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# ON THE STRUCTURE OF SPLIT INVOLUTIVE REGULAR HOM–LIE ALGEBRAS

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*Abstract.* We study the strucure of arbitrary split involutive regular Hom-Lie algebras. By developing techniques of connections of roots for this kind of algebras, we show that such an algebra $L$ is of the form $L = U + \sum_{[j]\in\Lambda/\sim} I_{[j]}$ with $U$ a subspace of the involutive abelian subalgebra $H$ and any $I_{[j]}$, a well described involutive ideal of $L$, satisfying $[I_{[j]}, I_{[k]}] = 0$ if $[j] \neq [k]$. Under certain conditions, in the case of $L$ being of maximal length, the simplicity of the algebra is characterized and it is shown that $L$ is the direct sum of the family of its minimal involutive ideals, each one being a simple split involutive regular Hom-Lie algebra.

## 1. Introduction

A Hom-algebra is an algebra in which the multiplication is twisted by a linear homomorphism. The notion of Hom-Lie algebras was introduced by Hartwig, Larsson and Silvestrov to describe the $q$-deformation of the Witt and the Virasoro algebras [1]. Since then, many authors have studied Hom-type algebras [2, 3, 4, 5, 6, 7]. Recently, the notion of the involutive Lie algebras was introduced in [8], and their structures of split involutive Lie algebras were studied in detail.

As is well-known, the class of the split algebras is specially related to addition quantum numbers, graded contractions, and deformations. For instance, for a physical system which displays a symmetry of $L$, it is interesting to know in detail the structure of the split decomposition because its roots can be seen as certain eigenvalues which are the additive quantum numbers characterizing the state of such system. Determining the structure of split algebras will become more and more meaningful in the area of research in mathematical physics. Recently, in [9, 10, 11, 12, 13, 8], the structure of arbitrary split Lie algebras, arbitrary split Leibniz algebras, arbitrary split Lie triple systems, arbitrary split Leibniz triple systems, arbitrary split regular Hom-Lie algebras and arbitrary split involutive Lie algebras have been determined by the techniques of connections of roots. The purpose of this paper is to consider the structure of arbitrary split involutive regular Hom-Lie algebras by the techniques of connections of roots based on some work in [8, 10].

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Throughout this paper, split involutive regular Hom-Lie algebras $L$ are considered of arbitrary dimension and over an arbitrary base field $\mathbb{K}$. This paper is organized as follows. In section 2, we establish the preliminaries on split involutive regular Hom-Lie algebras theory. In section 3, we develop techniques of connections of roots for split involutive regular Hom-Lie algebras. In section 4, we show that such an arbitrary split involutive regular Hom-Lie algebras $L$ with a symmetric root system is of the form $L = U + \sum_{[j]\in\Lambda/\sim} I_{[j]}$ with $U$ a subspace of the involutive abelian subalgebra $H$ and any $I_{[j]}$ a well described ideal of $L$, satisfying $[I_{[j]}, I_{[k]}] = 0$ if $[j] \neq [k]$. In section 5, we show that under certain conditions, in the case of $L$ being of maximal length, the simplicity of the algebra is characterized and it is shown that $L$ is the direct sum of the family of its minimal involutive ideals, each one being a simple split involutive regular Hom-Lie algebra.

## 2. Preliminaries

First we recall the definition of Hom-Lie algebras and give the definition of involutive regular Hom-Lie algebras.

DEFINITION 2.1. [10] A *Hom-Lie algebra* $L$ is a triple $(L, [\cdot,\cdot]_L, \phi)$ consisting of a vector space $L$, a bilinear map $[\cdot,\cdot]_L : L\times L \to L$ and a linear map $\phi\colon L \to L$ satisfying

$[x,y] = -[y,x]$,

$[\phi(x),[y,z]] + [\phi(y),[z,x]] + [\phi(z),[x,y]] = 0$, (Hom-Jacobi identity )

for $x,y,z \in L$. When $\phi$ furthermore is an algebra automorphism it is said that $L$ is a *regular Hom-Lie algebra*.

Let $L$ be a Hom-Lie algebra over the base field $\mathbb{K}$, and let $- : \mathbb{K} \to \mathbb{K}$ be an involutive automorphism, ( we say that $-$ is a conjugation on $\mathbb{K}$). An involution on $L$ is a conjugate-linear map, $* : L \to L$, $(x \mapsto x^*)$, such that $(x^*)^* = x$, $[x,y]^* = [y^*, x^*]$ and $(\phi(x))^* = \phi(x^*)$ for any $x,y \in L$. A regular Hom-Lie algebra endowed with an involution is an involutive regular Hom-Lie algebra. An involutive subset of an involutive algebra is a subset globally invariant by the involution.

Throughout this paper we will consider involutive regular Hom-Lie algebras $L$ being of arbitrary dimension and arbitrary base field $\mathbb{K}$. $\mathbb{N}$ denotes the set of all non-negative integers and $\mathbb{Z}$ denotes the set of all integers.

An involutive subalgebra $A$ of $L$ is a linear subspace such that $[A,A] \subset A$ and $\phi(A) = A$. A linear subspace $I$ of $L$ is called an involutive ideal if $[I,L] \subset I$ and $\phi(I) = I$. We say that $L$ is simple if the product is nonzero and its only ideals are $\{0\}$ and $L$. From now on, $(L,*)$ denotes an involutive regular Hom-Lie algebra.

Let us introduce the class of split algebras in the framework of involutive regular Hom-Lie algebras. Denote by $H$ a maximal involutive abelian subalgebra of $L$. For a linear functional commuting with the involution

$$\alpha : (H,*) \to (\mathbb{K}, -),$$

that is, $\alpha(h^*) = \overline{\alpha(h)}$ for any $h \in H$, we define the root space of $L$ (with respect to $H$) associated to $\alpha$ as the subspace

$$L_\alpha = \{v_\alpha \in L : [h, v_\alpha] = \alpha(h)\phi(v_\alpha) \text{ for any } h \in H\}.$$

The elements $\alpha : (H, *) \to (\mathbb{K}, -)$ satisfying $L_\alpha \neq 0$ are called roots of $L$ with respect to $H$. We denote $\Lambda := \{\alpha : (H, *) \to (\mathbb{K}, -) : L_\alpha \neq 0\}$.

DEFINITION 2.2. We say that $L$ is a *split involutive regular Hom-Lie algebra*, with respect to $H$, if

$$L = H \oplus (\oplus_{\alpha \in \Lambda} L_\alpha).$$

We also say that $\Lambda$ is the root system of $L$.

Observe that, we have $H^* = H$. For convenience, the mappings $\phi|_H$, $\phi|_H^{-1} : H \to H$ will be denoted by $\phi$ and $\phi^{-1}$ respectively.

LEMMA 2.3. *For any $\alpha$, $\beta \in \Lambda \cup \{0\}$, the following assertions hold.*
1. $\phi(L_\alpha) \subset L_{\alpha\phi^{-1}}$ *and* $\phi^{-1}(L_\alpha) \subset L_{\alpha\phi}$.
2. $[L_\alpha, L_\beta] \subset L_{\alpha\phi^{-1} + \beta\phi^{-1}}$.
3. $(L_\alpha)^* = L_{-\alpha}$.

*Proof.* 1. and 2. are analogous to [10, Lemma 1.3].

3. For any $h \in H$ and $v_\alpha \in L_\alpha$, we have $[h, v_\alpha]^* = (\alpha(h)\phi(v_\alpha))^* = \overline{\alpha(h)}\phi(v_\alpha^*)$. From here $[h^*, v_\alpha^*] = -\overline{\alpha(h)}\phi(v_\alpha^*) = -\alpha(h^*)\phi(v_\alpha^*)$. The facts $H^* = H$ and $*^2 = *$ conclude the proof. □

LEMMA 2.4. *The following assertions hold.*
1. *If $\alpha \in \Lambda$ then $\alpha\phi^{-z} \in \Lambda$ for any $z \in \mathbb{Z}$.*
2. $L_0 = H$.

*Proof.* It is analogous to [10, Lemma 1.4]. □

DEFINITION 2.5. A root system $\Lambda$ of a split involutive regular Hom-Lie algebra is called *symmetric* if it satisfies that $\alpha \in \Lambda$ implies $-\alpha \in \Lambda$.

## 3. Connections of roots

In the following, $L$ denotes a split involutive regular Hom-Lie algebra with a symmetric root system $\Lambda$ and $L = H \oplus (\oplus_{\alpha \in \Lambda} L_\alpha)$ the corresponding root decomposition. We begin by developing the techniques of connections of roots in this section.

DEFINITION 3.1. Let $\alpha$ and $\beta$ be two nonzero roots. We shall say that $\alpha$ is *connected* to $\beta$ if there exists $\alpha_1, \cdots, \alpha_k \in \Lambda$ such that

If $k = 1$, then
1. $\alpha_1 \in \{\Lambda\phi^{-n} : n \in \mathbb{N}\} \cap \{\pm\beta\phi^{-m} : m \in \mathbb{N}\}$.

If $k \geqslant 2$, then
1. $\alpha_1 \in \{\alpha\phi^{-n} : n \in \mathbb{N}\}$.
2. $\alpha_1\phi^{-1} + \alpha_2\phi^{-1} \in \Lambda$,

$$\alpha_1\phi^{-2}+\alpha_2\phi^{-2}+\alpha_3\phi^{-1}\in\Lambda,$$
$$\alpha_1\phi^{-3}+\alpha_2\phi^{-3}+\alpha_3\phi^{-2}+\alpha_4\phi^{-1}\in\Lambda,$$
$$\vdots$$
$$\alpha_1\phi^{-i}+\alpha_2\phi^{-i}+\alpha_3\phi^{-i+1}+\cdots+\alpha_{i+1}\phi^{-1}\in\Lambda,$$
$$\vdots$$
$$\alpha_1\phi^{-k+2}+\alpha_2\phi^{-k+2}+\alpha_3\phi^{-k+3}+\cdots+\alpha_i\phi^{-k+i}+\cdots+\alpha_{k-1}\phi^{-1}\in\Lambda.$$

3. $\alpha_1\phi^{-k+1}+\alpha_2\phi^{-k+1}+\alpha_3\phi^{-k+2}+\cdots+\alpha_i\phi^{-k+i-1}+\cdots+\alpha_k\phi^{-1}\in\{\pm\beta\phi^{-m}: m\in\mathbb{N}\}$.

We shall also say that $\{\alpha_1,\cdots,\alpha_k\}$ is a connection from $\alpha$ to $\beta$.

Observe that the case $k=1$ in Definition 3.1 is equivalent to the fact $\beta=\varepsilon\alpha\phi^z$ for some $z\in\mathbb{Z}$ and $\varepsilon\in\{\pm 1\}$.

LEMMA 3.2. *The following assertions hold.*

1. *For any $\alpha\in\Lambda$, we have that $\alpha\phi^{z_1}$ is connected to $\alpha\phi^{z_2}$ for every $z_1,z_2\in\mathbb{Z}$. We also have that $\alpha\phi^{z_1}$ is connected to $-\alpha\phi^{z_2}$ in case $-\alpha\phi^{z_2}\in\Lambda$.*

2. *Let $\{\alpha_1,\cdots,\alpha_k\}$ be a connection from $\alpha$ to $\beta$. Suppose $\alpha_1=\alpha\phi^{-n}$, $n\in\mathbb{N}$. Then for any $r\in\mathbb{N}$ such that $r\geqslant n$, there exists a connection $\{\overline{\alpha}_1,\cdots,\overline{\alpha}_k\}$ from $\alpha$ to $\beta$ such that $\overline{\alpha}_1=\alpha\phi^{-r}$.*

3. *Let $\{\alpha_1,\cdots,\alpha_k\}$ be a connection from $\alpha$ to $\beta$. Suppose that $\alpha_1=\varepsilon\beta\phi^{-m}$ in case $k=1$ or*

$$\alpha_1\phi^{-k+1}+\alpha_2\phi^{-k+1}+\alpha_3\phi^{-k+2}+\cdots+\alpha_k\phi^{-1}=\varepsilon\beta\phi^{-m}$$

*in case $k\geqslant 2$, with $m\in\mathbb{N}$ and $\varepsilon\in\{\pm1\}$. Then for any $r\in\mathbb{N}$ such that $r\geqslant m$, there exists a connection $\{\overline{\alpha}_1,\cdots,\overline{\alpha}_k\}$ from $\alpha$ to $\beta$ such that $\overline{\alpha}_1=\varepsilon\beta\phi^{-r}$ in case $k=1$ or*

$$\overline{\alpha}_1\phi^{-k+1}+\overline{\alpha}_2\phi^{-k+1}+\overline{\alpha}_3\phi^{-k+2}+\cdots+\overline{\alpha}_k\phi^{-1}=\varepsilon\beta\phi^{-r}$$

*in case $k\geqslant 2$.*

*Proof.* They are proof in [10, Lemma 2.2 and Lemma 2.3 ]. □

PROPOSITION 3.3. *The relation $\sim$ in $\Lambda$, defined by $\alpha\sim\beta$ if and only if $\alpha$ is connected to $\beta$, is of equivalence.*

*Proof.* This can be proved completely analogously to [10, Proposition 2.4]. □

## 4. Decompositions

Proposition 3.3 tells us the connection relation $\sim$ in $\Lambda$ is an equivalence relation. So we denote by

$$\Lambda/\sim:=\{[\alpha]:\alpha\in\Lambda\},$$

where $[\alpha]$ denotes the set of nonzero roots of $L$ which are connected to $\alpha$. Our next goal is to associate an adequate ideal $I_{[\alpha]}$ to any $[\alpha]$. For a fixed $\alpha\in\Lambda$, we define

$$I_{0,[\alpha]}:=\text{span}_{\mathbb{K}}\{[L_\beta,(L_\beta)^*]:\beta\in[\alpha]\}\subset H$$

and

$$V_{[\alpha]} := \oplus_{\beta\in[\alpha]} L_\beta.$$

Then we denote by $I_{[\alpha]}$ the direct sum of the two subspaces above, that is,

$$I_{[\alpha]} := I_{0,[\alpha]} \oplus V_{[\alpha]}.$$

PROPOSITION 4.1. *For any $\alpha \in \Lambda$, the linear subspace $I_{[\alpha]}$ is an involutive subalgebra of $L$.*

*Proof.* First, it is sufficient to check that $I_{[\alpha]}$ satisfies $[I_{[\alpha]}, I_{[\alpha]}] \subset I_{[\alpha]}$. By $I_{0,[\alpha]} \subset H$, it is clear that $[I_{0,[\alpha]}, I_{0,[\alpha]}]$=0 and we have

$$[I_{0,[\alpha]} \oplus V_{[\alpha]}, I_{0,[\alpha]} \oplus V_{[\alpha]}] \subset [I_{0,[\alpha]}, V_{[\alpha]}] + [V_{[\alpha]}, I_{0,[\alpha]}] + [V_{[\alpha]}, V_{[\alpha]}]. \tag{4.1}$$

Let us consider the first summand in Eq. (4.1). For $\beta \in [\alpha]$, by Lemma 2.3-2, one gets $[I_{0,[\alpha]}, L_\beta] \subset L_{\beta\phi^{-1}}$, where $\beta\phi^{-1} \in [\alpha]$. Hence

$$[I_{0,[\alpha]}, V_{[\alpha]}] \subset V_{[\alpha]}. \tag{4.2}$$

Similarly, we can also get

$$[V_{[\alpha]}, I_{0,[\alpha]}] \subset V_{[\alpha]}. \tag{4.3}$$

Next, we consider the third summand in Eq. (4.1). Given $\beta$, $\gamma \in [\alpha]$ such that $[L_\beta, L_\gamma] \neq 0$, if $\gamma = -\beta$, we have $[L_\beta, L_\gamma] = [L_\beta, L_{-\beta}] = [L_\beta, (L_\beta)^*] \subset I_{0,[\alpha]}$. Suppose $\gamma \neq -\beta$, by Lemma 2.3-2, one gets $\beta\phi^{-1} + \gamma\phi^{-1} \in \Lambda$. Therfore, we get $\{\beta, \gamma\}$ is a connection from $\beta$ to $\beta\phi^{-1} + \gamma\phi^{-1}$. The transitivity of $\sim$ gives that $\beta\phi^{-1} + \gamma\phi^{-1} \in [\alpha]$ and so $[L_\beta, L_\gamma] \subset L_{\beta\phi^{-1}+\gamma\phi^{-1}} \subset V_{[\alpha]}$. Hence

$$[\oplus_{\beta\in[\alpha]} L_\beta, \oplus_{\beta\in[\alpha]} L_\beta] \subset I_{0,[\alpha]} \oplus V_{[\alpha]}.$$

That is,

$$[V_{[\alpha]}, V_{[\alpha]}] \subset I_{[\alpha]}. \tag{4.4}$$

From Eqs. (4.1), (4.2), (4.3) and (4.4), we get $[I_{[\alpha]}, I_{[\alpha]}] \subset I_{[\alpha]}$.

Second, we have to verify that $\phi(I_{[\alpha]}) = I_{[\alpha]}$. It is a direct consequence of Lemmas 2.3-1 and 3.2-1.

Third, we have to verify that $I^*_{[\alpha]} = I_{[\alpha]}$. It is easy to have that $I^*_{0,[\alpha]} = I_{0,[\alpha]}$ and $V^*_{[\alpha]} = V_{[\alpha]}$. Taking into account that $I_{[\alpha]} := I_{0,[\alpha]} \oplus V_{[\alpha]}$, we have $I^*_{[\alpha]} = I_{[\alpha]}$. □

PROPOSITION 4.2. *If $[\alpha] \neq [\beta]$, then $[I_{[\alpha]}, I_{[\beta]}] = 0$.*

*Proof.* We have

$$[I_{0,[\alpha]} \oplus V_{[\alpha]}, I_{0,[\beta]} \oplus V_{[\beta]}] \subset [I_{0,[\alpha]}, V_{[\beta]}] + [V_{[\alpha]}, I_{0,[\beta]}] + [V_{[\alpha]}, V_{[\beta]}]. \tag{4.5}$$

Let us consider the third summand $[V_{[\alpha]}, V_{[\beta]}]$ in Eq. (4.5) and suppose there exist $\alpha_1 \in [\alpha]$ and $\alpha_2 \in [\beta]$ such that $[L_{\alpha_1}, L_{\alpha_2}] \neq 0$. By known condition $[\alpha] \neq [\beta]$, one gets $\alpha_1 \neq -\alpha_2$. So $\alpha_1\phi^{-1} + \alpha_2\phi^{-1} \in \Lambda$. Hence $\{\alpha_1, \alpha_2, -\alpha_1\phi^{-1}\}$ is a connection from $\alpha_1$ to $\alpha_2$. By the transitivity of the connection relation, we have $\alpha \in [\beta]$, a contradiction. Hence $[L_{\alpha_1}, L_{\alpha_2}] = 0$ and so

$$[V_{[\alpha]}, V_{[\beta]}] = 0. \tag{4.6}$$

Next we consider the first summand $[I_{0,[\alpha]}, V_{[\beta]}]$ in Eq. (4.5) and suppose there exist $\alpha_1 \in [\alpha]$ and $\alpha_2 \in [\beta]$ such that $[[L_{\alpha_1}, (L_{\alpha_1})^*], \phi(L_{\alpha_2})] \neq 0$. By Hom-Jacobi identity, we get either $[[(L_{\alpha_1})^*, L_{\alpha_2}], \phi(L_{\alpha_1})] = 0$ or $[[L_{\alpha_2}, L_{\alpha_1}], \phi((L_{\alpha_1})^*)] = 0$ for any $\alpha_1 \in [\alpha]$ and $\alpha_2 \in [\beta]$. Hence, we get $[(L_{\alpha_1})^*, L_{\alpha_2}] \neq 0$ or $[L_{\alpha_2}, L_{\alpha_1}] \neq 0$. From here, $[V_{[\alpha]}, V_{[\beta]}] \neq 0$ in any case, which contradicts Eq. (4.6). Hence, $[I_{0,[\alpha]}, V_{[\beta]}] = 0$. In a similar way, we get $[V_{[\alpha]}, I_{0,[\beta]}] = 0$ and we conclude, together with Eqs. (4.5) and (4.6), that $[I_{[\alpha]}, I_{[\beta]}] = 0$. □

THEOREM 4.3. *The following assertions hold.*

1. *For any $\alpha \in \Lambda$, the involutive subalgebra*

$$I_{[\alpha]} = I_{0,[\alpha]} \oplus V_{[\alpha]}$$

*of $L$ associated to $[\alpha]$ is an involutive ideal of $L$.*

2. *If $L$ is simple, then there exists a connection from $\alpha$ to $\beta$ for any $\alpha, \beta \in \Lambda$ and $H = \sum_{\alpha \in \Lambda} [L_\alpha, (L_\alpha)^*]$.*

*Proof.* 1. Since $[I_{[\alpha]}, H] = [I_{[\alpha]}, L_0] \subset V_{[\alpha]}$, taking into account Propositions 4.1 and 4.2, we have

$$[I_{[\alpha]}, L] = [I_{[\alpha]}, H \oplus (\oplus_{\beta \in [\alpha]} L_\beta) \oplus (\oplus_{\gamma \notin [\alpha]} L_\gamma)] \subset I_{[\alpha]}.$$

As we also have by Lemmas 2.3-1 and 3.2-1 that $\phi(I_{[\alpha]}) = I_{[\alpha]}$, we conclude that $I_{[\alpha]}$ is an ideal of $I$.

2. The simplicity of $L$ implies $I_{[\alpha]} = L$. From here, it is clear that $[\alpha] = \Lambda$ and $H = \sum_{\alpha \in \Lambda} [L_\alpha, (L_\alpha)^*]$. □

THEOREM 4.4. *For a vector space complement $U$ of $span_{\mathbb{K}}\{[L_\alpha, (L_\alpha)^*] : \alpha \in \Lambda\}$ in H, we have*

$$L = U + \sum_{[\alpha] \in \Lambda/\sim} I_{[\alpha]},$$

*where any $I_{[\alpha]}$ is one of the involutive ideas of $L$ described in Theorem 4.3-1, satisfying $[I_{[\alpha]}, I_{[\beta]}] = 0$, whenever $[\alpha] \neq [\beta]$.*

*Proof.* Each $I_{[\alpha]}$ is well defined and, by Theorem 4.3-1, an involutive ideal of $L$. It is clear that

$$L = H \oplus (\oplus_{\alpha \in \Lambda} L_\alpha) = U + \sum_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}.$$

Finally Proposition 4.2 gives us $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. □

Let us denote by $Z(L)$ the center of $L$, that is, $Z(L) = \{x \in L : [x, L] = 0\}$.

COROLLARY 4.5. *If $Z(L)=0$ and $[L,L]=L$, then $L$ is the direct sum of the involutive ideals given in Theorem 4.3,*

$$L=\oplus_{[\alpha]\in\Lambda/\sim}I_{[\alpha]}.$$

*Proof.* From $[L,L]=L$, it is clear that $L=\oplus_{[\alpha]\in\Lambda/\sim}I_{[\alpha]}$. Finally, the sum is direct because $Z(L)=0$ and $[I_{[\alpha]},I_{[\beta]}]=0$ if $[\alpha]\neq[\beta]$. □

## 5. The simple components

In this section we focus on the simplicity of split involutive regular Hom-Lie algebras $L$ by centering our attention in those of maximal length. From now on char($\mathbb{K}$)=0.

LEMMA 5.1. *Let $L=H\oplus(\oplus_{\alpha\in\Lambda}L_\alpha)$ be a split regular Hom-Lie algebra. If $I$ is an ideal of $L$ then $I=(I\cap H)\oplus(\oplus_{\alpha\in\Lambda}(I\cap L_\alpha))$.*

*Proof.* We can see $L=H\oplus(\oplus_{\alpha\in\Lambda}L_\alpha)$ as a weight module with respect to the split Hom-Lie algebra $L$, with maximal abelian subalgebra $H$, in the natural way. The character of ideal of $I$ gives us that $I$ is a submodule of $L$. It is well-known that a submodule of a weight module is again a weight module. From here, $I$ is a weight module with respect to $L$ (and $H$) and so $I=(I\cap H)\oplus(\oplus_{\alpha\in\Lambda}(I\cap L_\alpha))$. □

LEMMA 5.2. *Let $L$ be a split regular Hom-Lie algebra with $\mathrm{Z}(L)=0$ and $I$ an ideal of $L$. If $I\subset H$ then $I=\{0\}$.*

*Proof.* Suppose there exists a nonzero ideal $I$ of $L$ such that $I\subset H$. We get $[I,H]\subset[H,H]=0$. We also get $[I,\oplus_{\alpha\in\Lambda}L_\alpha]\subset I\subset H$. Then taking into account $H=L_0$, we have $[I,\oplus_{\alpha\in\Lambda}L_\alpha]\subset H\cap(\oplus_{\alpha\in\Lambda}L_\alpha)=0$. From here $I\subset \mathrm{Z}(L)=0$, which is a contradiction. □

Let us introduce the concepts of root-multiplicativity and maximal length in the framework of split involutive regular Hom-Lie algebras, in a similar way to the ones for split Hom-Lie algebras (see [10]).

DEFINITION 5.3. A split involutive regular Hom-Lie algebra $L$ is *root-multiplicative* if given $\alpha,\beta\in\Lambda$ such that $\alpha\phi^{-1}+\beta\phi^{-1}\in\Lambda$, then $[L_\alpha,L_\beta]\neq0$.

DEFINITION 5.4. A split involutive regular Hom-Lie algebra $L$ is of *maximal length* if for any $\alpha\in\Lambda$, we have $\dim L_\alpha=1$.

Observe that if $L$ is of maximal length, then Lemma 5.1 let us assert that, given any nonzero ideal $I$ of $L$, then

$$I=(I\cap H)\oplus(\oplus_{\alpha\in\Lambda_I}L_\alpha),\ \text{where } \Lambda_I\subset\Lambda. \tag{5.7}$$

DEFINITION 5.5. A Hom-Lie algebra $L$ is called *perfect* if $\mathrm{Z}(L)=0$ and $[L,L]=L$.

THEOREM 5.6. *Let $L$ be a perfect split involutive regular Hom-Lie algebra of maximal length and root-multiplicative. If $L$ has all its nonzero roots connected, then any ideal $I$ of $L$ satisfies $I^* = I$.*

*Proof.* Consider $I$ a nonzero ideal of $L$. By Lemma 5.2 and Eq. (5.7) we can write $I = (I \cap H) \oplus (\oplus_{\alpha \in \Lambda_I} L_\alpha)$ with $\Lambda_I \subset \Lambda$ and $\Lambda_I \neq \emptyset$. Let us fix some $\alpha_0 \in \Lambda_I$ so that $0 \neq L_{\alpha_0} \subset I$. The fact $\phi(I) = I$ together with Lemma 2.3-1 allows us to assert that

$$\text{if } \alpha \in \Lambda_I \text{ then } \{\alpha\phi^z : z \in \mathbb{Z}\} \subset \Lambda_I, \tag{5.8}$$

that is

$$\{L_{\alpha_0\phi^z} : z \in \mathbb{Z}\} \in I. \tag{5.9}$$

Let us show that $(L_{\alpha_0})^* \subset I$. Since $\alpha_0 \neq 0$, and taking into account that the facts $L = [L, L]$ and Corollary 4.5 imply $H = \sum_{\beta \in \Lambda}[L_\beta, (L_\beta)^*]$, we have that there exists $[e_\beta, e_{-\beta}] \neq 0$, $e_{\pm\beta} \in L_{\pm\beta}$, $\beta \in \Lambda$, such that $\alpha_0([e_\beta, e_{-\beta}]) \neq 0$. If $\beta \in \{\pm\alpha_0\phi^z : z \in \mathbb{Z}\}$, as $0 \neq [e_{\alpha_0\phi^z}, e_{-\alpha_0\phi^z}] \in I$ then $e_{-\alpha_0} = -\alpha_0([e_{\alpha_0\phi^z}, e_{-\alpha_0\phi^z}])^{-1}[[e_{\alpha_0\phi^z}, e_{-\alpha_0\phi^z}], e_{-\alpha_0}] \in I$ and so $L_{-\alpha_0} = (L_{\alpha_0})^* \subset I$. If $\beta \notin \{\pm\alpha_0\phi^z : z \in \mathbb{Z}\}$, as $\alpha_0$ and $\beta$ are connected, the root multiplicativity and maximal length of $L$, give us a connection $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$, $k \geqslant 2$, from $\alpha_0$ to $\beta$ satisfying:

$\alpha_1 = a_0\phi^{-n}$ for some $n \in \mathbb{N}$, and
$\alpha_1\phi^{-1} + \alpha_2\phi^{-1} \in \Lambda$,
$\alpha_1\phi^{-2} + \alpha_2\phi^{-2} + \alpha_3\phi^{-1} \in \Lambda$,
$\vdots$
$\alpha_1\phi^{-i} + \alpha_2\phi^{-i} + \alpha_3\phi^{-i+1} + \cdots + \alpha_{i+1}\phi^{-1} \in \Lambda$,
$\vdots$
$\alpha_1\phi^{-k+2} + \alpha_2\phi^{-k+2} + \alpha_3\phi^{-k+3} + \cdots + \alpha_i\phi^{-k+i} + \cdots + \alpha_{k-1}\phi^{-1} \in \Lambda$.
$\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \cdots + \alpha_i\phi^{-k+i-1} + \cdots + \alpha_k\phi^{-1} = \varepsilon\beta\phi^{-m}$ for some $m \in \mathbb{N}$ and $\varepsilon \in \{\pm 1\}$.

Taking into account that $\alpha_1, \alpha_2 \in \Lambda$ and $\alpha_1\phi^{-1} + \alpha_2\phi^{-1} \in \Lambda$, the root multiplicativity and maximal length of $L$ allow us to assert $0 \neq [L_{\alpha_1}, L_{\alpha_2}] = L_{\alpha_1\phi^{-1} + \alpha_2\phi^{-1}}$. Since $0 \neq L_{\alpha_1} \subset I$ as consequence of Eq. (5.9) we get

$$0 \neq L_{\alpha_1\phi^{-1} + \alpha_2\phi^{-1}} \subset I.$$

A similar argument applied to $\alpha_1\phi^{-1} + \alpha_2\phi^{-1}$, $\alpha_3$ and

$$(\alpha_1\phi^{-1} + \alpha_2\phi^{-1})\phi^{-1} + \alpha_3\phi^{-1} = \alpha_1\phi^{-2} + \alpha_2\phi^{-2} + \alpha_3\phi^{-1}$$

gives us $0 \neq L_{\alpha_1\phi^{-2} + \alpha_2\phi^{-2} + \alpha_3\phi^{-1}} \subset I$. We can follow this process with the connection $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ to get

$$0 \neq L_{\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \cdots + \alpha_i\phi^{-k+i-1} + \cdots + \alpha_k\phi^{-1}} \subset I$$

and then

$$\text{either } L_{\beta\phi^{-m}} \subset I \text{ or } L_{-\beta\phi^{-m}} \subset I.$$

From Lemma 2.3-1, we now get

$$L_\beta \subset I \text{ or } L_{-\beta} = (L_\beta)^* \subset I. \tag{5.10}$$

In both case

$$[L_\beta, (L_\beta)^*] \subset I \tag{5.11}$$

and so $[e_\beta, e_{-\beta}] \in I$. As given any $e_{-\alpha_0} \in L_{-\alpha_0}$, we have

$$e_{-\alpha_0} = -\alpha_0([e_\beta, e_{-\beta}])^{-1}[[e_\beta, e_{-\beta}], e_{-\alpha_0}] \in I,$$

we conclude $L_{-\alpha_0} \subset I$, then we get $(L_{\alpha_0})^* \subset I$. Hence, $(\oplus_{\alpha\in\Lambda_I} L_\alpha)^* = \oplus_{\alpha\in\Lambda_I} L_\alpha$. Finally, the fact $H = \sum_{\beta\in\Lambda}[L_\beta, (L_\beta)^*]$ and Eq. (5.11) give us

$$H \subset I. \tag{5.12}$$

As $H^* = H$, we get, in particular, $(I \cap H)^* = I \cap H$. From here, and taking into account $(\oplus_{\alpha\in\Lambda_I} L_\alpha)^* = \oplus_{\alpha\in\Lambda_I} L_\alpha$, Eq. (5.7) let us conlude $I^* = I$. □

THEOREM 5.7. *Let $L$ be a perfect split involutive regular Hom-Lie algebra of maximal length and root-multiplicative. Then $L$ is simple if and only if it has all its nonzero roots connected.*

*Proof.* The first implication is Theorem 4.3-2. To prove the converse, write $L = H \oplus (\oplus_{\alpha\in\Lambda} L_\alpha)$ and consider $I$ a nonzero ideal of $L$. By Eq. (5.12), we have $H \subset I$. Given any $\alpha \in \Lambda$ and taking into account $\alpha \neq 0$ and the maximal length of $L$, we have $[H, L_{\alpha\phi^{-1}}] = L_\alpha$ and so $L_\alpha \subset I$. We conclude $I = L$, and therefore $L$ is simple. □

THEOREM 5.8. *Let $L$ be a perfect split involutive regular Hom-Lie algebra of maximal length and root multiplicative. Then $L$ is the direct sum of the family of its minimal involutive ideals, each one a simple split involutive regular Hom-Lie algebra having all its nonzero roots connected.*

*Proof.* By corollary 4.5, we can write $L = \oplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$ as the direct sum of the involutive of ideals

$$I_{[\alpha]} = I_{0,[\alpha]} \oplus V_{[\alpha]} = \text{span}_{\mathbb{K}}\{[L_\beta, (L_\beta)^*] : \beta \in [\alpha]\} \oplus (\oplus_{\beta\in[\alpha]} L_\beta),$$

where each $I_{[\alpha]}$ is a split involutive regular Hom-Lie algebra having as roots system $\Lambda_{I_{[\alpha]}} = [\alpha]$. To apply Theorem 5.7 to each $I_{[\alpha]}$, we have to observe that the root-multiplicativity of $L$ and Proposition 4.2 show that $\Lambda_{I_{[\alpha]}}$ has all of its elements $\Lambda_{I_{[\alpha]}}$-connected, that is, connected through connections contained in $\Lambda_{I_{[\alpha]}}$. We also get that any of the $I_{[\alpha]}$ is root-multiplicative as consequence of the root-multiplicativity of $L$. Clearly $I_{[\alpha]}$ is of maximal length, and finally $Z_{I_{[\alpha]}}(I_{[\alpha]})$=0, (where $Z_{I_{[\alpha]}}(I_{[\alpha]})$ denotes the center of $I_{[\alpha]}$ in $I_{[\alpha]}$), as consequence of $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$, (Theorem 4.4), and $Z(L) = 0$. We can therefore apply Theorem 5.7 to any $I_{[\alpha]}$ so as to conclude $I_{[\alpha]}$ is simple. It is clear that the decomposition $L = \oplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$ satisfies the assertions of the theorem. □

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