

SECOND REGULARIZED TRACE OF A DIFFERENTIAL OPERATOR WITH SECOND ORDER UNBOUNDED OPERATOR COEFFICIENT GIVEN IN A FINITE INTERVAL

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Abstract. In this work, we establish a formula for the second regularized trace of a second order differential operator with unbounded operator coefficient given in a finite interval.

1. Introduction

Let H be a separable Hilbert space with infinite dimension. Let us consider the differential expression

$$\ell_0(y) = -y''(x) + Ay(x)$$

in the space $H_1 = L_2(0, \pi; H)$. In this expression, $A : D(A) \rightarrow H$ is an operator which satisfies the conditions

$$A = A^* > I, \quad A^{-1} \in \sigma_\infty(H)$$

where I is the identity operator on H . $\sigma_\infty(H)$ denotes the set of compact operators from H to H .

Let $\gamma_1 \leq \gamma_2 \leq \dots \leq \gamma_n \leq \dots$ be eigenvalues of the operator A and $\varphi_1, \varphi_2, \dots, \varphi_n$ be orthonormal eigenfunctions corresponding to these eigenvalues respectively. We write out each eigenvalue according to its multiplicity.

We denote by $D(L'_0)$ the set of functions of the space H_1 satisfying the following conditions:

i) $y(x)$ has second order continuous derivative with respect to the norm of the space H in the interval $[0, \pi]$,

ii) $y(x)$ is an element of $D(A)$ for every $x \in [0, \pi]$ and the function $Ay(x)$ is continuous with respect to the norm of the space H on $[0, \pi]$,

iii) $y'(0) = y(\pi) = 0$.

The manifold $D(L'_0)$ is dense in H_1 and the operator $L'_0 : D(L'_0) \rightarrow H_1$ defined by

$$L'_0 y = \ell_0(y)$$

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is symmetric. The eigenvalues of L'_0 are

$$\left(q + \frac{1}{2}\right)^2 + \gamma_r \quad (q = 0, 1, 2, \dots; r = 1, 2, \dots)$$

and the eigenfunctions that correspond to these eigenvalues are

$$\sqrt{\frac{2}{\pi}} \varphi_r \cos\left(q + \frac{1}{2}\right)x \quad (q = 0, 1, 2, \dots; r = 1, 2, \dots)$$

respectively. We denote the closure of the operator L'_0 by L_0 . The operator L_0 is symmetric and the sequence $\left\{ \varphi_r \cos\left(q + \frac{1}{2}\right)x \right\}_{q=0, r=1}^{\infty, \infty}$ is complete in H_1 .

Therefore the operator $L_0 : D(L_0) \rightarrow H_1$ is self-adjoint.

We denote inner products on spaces H and H_1 by $(\cdot, \cdot)_H$ and (\cdot, \cdot) respectively. Also we will denote kernel operators space in H by $\sigma_1(H)$ and denote trace of the kernel operator T by $\text{tr} T$ [1].

We consider the operator function $Q(x)$ and suppose that it satisfies following four conditions:

Q1) $Q(x)$ has fourth order weak derivative and

$$Q'(0) = Q'(\pi) = Q'''(0) = Q'''(\pi) = 0.$$

Q2) $Q^{(i)}(x) : H \rightarrow H$ ($i = 0, 1, 2, 3, 4$) are self-adjoint operators for every $x \in [0, \pi]$.

Q3) The functions $AQ(x)$, $AQ''(x)$, $Q^{iv}(x) \in \sigma_1(H)$ for all $x \in [0, \pi]$ and $\|AQ(x)\|_{\sigma_1(H)}$, $\|AQ''(x)\|_{\sigma_1(H)}$, $\|Q^{iv}(x)\|_{\sigma_1(H)}$ are bounded and measurable in the interval $[0, \pi]$.

Q4)

$$\int_0^\pi (Q(x)f, f)_H dx = 0$$

for every $f \in H$.

Let us consider the operator L formed by differential expression

$$\ell(y) = -y''(x) + Ay(x) + Q(x)y(x)$$

with the boundary conditions

$$y'(0) = y(\pi) = 0$$

in $H_1 = L_2(0, \pi; H)$.

Since the operator $Q : H_1 \rightarrow H_1$ defined by

$$Qy = Q(x)y(x)$$

is self-adjoint, the operator $L : D(L_0) \rightarrow H_1$ defined by

$$L = L_0 + Q$$

is also self-adjoint.

The operators L_0 and L are semi-bounded from below and have pure discrete spectrum.

Let $\mu_1 \leq \mu_2 \leq \dots \leq \mu_n \leq \dots$ and $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$ be eigenvalues of the operator L_0 and L , respectively.

If $\gamma_j \sim a j^\alpha$ as $j \rightarrow \infty$, where $0 < a, \alpha < \infty$, then there exists a positive constant d such that

$$\mu_n, \lambda_n \sim dn^{\frac{2\alpha}{2+\alpha}} \tag{1.1}$$

as $n \rightarrow \infty$ (see e.g. [2]).

By using asymptotic formula (1.1), it can be shown that there exist a subsequence $\{\mu_{n_m}\}_{m=1}^\infty$ of the sequence $\{\mu_n\}_{n=1}^\infty$ such that

$$\mu_k - \mu_{n_m} > d_0 \left(k^{\frac{2\alpha}{2+\alpha}} - n_m^{\frac{2\alpha}{2+\alpha}} \right), \quad (k = n_m + 1, n_m + 2, \dots) \tag{1.2}$$

where d_0 is a positive number. Let $R_\lambda^0 = (L_0 - \lambda I)^{-1}$ and $R_\lambda = (L - \lambda I)^{-1}$ be resolvents of the operators L_0 and L , respectively.

In this work, we find a regularized trace formula as following:

$$\begin{aligned} & \lim_{m \rightarrow \infty} \sum_{k=1}^{n_m} \left(\lambda_k^2 - \mu_k^2 - 2 \sum_{j=2}^p (-1)^j j^{-1} \text{Res}_{\lambda=\mu_k} \text{tr} [\lambda (QR_\lambda^0)^j] \right) \\ &= \frac{1}{8} \left[\text{tr} Q''(\pi) - \text{tr} Q''(0) \right] - \frac{1}{2} \left[\text{tr} A Q(\pi) - \text{tr} A Q(0) \right], \quad (i = 0, 1, 2, 3, 4) \end{aligned} \tag{1.3}$$

where $\{n_m\}_{m=1}^\infty$ is a natural number sequence satisfying condition (1.2). Here $\alpha > 2$ is a constant and $p = \left\lceil \left| \frac{5\alpha+6}{\alpha-2} \right| \right\rceil + 1$. We call the expression on the left-hand side of (1.3) the second regularized trace of L .

The research on the regularized trace of scalar differential operator began with Gelfand-Levitan [3]. In many other works such as [4]–[6] regularized traces of various scalar differential operators have been investigated. The list of the works on the subjects is given in [7, 8]. In [9], a formula for the regularized trace of difference of two Sturm-Liouville operators which is given in half-axis with the bounded operator coefficient is found. The regularized trace of differential operators with operator coefficient has been investigated in the works [10]–[15] and some other works. The asymptotic distribution of eigenvalues is studied and a trace formula for the Sturm-Liouville operator equation obtained in [16]. Furthermore the interested reader is referred to article [17] for more details on spectral theory of Schrödinger operators.

2. Some relations about eigenvalues and resolvents

From (1.1) we can see that the series $\sum_{k=1}^\infty |\mu_k - \lambda|^{-1}$ and $\sum_{k=1}^\infty |\lambda_k - \lambda|^{-1}$ are convergent for $\alpha > 2$ and $\lambda \neq \lambda_k, \mu_k$ ($k = 1, 2, \dots$). Hence, R_λ^0 and R_λ are kernel operators and

$$\text{tr} \left(R_\lambda - R_\lambda^0 \right) = \sum_{k=1}^\infty \left(\frac{1}{\lambda_k - \lambda} - \frac{1}{\mu_k - \lambda} \right)$$

[1]. Multiplying both sides of this equality by $\frac{\lambda^2}{2\pi i}$ and then integrating part by part on the circle $|\lambda| = b_m = \frac{1}{2}(\mu_{n_m} + \mu_{n_m+1})$, we obtain

$$\frac{1}{2\pi i} \int_{|\lambda|=b_m} \lambda^2 \operatorname{tr}(R_\lambda - R_\lambda^0) d\lambda = \sum_{k=1}^{\infty} \frac{1}{2\pi i} \int_{|\lambda|=b_m} \frac{\lambda^2}{\lambda_k - \lambda} d\lambda - \sum_{k=1}^{\infty} \frac{1}{2\pi i} \int_{|\lambda|=b_m} \frac{\lambda^2}{\mu_k - \lambda} d\lambda. \tag{2.1}$$

For large values of m , it can be easily shown that

$$\lambda_{n_m} < b_m < \lambda_{n_m+1}. \tag{2.2}$$

From (2.1) and (2.2), we find

$$\sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2) = -\frac{1}{2\pi i} \int_{|\lambda|=b_m} \lambda^2 \operatorname{tr}(R_\lambda - R_\lambda^0) d\lambda. \tag{2.3}$$

By using (2.3) and the formula $R_\lambda = R_\lambda^0 - R_\lambda Q R_\lambda^0$, the equality

$$\sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2) = \frac{1}{2\pi i} \int_{|\lambda|=b_m} \lambda^2 \operatorname{tr} \left[\sum_{j=1}^p (-1)^{j+1} R_\lambda^0 (Q R_\lambda^0)^j + (-1)^p R_\lambda (Q R_\lambda^0)^{p+1} \right] d\lambda$$

is obtained where $p \geq 1$ is any natural number. The last equality is equivalent to

$$\sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2) = \sum_{j=1}^p D_{mj} + D_m^{(p)} \tag{2.4}$$

where

$$D_{mj} = \frac{(-1)^j}{\pi i j} \int_{|\lambda|=b_m} \lambda \operatorname{tr} [(Q R_\lambda^0)^j] d\lambda, \tag{2.5}$$

$$D_m^{(p)} = \frac{(-1)^p}{2\pi i} \int_{|\lambda|=b_m} \lambda^2 \operatorname{tr} [R_\lambda (Q R_\lambda^0)^{p+1}] d\lambda. \tag{2.6}$$

Let $\{\psi_k(x)\}_{k=1}^\infty$ be orthonormal eigenvectors corresponding to the eigenvalues $\{\mu_k\}_{k=1}^\infty$ of the operator L_0 , respectively. Since the orthonormal eigenvectors of the operator L_0 , corresponding to the eigenvalues

$$\left(q + \frac{1}{2}\right)^2 + \gamma_r \quad (q = 0, 1, 2, \dots; r = 1, 2, \dots),$$

are of the form

$$\sqrt{\frac{2}{\pi}} \varphi_r \cos\left(q + \frac{1}{2}\right)x \quad (q = 0, 1, 2, \dots; r = 1, 2, \dots),$$

respectively, we have

$$\psi_k(x) = \sqrt{\frac{2}{\pi}} \varphi_{r_k} \cos\left(q_k + \frac{1}{2}\right)x \quad (k = 1, 2, \dots). \tag{2.7}$$

From (2.5), we have

$$D_{m1} = -\frac{1}{\pi i} \int_{|\lambda|=b_m} \lambda \operatorname{tr}(QR_\lambda^0) d\lambda. \tag{2.8}$$

Since QR_λ^0 is a kernel operator for all $\lambda \neq \mu_k$ ($k = 1, 2, \dots$) and $\{\psi_k(x)\}_{k=1}^\infty$ is an orthonormal base of the space H_1 we have

$$\operatorname{tr}(QR_\lambda^0) = \sum_{k=1}^\infty (QR_\lambda^0 \psi_k, \psi_k)$$

[1]. If we substitute the last equality in (2.8), we obtain

$$\begin{aligned} D_{m1} &= -\frac{1}{\pi i} \int_{|\lambda|=b_m} \lambda \left[\sum_{k=1}^\infty (QR_\lambda^0 \psi_k, \psi_k) \right] d\lambda \\ &= -\frac{1}{\pi i} \int_{|\lambda|=b_m} \lambda \left[\sum_{k=1}^\infty \frac{(Q\psi_k, \psi_k)}{\mu_k - \lambda} \right] d\lambda \\ &= \frac{1}{\pi i} \sum_{k=1}^\infty (Q\psi_k, \psi_k) \int_{|\lambda|=b_m} \frac{\lambda}{\lambda - \mu_k} d\lambda. \end{aligned} \tag{2.9}$$

From (2.7), (2.9) and the following formula

$$\frac{1}{2\pi i} \int_{|\lambda|=b_m} \frac{\lambda}{\lambda - \mu_k} d\lambda = \begin{cases} \lambda_k, & \text{if } k \leq n_m \\ 0, & \text{if } k > n_m \end{cases}$$

we obtain

$$\begin{aligned} D_{m1} &= 2 \sum_{k=1}^{n_m} \mu_k (Q\psi_k, \psi_k) \\ &= 2 \sum_{k=1}^{n_m} \mu_k \int_0^\pi (Q(x)\psi_k(x), \psi_k(x))_H dx \\ &= \frac{4}{\pi} \sum_{k=1}^{n_m} \mu_k \int_0^\pi (Q(x)\varphi_{r_k} \cos\left(q_k + \frac{1}{2}\right)x, \varphi_{r_k} \cos\left(q_k + \frac{1}{2}\right)x)_H dx \\ &= \frac{4}{\pi} \sum_{k=1}^{n_m} \mu_k \int_0^\pi \cos^2\left(q_k + \frac{1}{2}\right)x (Q(x)\varphi_{r_k}, \varphi_{r_k})_H dx \\ &= \frac{2}{\pi} \sum_{k=1}^{n_m} \mu_k \int_0^\pi [1 + \cos(2q_k + 1)x] (Q(x)\varphi_{r_k}, \varphi_{r_k})_H dx. \end{aligned} \tag{2.10}$$

Since the operator function $Q(x)$ satisfies the condition $Q4$), it follows from (2.4) and (2.10) that

$$\sum_{k=1}^{n_m} (\lambda_k^2 - \mu_k^2) = \frac{2}{\pi} \sum_{k=1}^{n_m} \mu_k \int_0^{\pi} \left(Q(x) \varphi_{r_k}, \varphi_{r_k} \right)_H \cos(2q_k + 1)x dx + \sum_{j=2}^p D_{mj} + D_m^{(p)}. \quad (2.11)$$

THEOREM 2.1. *Let us suppose that the operator function $Q(x)$ satisfies condition $Q1$). If $AQ''(x), Q^v(x) \in \sigma_1(H)$ for every $x \in [0, \pi]$ and the functions $\|AQ''(x)\|_{\sigma_1(H)}$, $\|Q^v(x)\|_{\sigma_1(H)}$ are bounded and measurable in the interval $[0, \pi]$, then we have*

$$\sum_{q=0}^{\infty} \sum_{r=1}^{\infty} \left| \left(\left(q + \frac{1}{2} \right)^2 + \gamma_r \right) \int_0^{\pi} \left(Q(x) \varphi_r, \varphi_r \right)_H \cos(2q + 1)x dx \right| < \infty.$$

Proof. Let $g_r(x) = \left(Q(x) \varphi_r, \varphi_r \right)_H$. Considering $g'_r(0) = g'_r(\pi)$ ($r = 1, 2, \dots$) we obtain

$$\begin{aligned} & \int_0^{\pi} g_r(x) \cos(2q + 1)x dx \\ &= \frac{1}{2q + 1} \int_0^{\pi} g_r(x) \left(\sin(2q + 1)x \right)' dx \\ &= \frac{1}{2q + 1} g_r(x) \sin(2q + 1)x \Big|_0^{\pi} - \frac{1}{2q + 1} \int_0^{\pi} g'_r(x) \sin(2q + 1)x dx \\ &= \frac{1}{(2q + 1)^2} \int_0^{\pi} g'_r(x) \left(\cos(2q + 1)x \right)' dx \\ &= \frac{1}{(2q + 1)^2} g'_r(x) \cos(2q + 1)x \Big|_0^{\pi} - \frac{1}{(2q + 1)^2} \int_0^{\pi} g''_r(x) \cos(2q + 1)x dx \\ &= -\frac{1}{(2q + 1)^2} \int_0^{\pi} g''_r(x) \cos(2q + 1)x dx. \end{aligned}$$

So, we have

$$\begin{aligned} & \left(\left(q + \frac{1}{2} \right)^2 + \gamma_r \right) \int_0^{\pi} g_r(x) \cos(2q + 1)x dx \\ &= -\frac{1}{4} \int_0^{\pi} g''_r(x) \cos(2q + 1)x dx - \frac{\gamma_r}{(2q + 1)^2} \int_0^{\pi} g''_r(x) \cos(2q + 1)x dx. \quad (2.12) \end{aligned}$$

This time if we assume that $g_r'''(0) = g_r'''(\pi) = 0$, then find that

$$\begin{aligned}
 & \int_0^\pi g_r''(x) \cos(2q+1)x dx \\
 &= \frac{1}{2q+1} g_r''(x) \sin(2q+1)x \Big|_0^\pi - \frac{1}{2q+1} \int_0^\pi g_r'''(x) \sin(2q+1)x dx \\
 &= \frac{1}{(2q+1)^2} g_r'''(x) \cos(2q+1)x \Big|_0^\pi - \frac{1}{(2q+1)^2} \int_0^\pi g_r^{iv}(x) \cos(2q+1)x dx \\
 &= -\frac{1}{(2q+1)^2} \int_0^\pi g_r^{iv}(x) \cos(2q+1)x dx. \tag{2.13}
 \end{aligned}$$

From (2.12) and (2.13), we obtain

$$\begin{aligned}
 & \left(\left(q + \frac{1}{2} \right)^2 + \gamma_r \right) \int_0^\pi g_r(x) \cos(2q+1)x dx \\
 &= \frac{1}{4(2q+1)^2} \int_0^\pi g_r^{iv}(x) \cos(2q+1)x dx - \frac{\gamma_r}{(2q+1)^2} \int_0^\pi g_r''(x) \cos(2q+1)x dx.
 \end{aligned}$$

By using the last equality, we find

$$\begin{aligned}
 & \sum_{q=0}^\infty \sum_{r=1}^\infty \left| \left(\left(q + \frac{1}{2} \right)^2 + \gamma_r \right) \int_0^\pi g_r(x) \cos(2q+1)x dx \right| \\
 & \leq \sum_{q=0}^\infty \sum_{r=1}^\infty \left[(q+1)^{-2} \int_0^\pi |g_r^{iv}(x)| dx + \gamma_r (q+1)^{-2} \int_0^\pi |g_r''(x)| dx \right] \\
 & = \sum_{r=1}^\infty \left[\int_0^\pi |g_r^{iv}(x)| dx + \gamma_r \int_0^\pi |g_r''(x)| dx \right] \sum_{q=1}^\infty q^{-2}. \tag{2.14}
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 \sum_{r=1}^\infty \int_0^\pi |g_r^{iv}(x)| dx &= \lim_{n \rightarrow \infty} \int_0^\pi \left[\sum_{r=1}^n |g_r^{iv}(x)| \right] dx \\
 & \leq \int_0^\pi \left[\sum_{r=1}^\infty |g_r^{iv}(x)| \right] dx \\
 & = \int_0^\pi \left[\sum_{r=1}^\infty |(\mathcal{Q}^{iv}(x) \varphi_r, \varphi_r)_H| \right] dx, \tag{2.15}
 \end{aligned}$$

$$\begin{aligned} \sum_{r=1}^{\infty} \gamma_r \int_0^{\pi} |g_r''(x)| dx &\leq \int_0^{\pi} \left[\sum_{r=1}^{\infty} \gamma_r |g_r''(x)| \right] dx \\ &= \int_0^{\pi} \left[\sum_{r=1}^{\infty} |(AQ''(x)\varphi_r, \varphi_r)_H| \right] dx. \end{aligned} \tag{2.16}$$

By the hypothesis, for every $x \in [0, \pi]$, $AQ''(x)$ and $Q^w(x)$ are kernel operators on $\sigma_1(H)$. In this case from [1], we know that

$$\sum_{r=1}^{\infty} |(Q^w(x)\varphi_r, \varphi_r)_H| \leq \|Q^w(x)\|_{\sigma_1(H)} \tag{2.17}$$

$$\sum_{r=1}^{\infty} |(AQ''(x)\varphi_r, \varphi_r)_H| \leq \|AQ''(x)\|_{\sigma_1(H)}. \tag{2.18}$$

From (2.14), (2.15), (2.16), (2.17) and (2.18), we obtain

$$\begin{aligned} &\sum_{q=0}^{\infty} \sum_{r=1}^{\infty} \left| \left(\left(q + \frac{1}{2} \right)^2 + \gamma_r \right) \int_0^{\pi} g_r(x) \cos(2q+1)x dx \right| \\ &< \left[\int_0^{\pi} \|Q^w(x)\|_{\sigma_1(H)} dx + \int_0^{\pi} \|AQ''(x)\|_{\sigma_1(H)} dx \right] \sum_{q=1}^{\infty} q^{-2}. \end{aligned} \tag{2.19}$$

Since the functions $\|Q^w(x)\|_{\sigma_1(H)}$ and $\|AQ''(x)\|_{\sigma_1(H)}$ are assumed to be bounded and measurable in the interval $[0, \pi]$, from (2.19), we find that

$$\sum_{q=0}^{\infty} \sum_{r=1}^{\infty} \left| \left(\left(q + \frac{1}{2} \right)^2 + \gamma_r \right) \int_0^{\pi} (Q(x)\varphi_r, \varphi_r)_H \cos(2q+1)x dx \right| < \infty. \quad \square$$

3. Second regularized trace of the operator L

In this section we establish a formula for second regularized trace of the operator L . Since the eigenvalues of the operator L_0 are of the form

$$\left(q + \frac{1}{2} \right)^2 + \gamma_r \quad (q = 0, 1, 2, \dots; r = 1, 2, \dots)$$

then, we have

$$\mu_k = \left(q_k + \frac{1}{2} \right)^2 + \gamma_{r_k} \quad (k = 1, 2, \dots). \tag{3.1}$$

Moreover, expression (2.5) can be written as follows

$$\begin{aligned} D_{mj} &= 2(-1)^j j^{-1} \frac{1}{2\pi i} \int_{|\lambda|=b_m} \text{tr} \left[\lambda (QR_{\lambda}^0)^j \right] d\lambda \\ &= 2(-1)^j j^{-1} \sum_{k=1}^{n_m} \text{Res}_{|\lambda|=\mu_k} \text{tr} \left[\lambda (QR_{\lambda}^0)^j \right]. \end{aligned} \tag{3.2}$$

From (2.11), (3.1) and (3.2), we obtain

$$\begin{aligned} & \sum_{k=1}^{n_m} \left(\lambda_k^2 - \mu_k^2 - 2 \sum_{j=2}^p (-1)^j j^{-1} \text{Res}_{\lambda=\mu_k} \text{tr} \left[\lambda (QR_\lambda^0)^j \right] \right) \\ &= \frac{2}{\pi} \sum_{k=1}^{n_m} \left(\left(q_k + \frac{1}{2} \right)^2 + \gamma_{r_k} \right) \int_0^\pi \left(Q(x) \varphi_{r_k}, \varphi_{r_k} \right)_H \cos(2q_k + 1)xdx + D_m^{(p)}. \end{aligned} \tag{3.3}$$

By using Theorem 2.1, we obtain

$$\begin{aligned} & \lim_{m \rightarrow \infty} \sum_{k=1}^{n_m} \left(\left(q_k + \frac{1}{2} \right)^2 + \gamma_{r_k} \right) \int_0^\pi \left(Q(x) \varphi_{r_k}, \varphi_{r_k} \right)_H \cos(2q_k + 1)xdx \\ &= \sum_{q=0}^\infty \sum_{r=1}^\infty \left(\left(q + \frac{1}{2} \right)^2 + \gamma_r \right) \int_0^\pi \left(Q(x) \varphi_r, \varphi_r \right)_H \cos(2q + 1)xdx. \end{aligned} \tag{3.4}$$

It is known that ST is element of $\sigma_1(H)$ for every operators $S \in B(H)$ and $T \in \sigma_1(H)$, and the inequalities

$$\left| \text{tr } T \right| \leq \|T\|_{\sigma_1(H)} \tag{3.5}$$

$$\|ST\|_{\sigma_1(H)} \leq \|S\|_H \|T\|_{\sigma_1(H)} \tag{3.6}$$

are satisfied (see [1]). By using (2.6), (3.5) and (3.6) we obtain

$$\begin{aligned} \left| D_m^{(p)} \right| &\leq \int_{|\lambda|=b_m} |\lambda|^2 \left| \text{tr} [R_\lambda (QR_\lambda^0)^{p+1}] \right| |d\lambda| \\ &\leq b_m^2 \int_{|\lambda|=b_m} \left\| R_\lambda (QR_\lambda^0)^{p+1} \right\|_{\sigma_1(H_1)} |d\lambda| \\ &\leq b_m^2 \int_{|\lambda|=b_m} \left\| R_\lambda \right\| \left\| (QR_\lambda^0)^{p+1} \right\|_{\sigma_1(H_1)} |d\lambda| \\ &\leq b_m^2 \int_{|\lambda|=b_m} \left\| R_\lambda \right\| \left\| (QR_\lambda^0)^p \right\| \left\| QR_\lambda^0 \right\|_{\sigma_1(H_1)} |d\lambda| \\ &\leq b_m^2 \int_{|\lambda|=b_m} \left\| R_\lambda \right\| \left\| Q \right\|^p \left\| R_\lambda^0 \right\|^p \left\| Q \right\| \left\| R_\lambda^0 \right\|_{\sigma_1(H_1)} |d\lambda|. \end{aligned} \tag{3.7}$$

It can be shown that the inequalities

$$\begin{aligned} \left\| R_\lambda^0 \right\|_{\sigma_1(H_1)} &\leq \text{const.} n_m^{-\delta}, \\ \left\| R_\lambda \right\| &\leq \text{const.} n_m^{-\delta} \quad \left(\delta = \frac{\alpha - 2}{\alpha + 2} \right) \end{aligned} \tag{3.8}$$

are satisfied similarly in work [13]. From (3.7) and (3.8), we obtain

$$|D_m^{(p)}| \leq \text{const} b_m^3 n_m^{-(1+p)\delta} n_m^{1-\delta}. \tag{3.9}$$

By recalling $b_m = 2^{-1}(\mu_{n_m} + \mu_{n_m+1})$, from (1.1) and (3.9) we obtain

$$|D_m^{(p)}| \leq \text{const} .n_m^{3(1+\delta)} n_m^{-(1+p)\delta} n_m^{1-\delta} = \text{const} .n_m^{4-(p-1)\delta}. \tag{3.10}$$

From (3.10), for $p = \lceil \frac{4}{\delta} + 1 \rceil + 1$ or $p = \lceil \frac{5\alpha+6}{\alpha-2} \rceil + 1$, we find

$$\lim_{m \rightarrow \infty} D_m^{(p)} = 0. \tag{3.11}$$

From (3.3), (3.4) and (3.11) we obtain

$$\begin{aligned} & \lim_{m \rightarrow \infty} \sum_{k=1}^{n_m} \left(\lambda_k^2 - \mu_k^2 - 2 \sum_{j=2}^p (-1)^j j^{-1} \text{Res}_{\lambda=\mu_k} \text{tr} [\lambda (QR_\lambda^0)^j] \right) \\ &= \frac{2}{\pi} \sum_{q=0}^{\infty} \sum_{r=1}^{\infty} \left(\left(q + \frac{1}{2} \right)^2 + \gamma_r \right) \int_0^\pi (Q(x) \varphi_r, \varphi_r)_H \cos(2q+1) x dx \end{aligned} \tag{3.12}$$

where $\alpha > 0$ and $p = \lceil \frac{5\alpha+6}{\alpha-2} \rceil + 1$.

THEOREM 3.1. *If the operator function $Q(x)$ satisfies conditions Q1)–Q4) and $\gamma_j \sim a j^\alpha$ as $j \rightarrow \infty$ ($0 < a < \infty$, $2 < \alpha < \infty$), then we have*

$$\begin{aligned} & \lim_{m \rightarrow \infty} \sum_{k=1}^{n_m} \left(\lambda_k^2 - \mu_k^2 - 2 \sum_{j=2}^p (-1)^j j^{-1} \text{Res}_{\lambda=\mu_k} \text{tr} [\lambda (QR_\lambda^0)^j] \right) \\ &= \frac{1}{8} [\text{tr} Q''(\pi) - \text{tr} Q''(0)] - \frac{1}{2} [\text{tr} A Q(\pi) - \text{tr} A Q(0)], \end{aligned}$$

where $p = \lceil \frac{5\alpha+6}{\alpha-2} \rceil$.

Proof. By using Theorem 2.1, the formula (3.12) can be written as follows

$$\begin{aligned} & \lim_{m \rightarrow \infty} \sum_{k=1}^{n_m} \left(\lambda_k^2 - \mu_k^2 - 2 \sum_{j=2}^p (-1)^j j^{-1} \text{Res}_{\lambda=\mu_k} \text{tr} [\lambda (QR_\lambda^0)^j] \right). \\ &= \frac{2}{\pi} \sum_{q=0}^{\infty} \sum_{r=1}^{\infty} \left(q + \frac{1}{2} \right)^2 \int_0^\pi (Q(x) \varphi_r, \varphi_r)_H \cos(2q+1) x dx \\ &+ \frac{2}{\pi} \sum_{q=0}^{\infty} \sum_{r=1}^{\infty} \gamma_r \int_0^\pi (Q(x) \varphi_r, \varphi_r)_H \cos(2q+1) x dx. \end{aligned} \tag{3.13}$$

Here, by using the equality

$$\int_0^\pi (Q(x)\varphi_r, \varphi_r)_H \cos(2q+1)x dx = -\frac{1}{(2q+1)^2} \int_0^\pi (Q''(x)\varphi_r, \varphi_r)_H \cos(2q+1)x dx$$

for the first sum on the right-hand side of (3.13), we obtain

$$\begin{aligned} & \frac{2}{\pi} \sum_{q=0}^\infty \sum_{r=1}^\infty \left(q + \frac{1}{2}\right)^2 \int_0^\pi (Q(x)\varphi_r, \varphi_r)_H \cos(2q+1)x dx \\ &= -\frac{1}{2\pi} \sum_{q=0}^\infty \sum_{r=1}^\infty \int_0^\pi (Q''(x)\varphi_r, \varphi_r)_H \cos(2q+1)x dx \\ &= -\frac{1}{2\pi} \sum_{q=0}^\infty \int_0^\pi \left[\sum_{r=1}^\infty (Q''(x)\varphi_r, \varphi_r)_H \right] \cos(2q+1)x dx \\ &= -\frac{1}{2\pi} \sum_{q=0}^\infty \int_0^\pi \text{tr} Q''(x) \cos(2q+1)x dx \\ &= -\frac{1}{4\pi} \sum_{q=1}^\infty \left[\int_0^\pi \text{tr} Q''(x) \cos qx dx - (-1)^q \int_0^\pi \text{tr} Q''(x) \cos qx dx \right] \\ &\quad - \frac{1}{8} \left\{ \sum_{q=1}^\infty \left[\frac{2}{\pi} \int_0^\pi \text{tr} Q''(x) \cos qx dx \right] \cos q0 + \left[\frac{1}{\pi} \int_0^\pi \text{tr} Q''(x) dx \right] \cos 0 \right\} \\ &\quad + \frac{1}{8} \left\{ \sum_{q=1}^\infty \left[\frac{2}{\pi} \int_0^\pi \text{tr} Q''(x) \cos qx dx \right] \cos q\pi + \left[\frac{1}{\pi} \int_0^\pi \text{tr} Q''(x) dx \right] \cos 0\pi \right\} \\ &= \frac{1}{8} [\text{tr} Q''(\pi) - \text{tr} Q''(0)] \tag{3.14} \end{aligned}$$

Moreover, we have

$$\begin{aligned} & \frac{2}{\pi} \sum_{q=0}^\infty \sum_{r=1}^\infty \gamma_r \int_0^\pi (Q(x)\varphi_r, \varphi_r)_H \cos(2q+1)x dx \\ &= \frac{2}{\pi} \sum_{q=0}^\infty \int_0^\pi \sum_{r=1}^\infty (AQ(x)\varphi_r, \varphi_r)_H \cos(2q+1)x dx \\ &= \frac{2}{\pi} \sum_{q=0}^\infty \int_0^\pi \text{tr} AQ(x) \cos(2q+1)x dx \\ &= \frac{1}{\pi} \sum_{q=1}^\infty \left[\int_0^\pi \text{tr} AQ(x) \cos qx dx - (-1)^q \int_0^\pi \text{tr} AQ(x) \cos qx dx \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left\{ \sum_{q=1}^{\infty} \left[\frac{2}{\pi} \int_0^{\pi} \operatorname{tr} A Q(x) \cos q x dx \right] \cos q 0 + \left[\frac{1}{\pi} \int_0^{\pi} \operatorname{tr} A Q(x) dx \right] \cos 0 \right\} \\
&\quad - \frac{1}{2} \left\{ \sum_{q=1}^{\infty} \left[\frac{2}{\pi} \int_0^{\pi} \operatorname{tr} A Q(x) \cos q x dx \right] \cos q \pi + \left[\frac{1}{\pi} \int_0^{\pi} \operatorname{tr} A Q(x) dx \right] \cos 0 \pi \right\} \\
&= -\frac{1}{2} \left[\operatorname{tr} A Q(\pi) - \operatorname{tr} A Q(0) \right]. \tag{3.15}
\end{aligned}$$

From (3.13), (3.14) and (3.15), we find the formula

$$\begin{aligned}
&\lim_{m \rightarrow \infty} \sum_{k=1}^{n_m} \left(\lambda_k^2 - \mu_k^2 - 2 \sum_{j=2}^p (-1)^j j^{-1} \operatorname{Res}_{\lambda=\mu_k} \operatorname{tr} \left[\lambda (Q R_{\lambda}^0)^j \right] \right) \\
&= \frac{1}{8} \left[\operatorname{tr} Q''(\pi) - \operatorname{tr} Q''(0) \right] - \frac{1}{2} \left[\operatorname{tr} A Q(\pi) - \operatorname{tr} A Q(0) \right]
\end{aligned}$$

for second regularized trace formula of the operator L . \square

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