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GRAM MATRICES OF REPRODUCING KERNEL HILBERT SPACES OVER GRAPHS III

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Abstract. We study reproducing kernel Hilbert spaces induced by inclusion $G_1 \subset G_2$ of two connected graphs having a common vertex set. Under a certain finiteness condition, it is shown that the dimensions of de Branges-Rovnyak complements associated with inclusion $G_1 \subset G_2$ are described by the language of graph theory.

1. Introduction

Let G_1 and G_2 be connected graphs with a common vertex set V, where V is assumed to be at most countable. Throughout this paper, we suppose that G_1 is a subgraph of G_2. Further, we assume that every graph appearing in this paper is locally finite, that is, number of edges connecting with x is finite for every x in V. For undefined terms of graphs, see Section 2. Let L_j (resp. $A^{(j)}$) be the Laplace matrix (resp. the adjacency matrix) of G_j for $j = 1,2$. Then we define a densely defined bi-linear form on $\ell^2(V)$, the set of all square summable real functions on V, as follows:

$$\begin{aligned}\langle u,v\rangle_{\mathscr{H}_{G_j}} &= \langle u,v\rangle_{\ell^2(V)} + \sum \langle L_j u, v\rangle_{\ell^2(V)}\\ &= \langle u,v\rangle_{\ell^2(V)} + \frac{1}{2}\sum_{x,y\in V} A^{(j)}_{x,y}(u(x)-u(y))(v(x)-v(y)),\end{aligned}$$

Let $\mathscr{H}_{G_j}$ denote the real Hilbert space induced by $\langle \cdot,\cdot\rangle_{\mathscr{H}_{G_j}}$. Further, it is well known that $\mathscr{H}_{G_j}$ has reproducing kernels, that is, for any x in V, there exists a unique vector $k_x^{(j)}$ in $\mathscr{H}_{G_j}$ such that

$$u(x) = \langle u, k_x^{(j)}\rangle_{\mathscr{H}_{G_j}} \quad (u \in \mathscr{H}_{G_j}).$$

$k_x^{(j)}$ is called the reproducing kernel of $\mathscr{H}_{G_j}$ at x, and

$$K_j = (\langle k_x^{(j)}, k_y^{(j)}\rangle_{\mathscr{H}_{G_j}})_{x,y\in V}$$

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will be called the Gram matrix of $\mathscr{H}_{G_j}$. It is not hard to see that $K_1 - K_2$ is positive semi-definite, for example, which is deduced from Lemma 3.1 in this paper. Let T be the canonical embedding map of $\mathscr{H}_{G_2}$ into $\mathscr{H}_{G_1}$, and let $\mathscr{H}(T)$ (resp. $\mathscr{H}(T^*)$) denote the de Branges-Rovnyak complement associated with T (resp. T^*). Some details of these spaces are given in Section 3.

Let V_0 be the set of vertices which are not isolated in $G_2 - G_1 = (V, E_2 \setminus E_1)$. In graph theory, $G_2 - G_1$ is called the relative complement of G_1 in G_2. Now, we set $G_2 \ominus G_1 = (V_0, E_2 \setminus E_1)$. Note that $G_2 \ominus G_1$ is the graph obtained by deleting isolated vertices from $G_2 - G_1$. The main result of this paper is the following formula holding under the condition that $|E_2 \setminus E_1|$ is finite:

$$\dim \mathscr{H}(T) = \dim \mathscr{H}(T^*) = |V_0| - \chi(G_2 \ominus G_1) < \infty \tag{1.1}$$

(Theorem 4.1 and Theorem 5.1).

We shall mention that there is a large number of researches on graphs from the view point of electric networks. One of standard references will be Doyle-Snell [3]. For recent progress, see Jorgensen-Tian [4, 5] and Jorgensen-Pearse [6]. In these papers, we will find that reproducing kernels play important role for graph theory.

2. Preliminaries from graph theory

In this section, we collect the terminology for graphs, see Bollobás [1] for more details. A graph G is a pair (V, E), V is called the vertex set (in this paper, we assume that V is at most countable), and E is a subset of the set of $\{\{x,y\} \mid x,y \in V, x \neq y\}$, called the edge set. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, G_1 is called a subgraph of G_2 if $V_1 \subset V_2$ and $E_1 \subset E_2$ hold. A graph G is said to be connected if for any distinct vertices x and y, there exist edges $\{z_1, z_2\}, \{z_2, z_3\}, \dots$ and $\{z_{l-1}, z_l\}$ such that $x = z_1$ and $z_l = y$. If G is not connected, then G is a disjoint union of connected subgraphs of G, and those connected subgraphs are called connected components of G. For a subgraph G_1 of G_2 with a common vertex set, the relative complement of G_1 in G_2, which will be denoted by $G_2 - G_1$, is defined to be $(V, E_2 \setminus E_1)$. For two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_1, E_2)$, a homomorphism $\iota : G_1 \to G_2$ is a map from V_1 to V_2 such that if $\{x,y\} \in E_1$ then $\{\iota(x), \iota(y)\} \in E_2$. In this paper, we deal only with the case $V_1 = V_2$ and $E_1 \subset E_2$, namely the case where G_1 is a subgraph of G_2 with a common vertex set. The degree d_x of G at $x \in V$ is the number of edges connecting with x. G is said to be locally finite if d_x is finite for every x in V. The adjacency matrix A and the Laplace matrix L of G are $|V| \times |V|$ matrices with rows and columns indexed by the elements of V such that for $x, y \in V$,

$$A_{xy} = \begin{cases} 1 & \text{if } \{x,y\} \in E, \\ 0 & \text{otherwise}, \end{cases} \qquad L_{xy} = \begin{cases} d_x & \text{if } x = y, \\ -1 & \text{if } \{x,y\} \in E, \\ 0 & \text{otherwise}. \end{cases}$$

Let us assign an orientation to any edge in the graph G. For an edge $e = \{x,y\} \in E$, the oriented edge e is a pair either (x,y) or (y,x). When an oriented edge is (x,y), we say x is the initial vertex and y is the terminal vertex. For the orientation, we define the incidence matrix B by a $|V|\times|E|$ matrix with rows indexed by the elements of V and columns indexed by the elements of E such that for $x\in V$ and $e\in E$,

$$B_{x,e}=\begin{cases}1 & \text{if } x \text{ is the initial vertex of the edge } e,\\ -1 & \text{if } x \text{ is the terminal vertex of the edge } e,\\ 0 & \text{otherwise.}\end{cases}$$

3. Preliminaries from operator theory

In this section, lemmas needed to prove (1.1) are given. Although some of them are quite elementary or have been given previously in our papers [8], [9] and [10], we shall give full proofs for general readers.

Let $\iota : G_1 \to G_2$ be the homomorphism such that ι fixes vertices, namely $\iota(x)=x$ for any x in V, and let T denote the embedding map $u\mapsto u\circ\iota=u$ from $\mathscr{H}_{G_2}$ to $\mathscr{H}_{G_1}$.

LEMMA 3.1. The following hold.

(i) $\|Tu\|_{\mathscr{H}_{G_1}}\leqslant\|u\|_{\mathscr{H}_{G_2}}$,

(ii) $T^*k_x^{(1)}=k_x^{(2)}$,

(iii) T and T^* are invertible if $|E_2\setminus E_1|$ is finite.

Proof. First, since

$$\begin{aligned}\|u\|^2_{\mathscr{H}_{G_2}}-\|Tu\|^2_{\mathscr{H}_{G_1}}&=\frac{1}{2}\sum_{x,y\in V}A^{(2)}_{x,y}|u(x)-u(y)|^2-\frac{1}{2}\sum_{x,y\in V}A^{(1)}_{x,y}|u(x)-u(y)|^2\\&=\frac{1}{2}\sum_{x,y\in V}(A^{(2)}_{x,y}-A^{(1)}_{x,y})|u(x)-u(y)|^2,\end{aligned}$$

we have (i). Next,

$$\langle u,T^*k_x^{(1)}\rangle_{\mathscr{H}_{G_2}}=\langle Tu,k_x^{(1)}\rangle_{\mathscr{H}_{G_1}}=\langle u,k_x^{(1)}\rangle_{\mathscr{H}_{G_1}}=u(x)=\langle u,k_x^{(2)}\rangle_{\mathscr{H}_{G_2}}.$$

This concludes (ii). If $|E_2\setminus E_1|$ is finite, then $\mathscr{H}_{G_1}=\mathscr{H}_{G_2}$ as linear spaces. By the open mapping theorem, we have (iii). $\square$

We note that $I_{\mathscr{H}_{G_2}}-T^*T$ and $I_{\mathscr{H}_{G_1}}-TT^*$ are positive semi-definite by (i) of Lemma 3.1. These two operators will play important roles in our study. We also note that K_1-K_2 is positive semi-definite. Indeed, for any finite sequence $(c_x)_{x\in V}$, we have that

$$\begin{aligned}\langle(K_1-K_2)(c_x),(c_y)\rangle_{\ell^2(V)}&=\sum_{x,y\in V}c_xc_y(\langle k_x^{(1)},k_y^{(1)}\rangle_{\mathscr{H}_{G_1}}-\langle k_x^{(2)},k_y^{(2)}\rangle_{\mathscr{H}_{G_2}})\\&=\langle(I-TT^*)\sum_{x\in V}c_xk_x^{(1)},\sum_{x\in V}c_yk_y^{(1)}\rangle_{\mathscr{H}_{G_1}}.\end{aligned}$$

For a subset $\mathscr{X}$ of a linear space, span $\mathscr{X}$ will denote the linear subspace generated by $\mathscr{X}$.

LEMMA 3.2. *If $|E_2 \setminus E_1|$ is finite, then*

$$\operatorname{ran}(I_{\mathscr{H}_{G_2}} - T^*T) = \operatorname{span}\{k_x^{(2)} - k_y^{(2)} : A_{x,y}^{(1)} < A_{x,y}^{(2)}\}.$$

Proof. We set

$$\mathscr{S} = \operatorname{span}\{k_x^{(2)} - k_y^{(2)} : A_{x,y}^{(1)} < A_{x,y}^{(2)}\}.$$

Then, since

$$\begin{aligned}\langle (I_{\mathscr{H}_{G_2}} - T^*T)u, u\rangle_{\mathscr{H}_{G_2}} &= \|u\|^2_{\mathscr{H}_{G_2}} - \|u\|^2_{\mathscr{H}_{G_1}} \\ &= \frac{1}{2}\sum_{x,y\in V}(A_{x,y}^{(2)} - A_{x,y}^{(1)})|u(x)-u(y)|^2,\end{aligned}$$

we have that $\ker(I_{\mathscr{H}_{G_2}} - T^*T) = \mathscr{S}^{\perp}$ in $\mathscr{H}_{G_2}$, and which is equivalent to that $\operatorname{ran}(I_{\mathscr{H}_{G_2}} - T^*T) = \mathscr{S}$. This concludes the proof. □

REMARK 3.1. We note that the value of inner product $\langle (I_{\mathscr{H}_{G_2}} - T^*T)u, u\rangle_{\mathscr{H}_{G_2}}$ is dependent only on Laplace matrices L_1 and L_2.

LEMMA 3.3. If $|E_2 \setminus E_1|$ is finite, then

$$\operatorname{ran}(I_{\mathscr{H}_{G_1}} - TT^*) = \operatorname{span}\{k_x^{(1)} - k_y^{(1)} : A_{x,y}^{(1)} < A_{x,y}^{(2)}\}.$$

Proof. Since $T^* : \mathscr{H}_{G_1} \to \mathscr{H}_{G_2}$ is invertible by (iii) of Lemma 3.1 and $T^*(I_{\mathscr{H}_{G_1}} - TT^*) = (I_{\mathscr{H}_{G_2}} - T^*T)T^*$,

$$T^*|_{\operatorname{ran}(I_{\mathscr{H}_{G_1}} - TT^*)} : \operatorname{ran}(I_{\mathscr{H}_{G_1}} - TT^*) \to \operatorname{ran}(I_{\mathscr{H}_{G_2}} - T^*T)$$

is a bijective linear mapping. By (ii) of Lemma 3.1 and Lemma 3.2, we have the conclusion. □

Now, we shall give another inner product on $\operatorname{ran}(I_{\mathscr{H}_{G_1}} - TT^*)^{1/2}$ as follows:

$$\langle (I_{\mathscr{H}_{G_1}} - TT^*)^{1/2}u, (I_{\mathscr{H}_{G_1}} - TT^*)^{1/2}v\rangle_{\mathscr{H}(T)} = \langle Pu, Pv\rangle_{\mathscr{H}_{G_1}},$$

where P denotes the orthogonal projection onto the orthogonal complement of $\ker(I_{\mathscr{H}_{G_1}} - TT^*)^{1/2}$ in $\mathscr{H}_{G_1}$. Let $\mathscr{H}(T)$ be the Hilbert space with the inner product defined above, that is, we set

$$\mathscr{H}(T) = (\operatorname{ran}(I_{\mathscr{H}_{G_1}} - TT^*)^{1/2}, \langle\cdot,\cdot\rangle_{\mathscr{H}(T)}).$$

$\mathscr{H}(T)$ is called the de Branges-Rovnyak complement induced by T. $\mathscr{H}(T^*)$ is defined similarly. See Ball-Bolotnikov [2] or Sarason [7] for details of general de Branges-Rovnyak space theory.

4. Structure of $\mathscr{H}(T^*)$

In this section we shall study the structure of $\mathscr{H}(T^*)$. Let V_0 be the set of vertices which are not isolated in $G_2 - G_1 = (V, E_2 \setminus E_1)$, and we set $G_2 \ominus G_1 = (V_0, E_2 \setminus E_1)$.

THEOREM 4.1. *If $|E_2 \setminus E_1|$ is finite, then*

$$\dim \mathscr{H}(T^*) = |V_0| - \chi(G_2 \ominus G_1).$$

Proof. If $|E_2 \setminus E_1|$ is finite, $I_{\mathscr{H}_{G_2}} - T^*T$ is a finite rank self-adjoint operator by Lemma 3.2. Then, by elementary spectral theory, we have that $\operatorname{ran}(I_{\mathscr{H}_{G_2}} - T^*T)^{1/2} = \operatorname{ran}(I_{\mathscr{H}_{G_2}} - T^*T)$. Hence it suffices to show that

$$\dim \operatorname{ran}(I_{\mathscr{H}_{G_2}} - T^*T) = |V_0| - \chi(G_2 \ominus G_1).$$

Let S be the linear operator induced by the canonical embedding map of $G_2 \ominus G_1$ into G_2, and j_x denote the reproducing kernel of $\mathscr{H}_{G_2 \ominus G_1}$. Then, by (ii) of Lemma 3.1, we have that $S^* j_x = k_x^{(2)}$ for any x in V_0. Hence we have that

$$S^* \operatorname{span}\{j_x - j_y : A_{x,y}^{(1)} < A_{x,y}^{(2)}\} = \operatorname{span}\{k_x^{(2)} - k_y^{(2)} : A_{x,y}^{(1)} < A_{x,y}^{(2)}\}.$$

Since $\ker S^*$ is trivial, it suffices to show that

$$\dim \operatorname{span}\{j_x - j_y : A_{x,y}^{(1)} < A_{x,y}^{(2)}\} = |V_0| - \chi(G_2 \ominus G_1).$$

However, this is trivial, because

$$\dim(\mathscr{H}_{G_2 \ominus G_1} \ominus \operatorname{span}\{j_x - j_y : A_{x,y}^{(1)} < A_{x,y}^{(2)}\}) = \chi(G_2 \ominus G_1). \quad \square$$

Here, the authors would like to state another proof of Theorem 4.1. Although this second proof having graph theoretical flavor is slightly complicated, it seems to give us further information on the structure of $\mathscr{H}(T^*)$.

Proof. We fix an orientation of $H = G_2 \ominus G_1 = (V_0, E_2 \setminus E_1)$. Let B be the incidence matrix of H. Since $H = G_2 \ominus G_1$ is a finite graph, setting $V_0 = \{x_1, \ldots, x_n\}$, $E_2 \setminus E_1 = \{e_1, \ldots, e_m\}$, B can be written as follows:

$$B = \begin{pmatrix} b_{x_1,e_1} & \cdots & b_{x_1,e_m} \\ b_{x_2,e_1} & \cdots & b_{x_2,e_m} \\ \vdots & & \vdots \\ b_{x_n,e_1} & \cdots & b_{x_n,e_m} \\ & O & \end{pmatrix}.$$

Then, the space $\operatorname{span}\{k_x^{(2)} - k_y^{(2)} : A_{x,y}^{(1)} < A_{x,y}^{(2)}\}$ coincides with the column space of

$$F_2 = K_2 B = \begin{pmatrix} k_{x_1}^{(2)}(x_1) & \cdots & k_{x_1}^{(2)}(x_n) & * \\ \vdots & & \vdots & \\ k_{x_n}^{(2)}(x_1) & \cdots & k_{x_n}^{(2)}(x_n) & * \\ & * & & * \end{pmatrix} \begin{pmatrix} b_{x_1,e_1} & \cdots & b_{x_1,e_m} \\ b_{x_2,e_1} & \cdots & b_{x_2,e_m} \\ \vdots & & \vdots \\ b_{x_n,e_1} & \cdots & b_{x_n,e_m} \\ & O & \end{pmatrix}.$$

Moreover, for $\mathbf{c} = (c_1, \ldots, c_n, 0, \ldots)$ in $\ker K_2|_{\mathbb{R}^n \oplus \{0\}}$, $K_2\mathbf{c} = 0$ is equivalent to that

$$\sum_{j=1}^{n} c_j k_x^{(2)}(x_j) = 0 \quad (x \in V_0)$$

and

$$\sum_{j=1}^{n} c_j k_y^{(2)}(x_j) = 0 \quad (y \in V \setminus V_0).$$

Hence we have that

$$\sum_{j=1}^{n} c_j k_{x_j}^{(2)}(z) = \sum_{j=1}^{n} c_j k_z^{(2)}(x_j) = 0 \quad (z \in V).$$

Since reproducing kernels $k_{x_1}^{(2)}, \ldots, k_{x_n}^{(2)}$ are linearly independent, the dimension of the range space of $F_2 = K_2 B$ is equal to $\operatorname{rank} B$, and we have

$$\dim \operatorname{span}\{k_x^{(2)} - k_y^{(2)} : A_{x,y}^{(1)} < A_{x,y}^{(2)}\} = \operatorname{rank} F_2 = \operatorname{rank} K_2 B = \operatorname{rank} B. \tag{4.1}$$

Moreover, it is known that the kernel of B coincides with the cycle space $Z(H)$ of H (see p. 55 of [1]). Then by Theorem 9 in Section II of [1], we have that

$$\operatorname{rank} B = |E(H)| - \dim Z(H) = |E(H)| - (|E(H)| - |V_0| + \chi(H)) = |V_0| - \chi(H). \tag{4.2}$$

Combining the equations (4.1) and (4.2), we have

$$\dim \operatorname{span}\{k_x^{(2)} - k_y^{(2)} : A_{x,y}^{(1)} < A_{x,y}^{(2)}\} = |V_0| - \chi(H).$$

This concludes the proof. $\square$

Next, we shall see that $k_x^{(2)} - k_y^{(2)}$ plays remarkable role in $\mathscr{H}(T^*)$. Fixing an orientation of $G_2 \ominus G_1 = (V_0, E_2 \setminus E_1)$, we set $f_e^{(2)} = k_x^{(2)} - k_y^{(2)}$ for $e = \{x, y\}$ in E_2.

THEOREM 4.2. *If $|E_2 \setminus E_1|$ is finite, then $\{f_e^{(2)} : e \in E_2 \setminus E_1\}$ is a Parseval frame for $\mathscr{H}(T^*)$, that is,*

$$\|u\|_{\mathscr{H}(T^*)}^2 = \sum_{e \in E_2 \setminus E_1} |\langle u, f_e^{(2)} \rangle_{\mathscr{H}(T^*)}|^2 \quad (u \in \mathscr{H}(T^*)).$$

Proof. For $u = (I_{\mathscr{H}_{G_2}} - T^*T)a$, we may take this a from $(\ker(I_{\mathscr{H}_{G_2}} - T^*T))^{\perp}$. Then we have that

$$\begin{aligned}
\|u\|^2_{\mathscr{H}(T^*)} &= \langle (I_{\mathscr{H}_{G_2}} - T^*T)a, a\rangle_{\mathscr{H}_{G_2}} \\
&= \|a\|^2_{\mathscr{H}_{G_2}} - \|a\|^2_{\mathscr{H}_{G_1}} \\
&= \sum_{\{x,y\}\in E_2\setminus E_1} |a(x) - a(y)|^2 \\
&= \sum_{e\in E_2\setminus E_1} |\langle a, f_e^{(2)}\rangle_{\mathscr{H}_{G_2}}|^2 \\
&= \sum_{e\in E_2\setminus E_1} |\langle (I_{\mathscr{H}_{G_2}} - T^*T)a, f_e^{(2)}\rangle_{\mathscr{H}(T^*)}|^2 \\
&= \sum_{e\in E_2\setminus E_1} |\langle u, f_e^{(2)}\rangle_{\mathscr{H}(T^*)}|^2. \quad \square
\end{aligned}$$

5. Structure of $\mathscr{H}(T)$

In this section, we shall study the structure of $\mathscr{H}(T)$. Let $\operatorname{dom}(K_1 - K_2)$ be the set of finite sequences in $\ell^2(V)$, and we set $h_x = k_x^{(1)} - k_x^{(2)}$. We define two linear spaces as follows:

$$\begin{aligned}
\mathscr{V} &= \{\sum_{x\in V} c_x h_x : (c_x)_{x\in V} \in \operatorname{dom}(K_1 - K_2)\}, \\
\mathscr{N} &= \{\sum_{x\in V} c_x h_x \in \mathscr{V} : \sum_{x,y\in V} c_x c_y h_x(y) = 0\}.
\end{aligned}$$

We note that

$$\langle (K_1 - K_2)(c_x), (c_y)\rangle_{\ell^2(V)} = \sum_{x,y\in V} c_x c_y h_x(y)$$

for any $(c_x)_{x\in V}$ in $\operatorname{dom}(K_1 - K_2)$. Then $\langle [h_x], [h_y]\rangle = h_x(y)$ defines an inner product on $\mathscr{V}/\mathscr{N}$, where $[\cdot]$ denotes an equivalence class in $\mathscr{V}/\mathscr{N}$. Taking the completion of $\mathscr{V}/\mathscr{N}$ with respect to the norm induced by the above inner product, we obtain reproducing kernel Hilbert space $\mathscr{H}_{K_1 - K_2}$.

THEOREM 5.1. *If $|E_2 \setminus E_1|$ is finite, then*

(i) $\dim \mathscr{H}(T) = |V_0| - \chi(G_2 \ominus G_1)$,

(ii) $\mathscr{H}(T)$ *is isomorphic to* $\mathscr{H}_{K_1 - K_2}$.

Proof. By Lemma 3.3 and Theorem 4.1, we have (i). We shall show (ii). Here, we note that $\mathscr{H}(T)$ has also a reproducing kernel Hilbert space structure. Let P denote the orthogonal projection onto the orthogonal complement of $\ker(I_{\mathscr{H}_{G_1}} - TT^*)^{1/2}$ in $\mathscr{H}_{G_1}$. If $u = (I_{\mathscr{H}_{G_1}} - TT^*)^{1/2}v$, then

$$\begin{aligned}\langle u, (I_{\mathscr{H}_{G_1}} - TT^*)k_x^{(1)}\rangle_{\mathscr{H}(T)} &= \langle Pv, P(I_{\mathscr{H}_{G_1}} - TT^*)^{1/2}k_x^{(1)}\rangle_{\mathscr{H}_{G_1}} \\ &= \langle (I_{\mathscr{H}_{G_1}} - TT^*)^{1/2}v, k_x^{(1)}\rangle_{\mathscr{H}_{G_1}} \\ &= u(x),\end{aligned}$$

that is, $(I_{\mathscr{H}_{G_1}} - TT^*)k_x^{(1)}$ is the reproducing kernel of $\mathscr{H}(T)$. Then the Gram matrix of $\mathscr{H}(T)$ is given as follows:

$$\begin{aligned}&\langle (I_{\mathscr{H}_{G_1}} - TT^*)k_x^{(1)}, (I_{\mathscr{H}_{G_1}} - TT^*)k_y^{(1)}\rangle_{\mathscr{H}(T)} \\ &= \langle P(I_{\mathscr{H}_{G_1}} - TT^*)^{1/2}k_x^{(1)}, P(I_{\mathscr{H}_{G_1}} - TT^*)^{1/2}k_y^{(1)}\rangle_{\mathscr{H}_{G_1}} \\ &= \langle (I_{\mathscr{H}_{G_1}} - TT^*)k_x^{(1)}, k_y^{(1)}\rangle_{\mathscr{H}_{G_1}} \\ &= \langle k_x^{(1)}, k_y^{(1)}\rangle_{\mathscr{H}_{G_1}} - \langle T^*k_x^{(1)}, T^*k_y^{(1)}\rangle_{\mathscr{H}_{G_2}} \\ &= \langle k_x^{(1)}, k_y^{(1)}\rangle_{\mathscr{H}_{G_1}} - \langle k_x^{(2)}, k_y^{(2)}\rangle_{\mathscr{H}_{G_2}}.\end{aligned}$$

Hence the Gram matrix of $\mathscr{H}(T)$ is equal to $K_1 - K_2$. It follows from this that $\mathscr{H}_{K_1-K_2}$ is isomorphic to $\mathscr{H}(T)$. □

COROLLARY 5.1. *For finite graphs,*

$$\dim\ker(K_1 - K_2) = \chi(G_2 - G_1).$$

Proof. It is easy to see that $\mathscr{H}_{K_1-K_2} = \mathscr{V}/\mathscr{N}$ as reproducing kernel Hilbert spaces and

$$\dim\ker(K_1 - K_2) = \dim\mathscr{N}.$$

Therefore, by Lemma 3.3, we have that

$$|V| - \dim\ker(K_1 - K_2) = |V| - \dim\mathscr{N} = \dim\mathscr{H}_{K_1-K_2} = \dim\mathscr{H}(T).$$

Further, since

$$|V| - \chi(G_2 - G_1) = |V_0| - \chi(G_2 \ominus G_1) \tag{5.1}$$

for finite graphs, we have the conclusion by (i) of Theorem 5.1. □

COROLLARY 5.2. *Let $G_1 \subset \cdots \subset G_n$ be a chain of connected graphs having a common vertex set. If $|E_n \setminus E_1|$ is finite, then*

$$|V(G_n \ominus G_1)| - \sum_{j=1}^{n-1} |V(G_{j+1} \ominus G_j)| \leqslant \chi(G_n \ominus G_1) - \sum_{j=1}^{n-1} \chi(G_{j+1} \ominus G_j).$$

Proof. Let $\iota_{j,j+1} : G_j \to G_{j+1}$ and $\iota_j : G_1 \to G_j$ be canonical embedding maps, and let $T_{j,j+1} : \mathscr{H}_{G_{j+1}} \to \mathscr{H}_{G_j}$ and $T_j : \mathscr{H}_{G_j} \to \mathscr{H}_{G_1}$ be operators corresponding to $\iota_{j,j+1}$ and ι_j, respectively. Since $\iota_j = \iota_{j-1,j} \circ \cdots \circ \iota_{1,2}$, it is trivial that $T_{j+1} = T_j T_{j,j+1}$. Further, we note that every T_j is invertible by (iii) of Lemma 3.1. Hence, by the following decomposition of $I_{\mathscr{H}_{G_1}} - T_n T_n^*$:

$$I_{\mathscr{H}_{G_1}} - T_n T_n^* = \sum_{j=1}^{n-1} T_j (I_{\mathscr{H}_{G_j}} - T_{j,j+1} T_{j,j+1}^*) T_j^* \quad (T_1 := I_{\mathscr{H}_{G_1}})$$

(see also Theorem A140 in Vasyunin-Nikol'skiĭ [11]), we have that

$$\dim \mathscr{H}(T_n) \leqslant \sum_{j=1}^{n-1} \dim \mathscr{H}(T_{j,j+1}).$$

Therefore, by Theorem 5.1, we have the conclusion. □

REMARK 5.1. For chains of finite graphs, from (5.1) and Corollary 5.2, we have the following inequality:

$$\sum_{j=1}^{n-1} \chi(G_{j+1} - G_j) \leqslant \chi(G_n - G_1) + (n-2)|V|. \tag{5.2}$$

(5.2) should be known and elementary among graph theorists. However, it might be worthwhile mentioning that (5.2) means a dimension inequality for reproducing kernel Hilbert spaces over graphs.

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