and $b,c \in \mathbb{C}^N$. Then $C_{\psi,\varphi}$ is a bounded normal operator on $\mathscr{F}^2$ if and only if the following conditions hold:*

(1) *A is a normal operator on $\mathbb{C}^N$ with $|A| \leqslant 1$,*

(2) *$Ac+b = A^*b+c$,*

(3) *$|c| = |b|$.*

Furthermore, $\langle A\xi, b+Ac\rangle = 0$ whenever $|A\xi| = |\xi|$ for $\xi \in \mathbb{C}^N$.

PROPOSITION 2.7. [10, Proposition2.5] *Let ψ be a nonzero analytic function on $\mathbb{C}^N$ and φ be an analytic self-map of $\mathbb{C}^N$ with $\varphi(p) = p$ for some $p \in \mathbb{C}^n$. Then $C_{\psi,\varphi}$ is a bounded normal operator on $\mathscr{F}^2$ if and only if*

$$\varphi(z) = Az+b, \quad \psi(z) = tK_c(z),\ z \in \mathbb{C}^N,$$

where A is a normal operator on $\mathbb{C}^N$ with $|A| \leqslant 1$, $b = (I-A)p$, $c = (I-A^*)p$ and t is a nonzero constant. Furthermore, $\langle A\xi, (I-AA^*)p\rangle = 0$ whenever $|A\xi| = |\xi|$ for $\xi \in \mathbb{C}^N$. Here I is the identity on $\mathbb{C}^N$.

Let $\varphi(z) = Az+b$, $z \in \mathbb{C}^N$ for some operator A on $\mathbb{C}^N$ with $|A| \leqslant 1$ and $b \in \mathbb{C}^N$. The condition that $\varphi(p) = p$ for some $p \in \mathbb{C}^N$ in Proposition 2.7 is equivalent to $b \in ran(I-A)$, where I is the identity on $\mathbb{C}^N$ and $ran(I-A)$ is the range of $I-A$. So for the complete characterization of normal weighted composition operators on $\mathscr{F}^2$, it is only left to the case that $b \notin ran(I-A)$. In the following, we consider the case that $b \perp ran(I-A)$, i.e., $b \in \ker(I-A^*)$, from which the general case follows.

LEMMA 2.8. *Let A be an operator on $\mathbb{C}^N$ with $|A| \leqslant 1$. If $A^*b = b$ for some $b \in \mathbb{C}^N$, then $Ab = b$.*

Proof. Assume $b \neq 0$. By Cauchy-Schwarz inequality, it follows from $|A| \leqslant 1$ that

$$|\langle Ab, b\rangle| \leqslant |Ab|\,|b| \leqslant |b|^2.$$

Since $A^*b = b$, we have

$$\langle Ab, b\rangle = \langle b, A^*b\rangle = \langle b, b\rangle = |b|^2.$$

So there exists a constant s such that $Ab = sb$. Obviously, $s = 1$ by the equation above. $\square$

PROPOSITION 2.9. *Let $\varphi(z) = Az + b$, $z \in \mathbb{C}^N$ with A an operator on $\mathbb{C}^N$, $|A| \leqslant 1$, $b \in \ker(I - A^*)$ and ψ be a nonzero analytic function on $\mathbb{C}^N$. If $C_{\psi,\varphi}$ is a bounded normal operator on $\mathscr{F}^2$, then there exists a nonzero constant t such that*

$$\psi(z) = tK_{-b}(z), \; z \in \mathbb{C}^N.$$

Proof. Since $C_{\psi,\varphi}$ is a bounded normal operator on $\mathscr{F}^2$ and ψ is nonzero, it follows from Lemma 2.5 that ψ has no zeros.

Since $b \in \ker(I - A^*)$, by Lemma 2.8, $Ab = b$. It follows from Lemma 2.4 that there exists a constant s such that

$$\psi_b(u) = s \, \exp\Big(-\frac{u\langle Ab, b\rangle}{2} - \frac{|b|^2}{4}\Big), \; u \in \mathbb{C}.$$

In particular, $\psi(0) = s \, \exp(-\frac{|b|^2}{4})$ and

$$\psi(b) = \psi(0) \, \exp\Big(-\frac{\langle Ab, b\rangle}{2}\Big) = \psi(0) \, \exp\Big(-\frac{|b|^2}{2}\Big). \tag{2.1}$$

Since $C_{\psi,\varphi}$ is a bounded normal operator on $\mathscr{F}^2$ and $C^*_{\psi,\varphi}K_b = \overline{\psi(b)}K_{\varphi(b)}$, we have

$$\|C_{\psi,\varphi}K_b\|^2 = \|C^*_{\psi,\varphi}K_b\|^2 = |\psi(b)|^2\|K_{\varphi(b)}\|^2.$$

i.e.,

$$\frac{1}{(2\pi)^N}\int_{\mathbb{C}^N} |\psi(z)K_b(\varphi(z))|^2 \exp\Big(-\frac{|z|^2}{2}\Big) dm_{2N}(z) = |\psi(b)|^2 \exp\Big(\frac{|\varphi(b)|^2}{2}\Big). \tag{2.2}$$

By $\varphi(b) = 2b$, $\varphi(0) = b$ and equation (2.1), we obtain

$$|\psi(b)|^2 \exp\Big(\frac{|\varphi(b)|^2}{2}\Big) = |\psi(b)|^2 \exp(2|b|^2) = |\psi(0)|^2 \exp(|b|^2) = \big|\psi(z)K_b(\varphi(z)) \,|_{z=0}\big|^2.$$

Hence it follows from equation (2.2) that

$$\psi(z)K_b(\varphi(z)) = \psi(0)K_b(\varphi(0)) = \psi(0)K_b(b), \; z \in \mathbb{C}^N,$$

which implies that

$$\begin{aligned}
\psi(z) &= \psi(0)K_b(b)K_{-b}(\varphi(z)) \\
&= \psi(0) \, \exp\Big(\frac{|b|^2}{2}\Big) \, \exp\Big(\frac{\langle \varphi(z), -b\rangle}{2}\Big) \\
&= \psi(0)\exp\Big(\frac{\langle Az, -b\rangle}{2}\Big) \\
&= \psi(0)\exp\Big(\frac{\langle z, -b\rangle}{2}\Big) \\
&= \psi(0)K_{-b}(z), \qquad z \in \mathbb{C}^N. \quad \square
\end{aligned}$$

Proof of Theorem 1.1. The sufficiency follows from Proposition 2.6.

Necessity. Since $C_{\psi,\varphi}$ is a bounded normal operator on $\mathscr{F}^2$ and ψ is nonzero, it follows from Lemma 2.5 that ψ has no zeros and then it follows from Lemma 2.4 that

$$\varphi(z) = Az + b, \ \ z \in \mathbb{C}^N$$

where A is an operator on $\mathbb{C}^N$, $|A| \leqslant 1$ and $b \in \mathbb{C}^N$.

Let $\sigma(A)$ denote the spectrum of A. We break the proof up into two cases.

Case 1: $1 \notin \sigma(A)$.

Let $p = (I-A)^{-1}b$, then $\varphi(p) = p$. The conclusion follows from Proposition 2.7.

Case 2: $1 \in \sigma(A)$.

Let $b = b_1 + b_2$ with $b_1 \in ran(I-A)$ and $b_2 \in \ker(I-A^*) = ran(I-A)^{\perp}$. Then $b_1 = (I-A)p$ for some $p \in \mathbb{C}^N$. By Lemma 2.3, $U_{-p}C_{\psi,\varphi}U_p = C_{\Psi_p,\Phi_p}$ is a bounded normal operator on $\mathscr{F}^2$ with

$$\Phi_p(z) = (\varphi_p \circ \varphi \circ \varphi_{-p})(z) = Az + b_2,$$

$$\Psi_p(z) = \big(k_{-p}(\psi \circ \varphi_{-p})(k_p \circ \varphi \circ \varphi_{-p})\big)(z), \ \ z \in \mathbb{C}^N.$$

It follows from Proposition 2.9 that $\Psi_p = tK_{-b_2}$ for some nonzero constant t. So we have

$$\begin{aligned}
\psi(z) &= \frac{\Psi_p(\varphi_p(z))}{k_{-p}(\varphi_p(z))k_p(\varphi(z))} \\
&= t\ \exp\Big(\frac{\langle z-p,-b_2\rangle}{2} + \frac{\langle z-p,p\rangle}{2} + \frac{|p|^2}{4} + \frac{\langle Az+b,-p\rangle}{2} + \frac{|p|^2}{4}\Big) \\
&= t\ \exp\Big(\frac{\langle z,(I-A^*)p-b_2\rangle}{2} + \frac{\langle p,b_2\rangle - \langle b,p\rangle}{2}\Big) \\
&= sK_c(z), \qquad z \in \mathbb{C}^N,
\end{aligned}$$

here $s = t\exp(\frac{\langle p,b_2\rangle - \langle b,p\rangle}{2})$ and $c = (I-A^*)p - b_2$. The conditions satisfied by A, b and c follow from Proposition 2.6. $\square$

Proof of Theorem 1.2. Necessity. If $C_{\psi,\varphi}$ is a bounded self-adjoint operator on $\mathscr{F}^2$, then $C_{\psi,\varphi}$ is a bounded normal operator on $\mathscr{F}^2$. By Theorem 1.1,

$$\varphi(z) = Az + b, \ \ \psi(z) = sK_c(z), z \in \mathbb{C}^N$$

where A is an operator on $\mathbb{C}^N$ with $|A| \leqslant 1$, $b, c \in \mathbb{C}^N$ and s is a nonzero constant. Furthermore, $\langle A\xi, b + Ac\rangle = 0$ whenever $|A\xi| = |\xi|$ for $\xi \in \mathbb{C}^N$.

Since $C^*_{\psi,\varphi} = C_{\psi,\varphi}$, we have

$$\overline{\psi(w)}K_{\varphi(w)}(z) = (C^*_{\psi,\varphi}K_w)(z) = (C_{\psi,\varphi}K_w)(z) = \psi(z)K_w(\varphi(z)), \ \ z, w \in \mathbb{C}^N.$$

It follows from $\varphi(z) = Az + b$ and $\psi(z) = sK_c(z)$ that

$$\bar{s}\exp\left(\frac{\langle c,w\rangle}{2}\right)\exp\left(\frac{\langle z,Aw+b\rangle}{2}\right) = s\exp\left(\frac{\langle z,c\rangle}{2}\right)\exp\left(\frac{\langle Az+b,w\rangle}{2}\right), \quad z,w \in \mathbb{C}^N.$$

So $s = \bar{s}$ and

$$\langle z,b-c\rangle + \langle z,Aw - A^*w\rangle + \langle c-b,w\rangle = i4n(z,w)\pi, \quad z,w \in \mathbb{C}^N, \tag{2.3}$$

where i is the imaginary unit and $n(z,w)$ is an integer-valued continuous function on $\mathbb{C}^{2N}$.

Let $z = w = 0$ in (2.3), then we obtain $n(z,w) = 0$ for $z,w \in \mathbb{C}^N$ and hence

$$\langle z,b-c\rangle + \langle z,Aw - A^*w\rangle + \langle c-b,w\rangle = 0, \quad z,w \in \mathbb{C}^N.$$

By the arbitrariness of z,w, we have

$$b = c, \quad A = A^*.$$

Correspondingly, we have $\langle A\xi,(I+A)b\rangle = 0$ whenever $|A\xi| = |\xi|$ for $\xi \in \mathbb{C}^N$.

Sufficiency. Let $\varphi(z) = Az + b$ and $\psi(z) = sK_b(z)$ with A a operator on $\mathbb{C}^N$, $|A| \leqslant 1$ and $\langle A\xi,(I+A)b\rangle = 0$ for $|A\xi| = |\xi|$. By [10, Lemma 2.3 (1)], $C_{\psi,\varphi}$ is bounded on $\mathscr{F}^2$.

It is easy to verify that

$$(C_{\psi,\varphi}K_w)(z) = s\exp\left(\frac{\langle z,b\rangle + \langle Az+b,w\rangle}{2}\right),$$

$$(C^*_{\psi,\varphi}K_w)(z) = \bar{s}\exp\left(\frac{\langle b,w\rangle + \langle z,Aw+b\rangle}{2}\right).$$

It follows from $A = A^*$ and $s = \bar{s}$ that $C_{\psi,\varphi}K_w = C^*_{\psi,\varphi}K_w$. So we have $C_{\psi,\varphi} = C^*_{\psi,\varphi}$. $\square$

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Liankuo Zhao
School of Mathematics and Computer Science
Shanxi Normal University
041004 Linfen, China
e-mail: liankuozhao@sina.com