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OPERATOR–VALUED MAPS ON HILBERT C^*–MODULES

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Abstract. We provide a characterization for operator-valued completely bounded linear maps on Hilbert C^*-modules in terms of φ-maps. Also, we show that for every operator-valued completely positive map φ on a C^*-algebra $\mathscr{A}$, there is a unique (up to multiplication by a unitary operator) non-degenerate φ-map on each Hilbert $\mathscr{A}$-module.

1. Introduction

The study of φ-maps on Hilbert C^*-modules has increased significantly during recent decades. In this context, several concepts such as representation theory of Hilbert C^*-modules, dilation theory of φ-maps and CP-extendable maps were studied ([1, 3, 4, 5, 6, 9, 11, 12, 16, 17]). Therefore, it becomes natural to concentrate on φ-maps as important maps on Hilbert C^*-modules. To confirm this statement, we show that an operator-valued map on a Hilbert C^*-module is completely bounded if and only if it can be decomposed to a bounded operator and a φ-map, for a completely positive map φ on the underlying C^*-algebra of the Hilbert C^*-module.

Moreover, for a given operator-valued completely positive map φ on a C^*-algebra $\mathscr{A}$ and its minimal Stinespring dilation π, we construct a φ-map and a π-representation for each Hilbert $\mathscr{A}$-module $\mathscr{E}$ and also we show that every non-degenerate φ-map (non-degenerate π-representation) on $\mathscr{E}$ is a unitary operator multiple of the above constructed φ-map (π-representation).

We denote Hilbert spaces by $\mathscr{H},\mathscr{K},\mathscr{L}$. The set of all bounded operators between Hilbert spaces $\mathscr{H},\mathscr{K}$ is denoted by $\mathscr{B}(\mathscr{H},\mathscr{K})$, and $\mathscr{B}(\mathscr{H}) := \mathscr{B}(\mathscr{H},\mathscr{H})$.

Assume $\mathscr{E}$ is a Hilbert C^*-module over a unital C^*-algebras $\mathscr{A}$. The linking C^*-algebra of $\mathscr{E}$ is denoted by $\mathscr{L}(\mathscr{E})$ and defined as $\mathscr{L}(\mathscr{E}) := \{\begin{bmatrix} T & x \\ y^* & a \end{bmatrix} : a \in \mathscr{A}, T \in \mathbb{K}(\mathscr{E}), x, y \in \mathscr{E}\}$, where $\mathbb{K}(\mathscr{E})$ is the set of compact operators on $\mathscr{E}$. Also, for arbitrary given maps $\rho : \mathscr{A} \to \mathscr{B}(\mathscr{H})$, $\sigma : \mathbb{K}(\mathscr{E}) \to \mathscr{B}(\mathscr{K})$ and $\Psi : \mathscr{E} \to \mathscr{B}(\mathscr{H},\mathscr{K})$, the map $\begin{bmatrix} T & x \\ y^* & a \end{bmatrix} \mapsto \begin{bmatrix} \sigma(T) & \Psi(x) \\ \Psi(y)^* & \rho(a) \end{bmatrix}$ from $\mathscr{L}(\mathscr{E})$ into $\mathscr{B}(\mathscr{K} \oplus \mathscr{H})$ is denoted by $\begin{bmatrix} \sigma & \Psi \\ \Psi^* & \rho \end{bmatrix}$.

If $\varphi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ is a completely positive map and $\Phi : \mathscr{E} \to \mathscr{B}(\mathscr{H},\mathscr{K})$ a map, then we say that

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(1) Φ is *non-degenerate*, if $[\Phi(\mathscr{E})\mathscr{H}] = \mathscr{K}$.

(2) Φ is a *φ-map*, if $\Phi(x)^*\Phi(y) = \varphi(\langle x,y\rangle)$, for all $x,y \in \mathscr{E}$.

(3) Φ is a *representation* (or *ρ-representation*), if there is a $*$-representation $\rho : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ such that Φ is a ρ-map.

(4) Φ is a *completely semi-φ-map*, if $\Phi_n(x)^*\Phi_n(x) \leqslant \varphi_n(\langle x,x\rangle)$ for every $n \in \mathbb{N}$ and $x \in \mathbb{M}_n(\mathscr{E})$.

(5) Φ is a *CP-extendable map*, if there exist completely positive maps $\phi_1 : \mathbb{K}(\mathscr{E}) \to \mathscr{B}(\mathscr{K})$ and $\phi_2 : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ such that $\begin{bmatrix} \phi_1 & \Phi \\ \Phi^* & \phi_2 \end{bmatrix} : \mathscr{L}(\mathscr{E}) \to \mathscr{B}(\mathscr{K} \oplus \mathscr{H})$, is a completely positive map.

(6) Φ is *dilatable* if there is a representation $\Psi : \mathscr{E} \to \mathscr{B}(\mathscr{H}', \mathscr{K}')$ and bounded operators $V : \mathscr{H} \to \mathscr{H}'$ and $W : \mathscr{K} \to \mathscr{K}'$ such that $\Phi(x) = W^*\Psi(x)V$, for every $x \in \mathscr{E}$.

Positive definite kernels are a non-linear version of completely positive maps which are older than their linear counterpart (see [2, 7, 8, 14]) for more details). A positive definite kernel on a set X is a two-variable function $\phi : X \times X \to \mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is a Hilbert space, such that $[\phi(x_i,x_j)] \in \mathbb{M}_n(\mathscr{B}(\mathscr{H}))_+$, for any choice of $x_1, \cdots, x_n$ in X. From now on we use PD kernel to abbreviate positive definite kernel.

For a given PD kernel $\phi : X \times X \to \mathscr{B}(\mathscr{H})$ there is a standard way to construct another Hilbert space $\mathscr{K}$ such that ϕ is decomposed into more tractable functions from X into $\mathscr{B}(\mathscr{H}, \mathscr{K})$ [7, 8].

DEFINITION 1.1. Let X be a non-empty set and $\phi : X \times X \to \mathscr{B}(\mathscr{H})$ be a PD kernel. A Kolmogorov decomposition pair for ϕ is a pair $(\nu, \mathscr{K})$ consists of a Hilbert space $\mathscr{K}$ a map $\nu : X \to \mathscr{B}(\mathscr{H}, \mathscr{K})$ such that $\phi(x,y) = \nu(x)^*\nu(y)$. A Kolmogorov decomposition pair is called minimal when $[\nu(X)\mathscr{H}] = \mathscr{K}$.

The existence of the Kolmogorov decomposition pair for a PD kernel is a well-known result:

THEOREM 1.2. *Let $\phi : X \times X \to \mathscr{B}(\mathscr{H})$ be a PD kernel, then there is a Hilbert space $\mathscr{K}$ and a map $\nu : X \to \mathscr{B}(\mathscr{H}, \mathscr{K})$ such that $\phi(x,y) = \nu(x)^*\nu(y)$, for all $x,y \in X$.*

REMARK 1.3. Minimal Kolmogorov decomposition pairs of ϕ are unique up to unitary equivalence. That is, if $(\nu, \mathscr{K})$ is a minimal Kolmogorov decomposition of ϕ and $(\upsilon, \mathscr{L})$ is an arbitrary Kolmogorov decomposition pair of ϕ, then there is a unique isometry $V : \mathscr{K} \to \mathscr{L}$ such that $V\nu(x) = \upsilon(x)$.

To every map $\Phi : X \to \mathscr{B}(\mathscr{H}, \mathscr{K})$, one can associate a PD kernel $\Lambda_\Phi : X \times X \to \mathscr{B}(\mathscr{H})$ by $\Lambda_\Phi(x,y) = \Phi(x)^*\Phi(y)$, which has Φ as its Kolmogorov decomposition. Also, a completely positive map $\varphi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ induces a PD kernel $\tilde{\varphi} : \mathscr{E} \times \mathscr{E} \to \mathscr{B}(\mathscr{H})$ by $\tilde{\varphi}(x,y) = \varphi(\langle x,y\rangle)$, on every Hilbert $\mathscr{A}$-module $\mathscr{E}$.

2. Main Theorems

The following theorem says that each operator-valued completely bounded map on a Hilbert C^*-module is an operator multiple of some φ-map. We mention that a similar discussion can be found in [17, Section 3]. In fact, the main idea of the proof is to use the fact that the space $B(K \oplus H)$ is injective in the category of operator systems.

THEOREM 2.1. *Let $\mathscr{E}$ be a right Hilbert C^*-module over a unital C^*-algebra $\mathscr{A}$ and $\Phi : \mathscr{E} \to \mathscr{B}(\mathscr{H},\mathscr{K})$ be a map. The following statements are equivalent*

(i) Φ is a completely bounded linear map.

(ii) There exist a completely positive map $\varphi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$, a Hilbert space $\mathscr{L}$, a φ-map $\Gamma : \mathscr{E} \to \mathscr{B}(\mathscr{H},\mathscr{L})$ and a bounded operator $S : \mathscr{L} \to \mathscr{K}$ such that $\Phi(x) = S\Gamma(x)$, for all $x \in \mathscr{E}$.

The following lemma provides a representation theorem for completely positive maps on C^*-algebras, in term of maps on Hilbert C^*-modules.

LEMMA 2.2. *Let $\mathscr{E}$ be a right Hilbert C^*-module over a unital C^*-algebra $\mathscr{A}$, $\varphi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ a completely positive map and $(\pi,\mathscr{K},V)$ be the minimal Stinespring dilation triple of φ. Then, there exists a triple $((\Phi_\varphi,\mathscr{H}_\varphi),(\Psi_\pi,\mathscr{K}_\pi),W_\varphi)$ consisting of Hilbert spaces $\mathscr{H}_\varphi$ and $\mathscr{K}_\pi$, a unitary operator $W_\varphi : \mathscr{H}_\varphi \to \mathscr{K}_\pi$, a non-degenerate φ-map $\Phi_\varphi : \mathscr{E} \to \mathscr{B}(\mathscr{H},\mathscr{H}_\varphi)$ and a non-degenerate π-representation $\Psi_\pi : \mathscr{E} \to \mathscr{B}(\mathscr{K},\mathscr{K}_\pi)$ such that $\Phi_\varphi(x) = W_\varphi^*\Psi_\pi(x)V$, for all $x \in \mathscr{E}$.*

Now, we summarize some results about φ-maps on Hilbert C^*-modules. In fact, in the following theorem, part (i) is the same as Bhat-Ramesh-Sumesh's theorem [5, Theorem 2.1] and also says that for every completely positive map on a C^*-algebra $\mathscr{A}$, such as $\varphi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$, there is a unique (up to multiplication by a unitary operator) non-degenerate φ-map on each Hilbert $\mathscr{A}$-module. Part (ii) strengthens [4, Theorem 3.4] and characterizes completely semi-φ-maps as operator multiples of φ-maps. Also, part (iii) is a similar result to (i) and finally, part (iv) exposes the relation between every pair of φ-maps and π-representations on a same Hilbert C^*-module.

THEOREM 2.3. *With the notations of the above lemma, one has*

(i) a map $\Phi : \mathscr{E} \to \mathscr{B}(\mathscr{H},\mathscr{H}')$ is a (non-degenerate) φ-map if and only if there exist a (unitary) isometry $S_\Phi : \mathscr{H}_\varphi \to \mathscr{H}'$ and a (unitary) coisometry $W : \mathscr{H}' \to \mathscr{K}_\pi$ such that $S_\Phi\Phi_\varphi(x) = \Phi(x)$ and $\Phi(x) = W^*\Psi_\pi(x)V$, for all $x \in \mathscr{E}$;

(ii) a map $\Phi : \mathscr{E} \to \mathscr{B}(\mathscr{H},\mathscr{H}')$ is a (non-degenerate) completely semi-φ-map if and only if there exist a (dense range) contraction $S : \mathscr{H}_\varphi \to \mathscr{H}'$, and a (injective) contraction $W : \mathscr{H}' \to \mathscr{K}_\pi$ such that $S\Phi_\varphi(x) = \Phi(x)$ and $\Phi(x) = W^*\Psi_\pi(x)V$, for all $x \in \mathscr{E}$;

(iii) a map $\Psi : \mathscr{E} \to \mathscr{B}(\mathscr{K},\mathscr{K}')$ is a (non-degenerate) π-representation if and only if there exists an (unitary) isometry $S_\Psi : \mathscr{K}_\pi \to \mathscr{K}'$ such that $\Psi(x) = S_\Psi\Psi_\pi(x)$, for all $x \in \mathscr{E}$;

(iv) if $\Psi : \mathscr{E} \to \mathscr{B}(\mathscr{K}, \mathscr{K}')$ *is a* π*-representation and* $\Phi : \mathscr{E} \to \mathscr{B}(\mathscr{H}, \mathscr{H}')$ *is a* φ*-map, then there exists a partial isometry* $W : \mathscr{H}' \to \mathscr{K}'$ *such that* $\Phi(x) = W^*\Psi(x)V$*, for all* $x \in \mathscr{E}$*. Moreover,* W *is unitary when* Φ *and* Ψ *are non-degenerate.*

Completely semi-φ-maps were introduced in [4] as generalizations of φ-maps. By the above theorem, every completely semi-φ-map can be dilated to a representation of the Hilbert C^*-module and therefore it is a linear map. Also, CP-extendable maps were introduced in [17], and the authors in [4, Theorem 4.2] showed that an operator-valued map on a Hilbert C^*-module is dilatable if and only if it is CP-extendable. Therefore, we have the following result.

COROLLARY 2.4. *Let* $\mathscr{E}$ *be a right Hilbert* C^**-module over a unital* C^**-algebra* $\mathscr{A}$ *and* $\Phi : \mathscr{E} \to \mathscr{B}(\mathscr{H}, \mathscr{K})$ *be a map. The following statements are equivalent*

(i) Φ *is a completely bounded linear map.*

(ii) There exist a completely positive map $\varphi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$*, a Hilbert space* $\mathscr{L}$*, a* φ*-map* $\Gamma : \mathscr{E} \to \mathscr{B}(\mathscr{H}, \mathscr{L})$ *and a bounded operator* $S : \mathscr{L} \to \mathscr{K}$ *such that* $\Phi(x) = S\Gamma(x)$*, for all* $x \in \mathscr{E}$*.*

(iii) There is a completely positive map $\psi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ *such that* Φ *is a completely semi-*ψ*-map.*

(iv) Φ *is dilatable.*

(v) Φ *is CP-extendable.*

The following corollary is a well known theorem on completely bounded maps on C^*-algebras [15]. However, we can conclude it as a special case of the above result, since each C^*-algebra is a right Hilbert C^*-module over itself and also $\mathbb{K}(\mathscr{A}) \cong \mathscr{A}$, $\mathbb{M}_2(\mathscr{A}) \cong \mathscr{L}(\mathscr{A})$.

COROLLARY 2.5. *Let* $\mathscr{A}$ *be a unital* C^**-algebra. If* $\psi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$ *is a completely bounded map, there exist completely positive maps* $\phi_i : \mathscr{A} \to \mathscr{B}(\mathscr{H})$*,* $i = 1, 2$*, such that the map* $\begin{bmatrix} \phi_1 & \psi \\ \psi^* & \phi_2 \end{bmatrix} : \mathbb{M}_2(\mathscr{A}) \to \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ *is completely positive.*

3. Proofs

Proof of Theorem 2.1. $(i) \Rightarrow (ii)$: Assume Φ is completely bounded. Since $\mathscr{B}(\mathscr{K} \oplus \mathscr{H}) = \begin{bmatrix} \mathscr{B}(\mathscr{K}) & \mathscr{B}(\mathscr{H}, \mathscr{K}) \\ \mathscr{B}(\mathscr{K}, \mathscr{H}) & \mathscr{B}(\mathscr{H}) \end{bmatrix}$, we can consider Φ as a map from $\mathscr{E}$ into $\mathscr{B}(\mathscr{K} \oplus \mathscr{H})$.

Let $\mathscr{L}_1(\mathscr{E}) := \{\begin{bmatrix} T & x \\ y^* & a \end{bmatrix} : a \in \mathscr{A}, T \in \mathbb{K}_1(\mathscr{E}) := \mathbb{K}(\mathscr{E}) + \mathbb{C}I_{\mathscr{E}}, x, y \in \mathscr{E}\}$, be the unitization of the linking C^*-algebra of $\mathscr{E}$, then φ can be extended to a completely bounded map $\Psi : \mathscr{L}_1(\mathscr{E}) \to \mathscr{B}(\mathscr{K} \oplus \mathscr{H})$ by Wittstock's extension theorem. Then there is a $*$-representation $\pi : \mathscr{L}_1(\mathscr{E}) \to \mathscr{B}(\mathscr{L})$ and bounded operators $V_i : K \oplus H \to \mathscr{L}$, $i = 1, 2$ such that $\Psi(X) = V_1^*\pi(X)V_2$ for every $X \in \mathscr{L}_1(\mathscr{E})$. Using [1, Proposition 3.1] $\mathscr{L}$ decomposes to $\mathscr{L}_2 \oplus \mathscr{L}_1$ for two orthogonal closed subspaces $\mathscr{L}_1$ and $\mathscr{L}_2$

and there exist $*$-representations $\rho : \mathscr{A} \to \mathscr{B}(\mathscr{L}_1)$, $\sigma : \mathbb{K}_1(\mathscr{E}) \to \mathscr{B}(\mathscr{L}_2)$ and a σ-ρ-representation $\Gamma_0 : \mathscr{E} \to \mathscr{B}(\mathscr{L}_1, \mathscr{L}_2)$ such that $\pi = \begin{bmatrix} \sigma & \Gamma_0 \\ \Gamma_0^* & \rho \end{bmatrix} : \mathscr{L}_1(\mathscr{E}) \to \mathscr{B}(\mathscr{L}_2 \oplus \mathscr{L}_1)$.

Since Ψ is an extension of Φ, and the operators V_i, $i = 1,2$ has the matrix decompositions $V_i = \begin{bmatrix} S_{i,1} & S_{i,2} \\ S_{i,3} & S_{i,4} \end{bmatrix} \in \mathscr{B}(\mathscr{K} \oplus \mathscr{H}, \mathscr{L}_2 \oplus \mathscr{L}_1)$, $i = 1,2$, one has

$$\begin{bmatrix} 0 & \Phi(x) \\ 0 & 0 \end{bmatrix} = \Psi\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} S_{1,1} & S_{1,2} \\ S_{1,3} & S_{1,4} \end{bmatrix}^* \begin{bmatrix} \sigma(0) & \Gamma_0(x) \\ \Gamma_0(0)^* & \rho(0) \end{bmatrix} \begin{bmatrix} S_{2,1} & S_{2,2} \\ S_{2,3} & S_{2,4} \end{bmatrix},$$

for every $x \in \mathscr{E}$. Thus $\Phi(x) = S_{1,1}^* \Gamma_0(x) S_{2,4}$ for every $x \in \mathscr{E}$.

Now, if we set $\varphi(\cdot) = S_{2,4}^* \rho(\cdot) S_{2,4}$ and $\Gamma(\cdot) = \Gamma_0(\cdot) S_{2,4}$, then φ is a completely positive map, Γ is a φ-map and $\Phi(\cdot) = S_{1,1}^* \Gamma(\cdot)$.

$(ii) \Rightarrow (i)$ Let $\Phi(\cdot) = S\Gamma(\cdot)$. Let $[x_{ij}] \in M_n(\mathscr{E})$, then

$$\begin{aligned} \Phi_n([x_{ij}])^* \Phi_n([x_{ij}]) &= [\Gamma(x_{ji})^* S^*][S\Gamma(x_{ij})] \\ &= [\Gamma(x_{ji})^*] diag(S^*, \cdots, S^*) diag(S, \cdots, S)[\Gamma(x_{ij})] \\ &\leqslant ||S||^2 [\Gamma(x_{ji})^*][\Gamma(x_{ij})] = ||S||^2 \varphi_n(\langle [x_{ij}], [x_{ij}] \rangle) \end{aligned}$$

Therefore,

$$\|\Phi_n([x_{ij}])\|^2 = \|\Phi_n([x_{ij}])^* \Phi_n([x_{ij}])\| \leqslant ||S||^2 \|\varphi\|_{cb} \|[x_{ij}]\|^2$$

and then Φ is a completely bounded map. $\square$

Proof of Lemma 2.2. Define $\tilde{\varphi} : \mathscr{E} \times \mathscr{E} \to \mathscr{B}(\mathscr{H})$ by

$$\tilde{\varphi}(x,y) := \varphi(\langle x,y \rangle_{\mathscr{A}})$$

for all $x,y \in \mathscr{E}$. Note that the $\mathscr{A}$-valued inner-product on $\mathscr{E}$, $\langle \cdot,\cdot \rangle_{\mathscr{A}} : \mathscr{E} \times \mathscr{E} \to \mathscr{A}$ is a PD kernel and φ is a completely positive map on $\mathscr{A}$, therefore $\tilde{\varphi}$ is a PD kernel on $\mathscr{E}$. There is a (unique) minimal Kolmogorov decomposition $(\Phi_\varphi, \mathscr{H}_\varphi)$ for $\tilde{\varphi}$, consisting of a Hilbert space $\mathscr{H}_\varphi$ and a map $\Phi_\varphi : \mathscr{E} \to \mathscr{B}(\mathscr{H}, \mathscr{H}_\varphi)$ such that the linear span of $\Phi_\varphi(\mathscr{E})\mathscr{H}$ is a dense subspace of $\mathscr{H}_\varphi$ and $\tilde{\varphi}(x,y) = \Phi_\varphi(x)^* \Phi_\varphi(y)$ for all $x,y \in \mathscr{E}$. Thus Φ_φ is a non-degenerate φ-map from $\mathscr{E}$ into $\mathscr{B}(\mathscr{H}, \mathscr{H}_\varphi)$.

Similarly, define $\tilde{\pi} : \mathscr{E} \times \mathscr{E} \to \mathscr{B}(\mathscr{K})$ by $\tilde{\pi}(x,y) := \pi(\langle x,y \rangle_{\mathscr{A}})$, for all $x,y \in \mathscr{E}$. By the same reasoning as above, one shows the existence of a (unique) minimal Kolmogorov decomposition pair $(\Psi_\pi, \mathscr{K}_\pi)$ for $\tilde{\pi}$, consisting of a Hilbert space $\mathscr{K}_\pi$ and a map $\Psi_\pi : \mathscr{E} \to \mathscr{B}(\mathscr{H}, \mathscr{K}_\pi)$ such that the linear span of $\Psi_\pi(\mathscr{E})\mathscr{H}$ is a dense subspace of $\mathscr{K}_\pi$ and $\tilde{\pi}(x,y) = \Psi_\pi(x)^* \Psi_\pi(y)$ for all $x,y \in \mathscr{E}$. Thus Ψ_π is a non-degenerate π-map from $\mathscr{E}$ into $\mathscr{B}(\mathscr{H}, \mathscr{K}_\pi)$.

Since $(\pi, V, \mathscr{K})$ is a dilation triple for φ, and Φ_φ is a φ-map, for every $x,y \in \mathscr{E}$, we have

$$\Phi_\varphi(x)^* \Phi_\varphi(y) = \varphi(\langle x,y \rangle_{\mathscr{A}}) = V^* \pi(\langle x,y \rangle_{\mathscr{A}}) V = V^* \Psi_\pi(x)^* \Psi_\pi(y) V. \tag{1}$$

The above equation implies that for every $x_1,\dots,x_n \in \mathscr{E}$ and $h_1,\dots,h_n \in \mathscr{H}$

$$\|\sum_{i=1}^{n}\Phi_\varphi(x_i)h_i\|_{\mathscr{H}_\varphi} = \|\sum_{i=1}^{n}\Psi_\pi(x_i)Vh_i\|_{\mathscr{K}_\pi}.$$

Since Φ_φ is a non-degenerate φ-map, the above equality guarantees the existence of a unique isometry $W_\varphi : \mathscr{H}_\varphi \to \mathscr{K}_\pi$ such that $W_\varphi\Phi_\varphi(x) = \Psi_\pi(x)V$ satisfies for all $x \in \mathscr{E}$. Since Φ_φ and Ψ_π are non-degenerate continuous linear maps and $(\pi,\mathscr{K},V)$ is a minimal Stinespring dilation for φ,

$$\begin{aligned}W_\varphi(\mathscr{H}_\varphi) &= W_\varphi([\Phi_\varphi(\mathscr{E})\mathscr{H}]) = [W_\varphi\Phi_\varphi(\mathscr{E})\mathscr{H}] = [\Psi_\pi(\mathscr{E})V\mathscr{H}]\\ &= [\Psi_\pi(\mathscr{E})\pi(\mathscr{A})V\mathscr{H}] = [\Psi_\pi(\mathscr{E})[\pi(\mathscr{A})V\mathscr{H}]] = [\Psi_\pi(\mathscr{E})\mathscr{K}] = \mathscr{K}_\pi\end{aligned}$$

so W_φ is a unitary operator with the desired property. □

Proof of Theorem 2.3. (i) As in the proof of Lemma 2.2, for every $x_1,\dots,x_n \in \mathscr{E}$ and $h_1,\dots,h_n \in \mathscr{H}$ we have

$$\|\sum_{i=1}^{n}\Phi(x_i)h_i\|_{\mathscr{H}'} = \|\sum_{i=1}^{n}\Phi_\varphi(x_i)h_i\|_{\mathscr{H}_\varphi}.$$

Thus there is an (onto) isometry $S_\Phi : \mathscr{H}_\varphi \to \mathscr{H}'$ such that $S_\Phi\Phi_\varphi(x)h = \Phi(x)h$ for every $x \in \mathscr{E}$ and $h \in \mathscr{H}_\Phi$. Then $\Phi(x) = S_\Phi W_\varphi^*\Psi_\pi(x)V$ for every $x \in \mathscr{E}$. Put $W := W_\varphi S_\Phi^*$. Since W_φ is a unitary and S_Φ is an isometry, W is a coisometry. Besides, $\Phi(x) = W^*\Psi_\pi(x)V$ and $S_\Phi\Phi_\varphi(x) = \Phi(x)$ for every $x \in \mathscr{E}$.

For the non-degenerate case, we have $[\Phi(\mathscr{E})\mathscr{H}] = \mathscr{H}'$. Hence, the isometry S_Φ is onto and so it is unitary. Consequently, W is a unitary operator, too.

Conversely, each of the equations $\Phi(\cdot) = W^*\Psi_\pi(\cdot)V$ when W is a coisometry and $\Phi(\cdot) = S_\Phi\Phi_\varphi(\cdot)$ when S_Φ is an isometry, imply that Φ is a φ-map.

(ii) Let Φ be a (non-degenerate) completely semi-φ-map. For every $x_1,\dots,x_n \in \mathscr{E}$, we have

$$[\Phi(x_i)^*\Phi(x_j)]_{i,j} \leqslant [\varphi(\langle x_i,x_j\rangle)]_{i,j}.$$

Consequently, for every $x_1,\dots,x_n \in \mathscr{E}$ and $h_1,\dots,h_n \in \mathscr{H}$ we have

$$\|\sum_{i=1}^{n}\Phi(x_i)h_i\|^2 \leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\langle\varphi(\langle x_j,x_i\rangle)h_i,h_j\rangle = \|\sum_{i=1}^{n}\Phi_\varphi(x_i)h_i\|^2.$$

Thus there is a (dense range) contractive linear operator $S : \mathscr{H}_\varphi \to \mathscr{H}'$ such that $S\Phi_\varphi(x) = \Phi(x)$ for every $x \in \mathscr{E}$. Therefore $\Phi(x) = SW_\varphi^*\Psi_\pi(x)V$ for every $x \in \mathscr{E}$. Put $W := W_\varphi S^*$. Since W_φ is a unitary and S is a (dense range) contractive operator, W is a (injective) contraction, and $\Phi(x) = W^*\Psi_\pi(x)V$ for every $x \in \mathscr{E}$.

Conversely, when $W : \mathscr{H}' \to \mathscr{K}_\pi$ is a contraction, then $WW^* \leqslant \|W\|^2 id_{\mathscr{K}_\pi} \leqslant$

$id_{\mathscr{K}_\pi}$. Hence, the equation $\Phi(\cdot) = W^*\Psi_\pi(\cdot)V$ implies that

$$\begin{aligned}
\Phi_n([x_{ij}])^*\Phi_n([x_{ij}]) &= [W^*\Psi_\pi(x_{ij})V]^*[W^*\Psi_\pi(x_{ij})V] = [(\Psi_\pi(x_{ji})V)^*W][W^*(\Psi(x_{ij})V)] \\
&= [(\Psi_\pi(x_{ji})V)^*]diag(W,\dots,W)diag(W^*,\dots,W^*)[\Psi_\pi(x_{ij})V] \\
&\leqslant [V^*\Psi_\pi(x_{ji})^*]diag(id_{\mathscr{K}_\pi},\dots,id_{\mathscr{K}\pi})[(\Psi(x_{ij})V)] \\
&\leqslant [V^*\Psi_\pi(x_{ji})^*][\Psi_\pi(x_{ij})V] \\
&\leqslant diag(V^*,\dots,V^*)[\Psi_\pi(x_{ji})^*][\Psi_\pi(x_{ij})]diag(V,\dots,V) \\
&= diag(V^*,\dots,V^*)\pi_n(\langle [x_{ij}],[x_{ij}]\rangle)diag(V,\dots,V) = \varphi_n(\langle [x_{ij}],[x_{ij}]\rangle),
\end{aligned}$$

for every $n \in \mathbb{N}$ and every $[x_{ij}] \in \mathbb{M}_n(\mathscr{E})$. Thus, Φ is a completely semi-φ-map.

(iii) We note that every π-representation is a π-map. Therefore, part (iii) is a special case of part (i).

(iv) In order to prove this, it is sufficient to set $W := S_\Psi W_\varphi S_\Phi^*$. $\square$

Proof of Corollary 2.4. $(i \Leftrightarrow ii)$ By Theorem 2.1.

$(ii \Rightarrow iii)$ Let $\psi := \|S\|^2\varphi$. Let $[x_{ij}] \in M_n(\mathscr{E})$. Thus

$$\begin{aligned}
\Phi_n([x_{ij}])^*\Phi_n([x_{ij}]) &= [\Gamma(x_{ji})^*S^*][S\Gamma(x_{ij})] \\
&= [\Gamma(x_{ji})^*]diag(S^*,\cdots,S^*)diag(S,\cdots,S)[\Gamma(x_{ij})] \\
&\leqslant ||S||^2[\Gamma(x_{ji})^*][\Gamma(x_{ij})] = ||S||^2\varphi_n(\langle [x_{ij}],[x_{ij}]\rangle)
\end{aligned}$$

and then Φ is a completely semi-ψ-map.

$(iii \Rightarrow iv)$ By part (ii) of Theorem 2.3.

$(iv \Rightarrow v)$ See [4, Theorem 4.2]

$(v \Rightarrow i)$ As Φ is the $1-2$ corner of some completely positive mapping on the linking C^*-algebra $\mathscr{L}(\mathscr{E})$, then Φ is a completely bounded map. $\square$

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