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DILATIONS SIMILAR TO A SELF–ADJOINT OPERATOR

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Dedicated to the memory of Professor Leiba Rodman

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Abstract. It is shown that every bounded linear operator T in a complex Hilbert space H is the $(1,1)$-compression of an operator in $H\oplus H$ that is similar to a self-adjoint operator.

1. Introduction and notation

Let H_1, H_2 denote complex Hilbert spaces, and $K := H_1 \oplus H_2$ denote their orthogonal sum. An operator matrix (or matrix operator) with respect to this decomposition of K is a 2×2 matrix

$$\mathbf{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix},$$

where T_{jk} is a bounded linear operator mapping H_k into H_j $(j,k=1,2)$, in sign: $T_{jk} \in L(H_k, H_j)$.

Denote the orthogonal projection of K onto H_j by P_j, and the canonical injection of the space H_k into K by J_k, i.e., let

$$P_j(x_1 \oplus x_2) := x_j \in H_j, \qquad J_1x_1 := (x_1 \oplus 0),\ J_2x_2 := (0 \oplus x_2).$$

Then we see that $T_{jk} = P_j\mathbf{T}J_k$, and we have $P_k = J_k^*$, where * denotes the Hilbert space adjoint $(j,k=1,2)$. In the general situation described above, we shall say that $\mathbf{T}$ *is a (j,k)-dilation of the operator T_{jk} and, equivalently, T_{jk} is a (j,k)-compression of the operator* $\mathbf{T}$.

We shall show that though it may *not* have a self-adjoint (1,1)-dilation, every $T \in L(H)$ has a (1,1)-dilation that is *similar to a self-adjoint operator*.

The *notation* is standard with mild exceptions. An (either classical or generalized) resolution of the identity (operator-valued measure) of the operator T at the Borel set b is denoted by $E(T;b)$ or $G(T;b)$, integration with respect to it (in the spirit of [1]) by $\int f(z)G(T;dz)$. The *dilation* $\mathbf{T}$ of the operator $T \in L(H)$ is written in boldface. $\langle,\rangle$ denotes scalar product, $\oplus$ denotes *orthogonal sum,* $\ominus$ *orthogonal complement*. For $T \in L(H)$, $|T|$ will denote the operator $(T^*T)^{1/2}$, and $||T||$ the norm of T. $T = U(T)|T|$ will denote the polar decomposition of the operator T. $\mathbf{R}_+$ and $\mathbf{R}_-$ denote the open intervals $(0,\infty)$ and $(-\infty,0)$, respectively.

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2. Completely bounded measures and dilations

Recall the following concepts and facts (see, e.g., [3]).

Let X be a compact subset of $\mathbf{C}$, and let B be the σ-algebra of all Borel subsets of X. Let H be a complex Hilbert space, and $L(H)$ denote the C^*-algebra of all bounded linear operators in H. An $L(H)$-valued *measure* E on X is a map $B \to L(H)$ which is weakly countably additive. It is called *bounded* if $||E|| := \sup\{||E(b)|| : b \in B\} < \infty$. The measure E is *regular* if for every $x,y \in H$ the complex valued measure $m_{x,y}(b) := \langle E(b)x,y\rangle$ is regular. For such a measure E the map

$$\phi : C(X) \to L(H), \qquad \phi(f) := \int_X f(z)E(dz),$$

where $C(X)$ is the unital C^*-algebra of all continuous complex valued functions on X, is bounded and linear. In the converse direction: a bounded, linear map $\phi : C(X) \to L(H)$ determines uniquely operators $E(b) \in L(H)$ for $b \in B$ such that the map $b \mapsto E(b)$ is a bounded regular $L(H)$-valued measure. Such measures are called

(1) *spectral* if $E(a \cap b) = E(a)E(b)$,

(2) *positive* if $E(b) \geqslant 0$,

(3) *self-adjoint* if $E(b)^* = E(b)$,

(4) *normalized* if $E(X) = I$

(for all Borel sets a,b). It is clear that if E is spectral and self-adjoint, then its values are orthogonal projections, hence E is positive. The basic relationships between properties of a pair E,ϕ are listed in the following

SCHOLIUM. (cf. [3, p. 49]) *Let E,ϕ be as above.*

(i) E is self-adjoint $\Leftrightarrow$ ϕ is a self-adjoint map,

(ii) E is positive $\Leftrightarrow$ ϕ is a positive map,

(iii) E is spectral $\Leftrightarrow$ ϕ is a homomorphism,

(iv) E is spectral and self-adjoint $\Leftrightarrow$ ϕ is a *-homomorphism,

(v) E is completely bounded $\Leftrightarrow$ ϕ is a completely bounded map.

The definition of a completely bounded map can be found in [3, pp. 4–5], and we can accept (v) as the definition of a completely bounded $L(H)$-valued measure.

The basic characterization of a completely bounded $L(H)$-valued measure was obtained by Hadwin [2, Theorems 3, 20] (cf. also Wittstock [6]), and was completed later by Suen [5, Theorem 3.1]. We cite it in the latter form, and bring only one proof which is useful in our study. We readily acknowledge that the main ideas in this part come from Hadwin [2, Lemma 2, Theorem 3].

THEOREM (HWS). *Let E be a regular, bounded $L(H)$-valued measure. The following are equivalent:*

(i) E has a Hahn decomposition $E = (E_1 - E_2) + i(E_3 - E_4)$, where each E_k is a positive measure on B;

(ii) there exist positive measures F_1, F_2 such that the $L(H \oplus H)$-valued operator measure

$$\begin{pmatrix} F_1(\cdot) & E(\cdot) \\ E(\cdot)^* & F_2(\cdot) \end{pmatrix}$$

is positive;

(iii) there exist a Hilbert space K, a self-adjoint, spectral, $L(K)$-valued measure F on X, and linear operators $A : K \to H$, $V : H \to K$ such that $E(\cdot) = AF(\cdot)V$;

(iv) there exist a Hilbert space $M \supset H$ and a (not necessarily self-adjoint) spectral, $L(M)$-valued measure G such that

$$E(\cdot) = P[G(\cdot)|H],$$

where P is the orthogonal projection of M onto H;

(v) E is completely bounded.

Proof. As mentioned before, we shall prove here only that $(iii) \Rightarrow (iv)$. At first we show that in (iii) we can assume that $K = H$, without restricting the generality.

Indeed, under the conditions of (iii) define the following objects:

$$\begin{aligned}
&\mathbf{E}(\cdot) : H \oplus K \to H \oplus K, \quad && \mathbf{E}(\cdot) := E(\cdot)|H \oplus 0|K,\\
&\mathbf{A} : H \oplus K \to H \oplus K, && \mathbf{A} := 0|H + A|K,\\
&\mathbf{F}(\cdot) : H \oplus K \to H \oplus K, && \mathbf{F}(\cdot) := 0|H \oplus F(\cdot)|K,\\
&\mathbf{V} : H \oplus K \to H \oplus K, && \mathbf{V} := V|H + 0|K.
\end{aligned}$$

We have then for every $h \oplus k \in H \oplus K$

$$\begin{aligned}
\mathbf{AF}(\cdot)\mathbf{V}[h \oplus k] &= \mathbf{AF}(\cdot)[0 \oplus Vh] = \mathbf{A}[0 \oplus F(\cdot)Vh] = AF(\cdot)Vh \oplus 0 = E(\cdot)h \oplus 0\\
&= \mathbf{E}(\cdot)[h \oplus k],
\end{aligned}$$

each of the four objects map $H \oplus K$ into $H \oplus K$, $\mathbf{F}$ is a self-adjoint, spectral, $L(H \oplus K)$-valued (not necessarily normalized!) measure on X, and $\mathbf{E}$ is a regular, bounded, $L(H \oplus K)$-valued (not necessarily normalized) measure on X.

Hence we see that in (iii) we can assume that the two Hilbert spaces H, K occurring there are identical.

We shall denote *this Hilbert space by* Z, and apply the notation in (iii).

Let $M := Z_1 \oplus Z_2$, where $Z_k = Z$ $(k = 1, 2)$, let P_k denote the orthogonal projection of M onto Z_k, and let $J_k := P_k^*$ be the injection of Z_k into M $(k = 1, 2)$. Define the operator valued measure

$$F_M : B \to M, \quad F_M(\cdot) := F(\cdot) \oplus 0.$$

This measure is self-adjoint, spectral, and not normalized. Define the matrix operators in $L(M)$:

$$S := \begin{pmatrix} V & I \\ I - AV & -A \end{pmatrix}, \qquad W := \begin{pmatrix} A & I \\ I - VA & -V \end{pmatrix}.$$

Then $W = S^{-1}$, and it is easy to check that

$$P_1 S^{-1} F_M(\cdot) S J_1 = AF(\cdot)V = E(\cdot) \in L(Z).$$

Define now $G(\cdot) := S^{-1}F_M(\cdot)S$. In general this is an $L(M)$-valued, spectral, not self-adjoint and not normalized measure, which is clearly similar to the measure F_M. For any $z \in Z$ we have

$$E(\cdot)z = P_1 G(\cdot)J_1 z = P_1 G(\cdot)[z \oplus 0],$$

which is statement (iv). □

REMARK. With the notation above consider the polar decomposition $S = U(S)|S|$. Since S is invertible, the positive operator $|S|$ is invertible and $U(S)$ is unitary in $L(M)$. The measure $\tilde{F}_M(\cdot) := U(S)^* F_M(\cdot)U(S)$ is self-adjoint, spectral, and not necessarily normalized. Since

$$G(\cdot) = S^{-1}F_M(\cdot)S = |S|^{-1}U(S)^*F_M(\cdot)U(S)|S| = |S|^{-1}\tilde{F}_M(\cdot)|S|,$$

the measure G is similar to a self-adjoint, spectral measure *via the positive invertible operator* $|S|$.

COROLLARY 1. *Consider the polar decomposition $U(T)|T|$ of the operator $T \in L(H)$, and for every Borel set $b \subset [0,\infty)$ let*

$$F(b) := U(T)E(|T|;b)I.$$

Applying the method of the preceding proof, define

$$S := \begin{pmatrix} I & I \\ I - U(T) & -U(T) \end{pmatrix} \in L(H \oplus H).$$

Then S is invertible, and we obtain

$$F(b) = P_1 S^{-1}(E(|T|;b) \oplus 0)SJ_1,$$

i.e. F is the $(1,1)$-compression of a measure in $L(H \oplus H)$ that is similar to a self-adjoint, spectral, not normalized measure.

Proof. The (constructive) proof of the statement is contained in the proof of the preceding theorem. □

A modification of the method above yields the proof of

THEOREM 1. *Let $T \in L(H)$ with polar decomposition $U(T)|T|$, and define the operator matrix S as above. Then*

$$T = P_1 S^{-1}(|T| \oplus 0)SJ_1 = P_1|S|^{-1}U(S)^*[|T| \oplus 0]U(S)|S|J_1,$$

i.e. T is the $(1,1)$-compression of an operator in $L(H \oplus H)$ that is similar to a self-adjoint operator. Further, T is the $(1,1)$-compression of an operator in $L(H \oplus H)$ that is similar to a self-adjoint operator via a positive invertible operator.

Proof. Define the operator $S \in L(H \oplus H)$ as in the preceding proof. Then $F(b) = P_1 S^{-1}(E(|T|;b) \oplus 0)SJ_1,$, hence

$$\begin{aligned} T &= U(T)\int_{\mathbf{R}_+} zE(|T|;dz) = \int_{\mathbf{R}_+} zF(dz) = P_1 S^{-1}[\int_{\mathbf{R}_+} zE(|T|;dz) \oplus 0]SJ_1 \\ &= P_1 S^{-1}[|T| \oplus 0]SJ_1 = P_1|S|^{-1}U(S)^*[|T| \oplus 0]U(S)|S|J_1. \quad \square \end{aligned}$$

Let $H^k := \oplus_{j=1}^k H$. Assume that the self-adjoint $L(H^2)$-valued spectral measure $F(\cdot)$ is a (1,2)-dilation of a regular, completely bounded $L(H)$-valued measure $E(\cdot)$, i.e.,

$$E(\cdot) = P(H^2;1)F(\cdot)J(H \to H^2;2) \in L(H).$$

Here $P(H^2;1)$ denotes the (orthogonal) projection of H^2 onto the first orthogonal summand space H (parallel to the second space H), $J(H \to H^2;2)$ denotes the canonical injection of H onto the second summand of H^2, and we shall employ similar notation in what follows. The results above have the following

COROLLARY 2. *In the situation (and with the notation) described above, there is a spectral, $L(H^6)$-valued measure G_6 that is similar to a self-adjoint measure, and satisfies for every $h \in H$*

$$E(\cdot)h = P(H^6;1)G_6(\cdot)J(H \to H^6;1)h.$$

Proof. The proof of the theorem (HWS) $(iii) \Longrightarrow (iv)$ shows that we can choose $Z := H \oplus H^2 = H^3$, and there is an $L(H^6)$-valued measure G_6 with the indicated properties such that

$$\mathbf{E}(\cdot) = P(H^3 \oplus H^3;1)G_6(\cdot)J(H^3 \to H^3 \oplus H^3;1).$$

Pre- and postmultiplying with two suitable operators, we obtain

$$\begin{aligned} E(\cdot) &= P(H^3;1)\mathbf{E}(\cdot)J(H \to H^3;1) \\ &= P(H^3;1)P(H^3 \oplus H^3;1)G_6(\cdot)J(H^3 \to H^3 \oplus H^3;1)J(H \to H^3;1) \\ &= P(H^6;1)G_6(\cdot)J(H \to H^6;1). \quad \square \end{aligned}$$

For the basics on equivalent scalar products we refer to [4]. From Theorem 1 we obtain the following

THEOREM 2. *Let $T \in L(H)$, and apply the notation of Theorem 1. Let $H^2 := H_1 \oplus H_2$, where $H_1 = H_2 = H$, and $\hat{T} := S^{-1}(|T| \oplus 0)S \in L(H^2)$. Then there is a scalar product $(,)$ in H^2 that is equivalent to the original scalar product $\langle , \rangle$ in $H^2 := H \oplus H$, with respect to which the operator $\hat{T}$ is self-adjoint. In the relation*

$$T = P_1 \hat{T} J_1$$

the operators $J_1 : H_1 \to H^2$, $P_1 : H^2 \to H_1$ have the meanings as before, and are linear and bounded (also) with respect to the new norm in H^2. In other words: T is the classical $(1,1)$-compression of the operator $\hat{T} \in L(H^2)$, which is self-adjoint with respect to the new scalar product $(,)$, and the operators J_1, P_1 correspond to the direct sum decomposition $H^2 := H_1 \oplus H_2$, which need not be orthogonal in the scalar product $(,)$.

Proof. Since $S : H^2 \to H^2$ is a bijection, in its polar decomposition $S = U(S)|S|$ the operator $U(S)$ is a unitary, and $|S|$ is a strictly positive operator in H^2. Further,

$\hat{T} = |S|^{-1}U(S)^*(|T| \oplus 0)U(S)|S|$, and the middle operator $V := U(S)^*(|T| \oplus 0)U(S)$ is self-adjoint in the scalar product $\langle , \rangle$ in H^2. Define the new scalar product $(,)$ in H^2 with the notation $B := |S|$ by

$$(h,k) := \langle Bh, Bk \rangle \qquad (h,k \in H^2).$$

This induces the B-norm $|h|_B = (Bh,Bh)^{1/2}$ in H^2, which is equivalent to the old norm and, together with the original Hilbert space $X := [H^2, \langle , \rangle]$, we also consider the new $Z := [H^2, (,)]$. Any operator $W \in L(X)$ clearly lies in $L(Z)$, and conversely. Further, the adjoints $W^* \in L(X)$ and $W_B \in L(Z)$ are connected as follows. For any $x, y \in H^2$ we have

$$\langle BW_B B^{-1} Bx, By \rangle = \langle BW_B x, By \rangle = (W_B x, y) = (x, Wy) = \langle Bx, BWy \rangle = \langle Bx, BWB^{-1} By \rangle.$$

It implies $BW_B B^{-1} = [BWB^{-1}]^*$, hence

$$W_B = B^{-2} W^* B^2.$$

Since $\hat{T} = B^{-1} V B$, we obtain that

$$\hat{T}_B = B^{-2} \hat{T}^* B^2 = B^{-2} B V B^{-1} B^2 = B^{-1} V B = \hat{T},$$

i.e., $\hat{T}$ is self-adjoint in the new scalar product $(,)$.

It is clear that the direct sum decomposition $H^2 := H_1 \oplus H_2$ need not be orthogonal in the new scalar product $(,)$. Further, the definitions of the operators J_1, P_1 remain formally the same, e.g., $J_1 h_1 = h_1 \oplus 0 \in H^2$. Since the old and the new norms in H^2 are equivalent, the operators J_1, P_1 remain bounded also with respect to the new norms in H^2. $\square$

REFERENCES

[1] N. DUNFORD AND J. T. SCHWARTZ, *Linear Operators*, Parts II and III, Wiley-Interscience, New York, 1963, 1971.

[2] D. W. HADWIN, *Dilations and Hahn decompositions for linear maps*, Can. J. Math. **33** (1981), 826–839.

[3] V. I. PAULSEN, *Completely Bounded Maps and Operator Algebras*, Cambridge Univ. Press, Cambridge, 2002.

[4] J. RÄTZ, *Comparison of inner products*, Aeq. Math. **57** (1999), 312–321.

[5] C.-Y. SUEN, *Completely bounded linear extensions of operator-valued functions on *-semigroups*, Proc. Amer. Math. Soc. **105** (1989), 330–334.

[6] G. WITTSTOCK, *Ein operatorwertiger Hahn-Banach Satz*, J. Funct. Anal. **40** (1981), 127–150.

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