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(∞, C)–ISOMETRIC OPERATORS

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Abstract. In this paper we study properties of (∞, C)-isometric operators. In particular, we prove that if T is an (∞, C)-isometry and Q is a quasinilpotent operator, then $T+Q$ is an (∞, C)-isometry under suitable conditions. Moreover, we show that the class of (∞, C)-isometric operators is norm closed. Finally, we investigate properties of products and tensor products of (∞, C)-isometric operators.

1. Introduction

Agler and Stankus [1] studied the theory of m-isometric operators which are connected to Topelitz operators, classical function theory, ordinary differential equations, distributions, classical conjugate point theory, Fejer-Riesz factorization, stochastic processes, and other topics. Recently, the authors [3] have introduced (m, C)-isometric operators and studied properties of such operators. So it is natural to consider and study the classes, named (∞, C)-isometric operators, which contains every finite-isometric operators with conjugation C.

Let $\mathscr{L}(\mathscr{H})$ be the algebra of bounded linear operators on a separable complex Hilbert space $\mathscr{H}$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{C}$ be the set of complex numbers. In 1990s, Agler and Stankus [1] intensively studied the following operator; for a fixed $m \in \mathbb{N}$, an operator $T \in \mathscr{L}(\mathscr{H})$ is said to be an m-*isometric* operator if it satisfies an identity;

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*m-j} T^{m-j} = 0. \tag{1}$$

A *conjugation* on $\mathscr{H}$ is an antilinear operator $C : \mathscr{H} \to \mathscr{H}$ with $C^2 = I$ which satisfies $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathscr{H}$. Moreover, since $\|Cx\|^2 = \langle Cx, Cx \rangle = \langle x, x \rangle = \|x\|^2$ for all $x \in \mathscr{H}$, it follows that $\|C\| = 1$. For a conjugation C, there is an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathscr{H}$ such that $Ce_n = e_n$ for all n. Recall that if C is a conjugation on $\mathscr{H}$ and $T \in \mathscr{L}(\mathscr{H})$, then, since $C^2 = I$, $(CTC)^k = CT^kC$ and $(CTC)^* = CT^*C$ for every $k \in \mathbb{N}$ (see [8] or [9] for more details).

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Using the identity (1) and a conjugation C, we define (m,C)-isometric operators as follows; an operator $T \in \mathscr{L}(\mathscr{H})$ is said to be an *(m,C)-isometric operator* if there exists some conjugation C such that

$$\sum_{j=0}^{m}(-1)^j \binom{m}{j} T^{*m-j} C T^{m-j} C = 0$$

for some $m \in \mathbb{N}$. Put $\Lambda_m(T) := \sum_{j=0}^{m}(-1)^j \binom{m}{j} T^{*m-j} C T^{m-j} C$. Then T is an (m,C)-isometric operator if and only if $\Lambda_m(T) = 0$. Note that

$$T^* \Lambda_m(T)(CTC) - \Lambda_m(T) = \Lambda_{m+1}(T). \tag{2}$$

Hence, if $\Lambda_m(T) = 0$, then $\Lambda_n(T) = 0$ for all $n \geqslant m$. Moreover, it is obvious that T is an (m,C)-isometry if and only if CTC is an (m,C)-isometry (see [3]). We now introduce the concept of (∞,C)-isometric operators. An operator $T \in \mathscr{L}(\mathscr{H})$ is called an *(∞,C)-isometric operator with conjugation C* if

$$\limsup_{m\to\infty} \|\Lambda_m(T)\|^{\frac{1}{m}} = 0.$$

An operator $T \in \mathscr{L}(\mathscr{H})$ is called a finite-isometric operator with conjugation C if T is an (m,C)-isometry for some $m \geqslant 1$. The class of (∞,C)-isometric operators is a large class which contains finite-isometric operators with conjugation C.

In this paper we study properties of (∞,C)-isometric operators. In particular, we show that if T is an (∞,C)-isometry and Q is a quasinilpotent operator, then $T+Q$ is an (∞,C)-isometry where $TQ = QT$ and $T^*CQC = CQCT^*$. Moreover, we verify that the class of (∞,C)-isometric operators is norm closed. Finally, we examine properties of products and tensor products of (∞,C)-isometric operators.

2. (∞,C)-isometric operators

In this section, we give properties of (∞,C)-isometric operators. It is known from [8] that if C is a conjugation on a Hilbert space $\mathscr{H}$, then there exists an orthonormal basis $\{e_n\}$ of $\mathscr{H}$ such that

$$C(\sum_{n=1}^{\infty} a_n e_n) = \sum_{n=1}^{\infty} \overline{a_n} e_n$$

whenever $\sum |a_n|^2 < \infty$ and, specifically

$$C(e_n) = e_n$$

for all $n \in \mathbb{N}$. This means that every conjugation is unitarily equivalent to the canonical conjugation on an l^2-space with the appropriate dimension (see [8]). We refer to such a basis as a *C-real orthonormal basis* for $\mathscr{H}$. We start with the following example.

EXAMPLE 2.1. *Let C_n be the conjugation on $\mathbb{C}^n$ defined by*

$$C_n(z_1, z_2, \cdots, z_n) := (\overline{z_1}, \overline{z_2}, \cdots, \overline{z_n}).$$

Assume that $T = \oplus_{n=1}^{\infty} T_n$ where T_n is an $n \times n$ matrix;

$$T_n = I_n + N_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{n} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{n} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \frac{1}{n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Since N_n is nilpotent of order n, it obvious that T_n is a $(2n-1, C_n)$-isometric operator. Hence T is an $(\infty, \mathscr{C})$-isometric operator with a conjugation $\mathscr{C} = \oplus_{n=1}^{\infty} C_n$. Indeed, if $R_n = T_1 \oplus \cdots \oplus T_n \oplus I \oplus I \oplus \cdots$, then R_n is a $(2n-1, C_n)$-isometric operator and $R_n R_k = R_k R_n$ for all $n, k \geqslant 1$. Thus $R_n \to T$ in the operator norm. Hence T is an $(\infty, \mathscr{C})$-isometric operator with a conjugation $\mathscr{C} = \oplus_{n=1}^{\infty} C_n$ from Theorem 2.7(ii).

We next examine properties of (∞, C)-isometric operators.

THEOREM 2.2. *Let $T \in \mathscr{L}(\mathscr{H})$ be an (∞, C)-isometric operator where C is a conjugation on $\mathscr{H}$. Then the following statements hold;*

(a) *If $(T-\alpha)x = 0$ and $(T-\beta)y = 0$ with $\alpha\beta \neq 1$, then $\langle Cx, y \rangle = 0$. In particular, if x or y is nonzero vectors in* $\ker T$, *then $\langle Cx, y \rangle = 0$.*

(b) *If $(T-\alpha)x = 0$ and $(T-\beta)Cx = 0$ where x is nonzero, then $\alpha\beta = 1$.*

(c) *If $\{x_n\}$ and $\{y_n\}$ are sequences of unit vectors such that* $\lim_{n\to\infty}(T-\alpha)x_n = 0$ *and* $\lim_{n\to\infty}(T-\beta)y_n = 0$ *with $\alpha\beta \neq 1$, then a sequence $\{\langle Cx_n, y_n \rangle\}$ has a subsequence $\{\langle Cx_{n_l}, y_{n_l} \rangle\}$ which converges to 0.*

(d) *If $\{x_n\}$ is a sequence of unit vectors such that* $\lim_{n\to\infty}(T-\alpha)x_n = 0$ *and* $\lim_{n\to\infty}(T-\beta)Cx_n = 0$, *then $\alpha\beta = 1$.*

Proof. (a) Let $\alpha, \beta \in \mathbb{C}$ be distinct eigenvalues of T with $\alpha\beta \neq 0, 1$ and let x, y be the unit eigenvectors such that $Tx = \alpha x$ and $Ty = \beta y$. Then it follows that $CTC(Cx) = \overline{\alpha} Cx$ and so

$$\begin{aligned} \langle \Lambda_m(T)Cx, y \rangle &= \langle \left(\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*m-j} C T^{m-j} C \right) Cx, y \rangle \\ &= \langle \left(\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*m-j} \overline{\alpha}^{m-j} \right) Cx, y \rangle \\ &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \overline{\alpha}^{m-j} \langle T^{*m-j} Cx, y \rangle \\ &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \overline{\alpha}^{m-j} \langle Cx, T^{m-j} y \rangle \\ &= \langle (\overline{\alpha\beta} - 1)^m Cx, y \rangle = (\overline{\alpha\beta} - 1)^m \langle Cx, y \rangle. \end{aligned} \tag{3}$$

Moreover, since $\|C\|=1$, it follows from (3) that

$$|(\overline{\alpha\beta}-1)||\langle Cx,y\rangle|^{\frac{1}{m}}=|\langle\Lambda_m(T)Cx,y\rangle|^{\frac{1}{m}}\leqslant\|\Lambda_m(T)Cx\|^{\frac{1}{m}}\|y\|^{\frac{1}{m}}\leqslant\|\Lambda_m(T)\|^{\frac{1}{m}}. \tag{4}$$

Since T is an (∞,C)-isometric operator, it follows from (4) that

$$|(\overline{\alpha\beta}-1)|\lim_{m\to\infty}|\langle Cx,y\rangle|^{\frac{1}{m}}\leqslant\limsup_{m\to\infty}\|\Lambda_m(T)\|^{\frac{1}{m}}=0. \tag{5}$$

This implies that $\lim_{m\to\infty}|\langle Cx,y\rangle|^{\frac{1}{m}}=0$ is due to the fact that $\alpha\beta\neq 1$.

Since $\lim_{m\to\infty}|\langle Cx,y\rangle|^{\frac{1}{m}}=1$ if $\langle Cx,y\rangle\neq 0$, we conclude that $\langle Cx,y\rangle=0$.

On the other hand, if $\alpha=0$ or $\beta=0$ or $\alpha=\beta$, then we know $\langle Cx,y\rangle=0$ from (5).

(b) Assume that $\alpha\beta\neq 1$. Set $y=Cx$. Then it is a nonzero and (a) implies that $\|x\|^2=\langle Cx,Cx\rangle=0$, which is a contradiction. Hence $\alpha\beta=1$.

(c) Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences of unit vectors such that

$$\lim_{n\to\infty}(T-\alpha)x_n=0 \text{ and } \lim_{n\to\infty}(T-\beta)y_n=0.$$

Then $\lim_{n\to\infty}(CTC-\overline{\alpha})Cx_n=0$ and $\lim_{n\to\infty}(T^k-\beta^k)y_n=0$. Thus we have $\lim_{n\to\infty}(CT^kC-\overline{\alpha}^k)Cx_n=0$ for every $k\in\mathbb{N}$. Since $\{\langle Cx_n,y_n\rangle\}_{n=1}^{\infty}$ is bounded, $\{\langle Cx_n,y_n\rangle\}_{n=1}^{\infty}$ has a convergent subsequence $\{\langle Cx_{n_l},y_{n_l}\rangle\}$. If $\lim_{l\to\infty}\langle Cx_{n_l},y_{n_l}\rangle=\mu$, then it suffices to show that $\mu=0$. Note that for each fix $m\geqslant 1$, the following relations hold;

$$\begin{aligned}
|(\overline{\alpha\beta}-1)^m\mu| &= \lim_{l\to\infty}|(\overline{\alpha\beta}-1)^m\langle Cx_{n_l},y_{n_l}\rangle| \\
&= |\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\overline{\alpha\beta}^{m-j}\lim_{l\to\infty}\langle Cx_{n_l},y_{n_l}\rangle| \\
&= |\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\lim_{l\to\infty}\langle (CT^{m-j}C)Cx_{n_l},T^{m-j}y_{n_l}\rangle| \\
&= |\lim_{l\to\infty}\langle\left(\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{*m-j}CT^{m-j}C\right)Cx_{n_l},y_{n_l}\rangle| \\
&= \lim_{l\to\infty}|\langle\Lambda_m(T)Cx_{n_l},y_{n_l}\rangle|\leqslant\|\Lambda_m(T)\|.
\end{aligned} \tag{6}$$

Since T is an (∞,C)-isometric operator, it follows from (6) that

$$|(\overline{\alpha\beta}-1)|\lim_{m\to\infty}|\mu|^{\frac{1}{m}}=\limsup_{m\to\infty}|(\overline{\alpha\beta}-1)^m\mu|^{\frac{1}{m}}\leqslant\limsup_{m\to\infty}\|\Lambda_m(T)\|^{\frac{1}{m}}=0.$$

Since $\alpha\beta\neq 1$, it follows that $\mu=0$. Hence $\lim_{l\to\infty}\langle Cx_{n_l},y_{n_l}\rangle=0$.

(d) Assume that $\alpha\beta\neq 1$. Set $y_n=Cx_n$ and $y_{n_l}=Cx_{n_l}$ in (c). Then $\{\langle Cx_n,Cx_n\rangle\}=\{1\}$ has a subsequence $\{\langle Cx_{n_l},Cx_{n_l}\rangle\}=\{1\}$ which converges to 0 by (c). This is a contradiction. Hence $\alpha\beta=1$. □

Recall that a vector $x\in\mathscr{H}$ is said to be *isotropic* if $\langle x,Cx\rangle=0$ (see [7, Page 16]).

THEOREM 2.3. *Let $T \in \mathscr{L}(\mathscr{H})$. Then the following assertions hold:*
(i) *If T is complex symmetric with a conjugation C, then*

$$\limsup_{m\to\infty} \|\Lambda_m(T)\|^{\frac{1}{m}} \leqslant r(T^2 - I)$$

where $r(A)$ denotes the spectral radius of A. In particular, if $r(T^2-I)=0$, then T is an (∞, C)-isometric operator.

(ii) *If T is an (∞, C)-isometric operator and $x \in \ker(T-\lambda)$, then $\lambda = 1$ or x is isotropic.*

(iii) *If T is a strict contraction, i.e., $\|T\| < 1$, then T is not an (∞, C)-isometric operator.*

Proof. (i) Since $T = CT^*C$, it follows that

$$\Lambda_m(T) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*m-j} C T^{m-j} C = C\Big(\sum_{j=0}^{m} (-1)^j \binom{m}{j} (T^2)^{m-j}\Big) C \tag{7}$$

and therefore

$$\|\Lambda_m(T)\| = \|C\Big(\sum_{j=0}^{m} (-1)^j \binom{m}{j} (T^2)^{m-j}\Big) C\| \leqslant \|(T^2 - I)^m\| \tag{8}$$

and hence $\|\Lambda_m(T)\|^{\frac{1}{m}} \leqslant \|(T^2-I)^m\|^{\frac{1}{m}}$. Thus we obtain that

$$\limsup_{m\to\infty} \|\Lambda_m(T)\|^{\frac{1}{m}} \leqslant \limsup_{m\to\infty} \|(T^2-I)^m\|^{\frac{1}{m}} = r(T^2 - I).$$

In particular, if $r(T^2-I)=0$, then T is an (∞, C)-isometric operator.

(ii) Let $x \in \ker(T-\lambda)$. Then $(T-\lambda)x = 0$. Therefore, $(CT^kC - \overline{\lambda}^k)Cx = 0$ and so $(T^k - \lambda^k)x = 0$ for every $k \in \mathbb{N}$. Then it holds that

$$\begin{aligned}
\langle \Lambda_m(T)Cx, x\rangle &= \Big\langle \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} T^{*m-j} C T^{m-j} C\right) Cx, x\Big\rangle \\
&= \langle \sum_{j=0}^{m} (-1)^j \binom{m}{j} (CT^{m-j}C)Cx, T^{m-j}x\rangle \\
&= \sum_{j=0}^{m} (-1)^j \binom{m}{j} \langle (CT^{m-j}C)Cx, T^{m-j}x\rangle \\
&= \sum_{j=0}^{m} (-1)^j \binom{m}{j} \overline{\lambda}^{2(m-j)} \langle Cx, x\rangle = (\overline{\lambda}^2 - 1)^m \langle Cx, x\rangle.
\end{aligned}$$

This gives that

$$\begin{aligned}
|\overline{\lambda}^2 - 1|^m \cdot |\langle Cx, x\rangle| &= |\langle \Lambda_m(T)Cx, x\rangle| \\
&\leqslant \|\Lambda_m(T)\| \|Cx\| \|x\| = \|\Lambda_m(T)\| \|x\|^2.
\end{aligned}$$

Since T is an (∞,C)-isometric operator, it follows that $\lambda=1$ or $\langle Cx,x\rangle=0$. Hence $\lambda=1$ or x is isotropic.

(iii) Assume that T is an (∞,C)-isometric operator. Then $T^*CTC\neq I$. Indeed, if T is a $(1,C)$-isometry, then

$$1>\|T\|^2=\|T^*\|\|C\|\|T\|\|C\|\geqslant\|T^*CTC\|=\|I\|=1,$$

which is a contradiction. By the structure of $\Lambda_m(T)$, (2) implies that

$$\|\Lambda_m(T)\|\leqslant\|T\|^2\|\Lambda_m(T)\|+\|\Lambda_{m+1}(T)\|.$$

Thus we have $(1-\|T\|^2)\|\Lambda_m(T)\|\leqslant\|\Lambda_{m+1}(T)\|$ for some $m\in\mathbb{N}$. Therefore, we get that $(1-\|T\|^2)^m\|\Lambda_1(T)\|\leqslant\|\Lambda_{m+1}(T)\|$ and so

$$(1-\|T\|^2)^{\frac{m}{m+1}}\|\Lambda_1(T)\|^{\frac{1}{m+1}}\leqslant\|\Lambda_{m+1}(T)\|^{\frac{1}{m+1}}. \tag{9}$$

Since T is an (∞,C)-isometric operator and $\Lambda_1(T)\neq 0$, by taking limsup as $m\to\infty$, we obtain that $1-\|T\|^2\leqslant 0$. Thus $\|T\|\geqslant 1$. So we have a contradiction. □

COROLLARY 2.4. *Let $T\in\mathscr{L}(\mathscr{H})$. Then the following statements hold.*

(i) *The inequality*

$$\limsup_{m\to\infty}\|\Lambda_m(\Lambda_k(T))\|^{\frac{1}{m}}\leqslant r(\Lambda_k(T)^2-I)$$

holds for any $k\in\mathbb{N}$ where $r(A)$ denotes the spectral radius of A.

(ii) *If $T^2=I$, then T is an (m,C)-isometric operator and if $T^2=I+Q$ where Q is quasinilpotent, then T is an (∞,C)-isometric operator.*

Proof. (i) Since

$$\Lambda_k(T)^*=\sum_{j=0}^{k}(-1)^j\binom{k}{j}CT^{*k-j}CT^{k-j},$$

it follows that $C\Lambda_k(T)^*C=\sum_{j=0}^{k}(-1)^j\binom{k}{j}T^{*k-j}CT^{k-j}C=\Lambda_k(T)$. Therefore, $\Lambda_k(T)$ is a complex symmetric operator with the conjugation C for any $k\in\mathbb{N}$.

Hence $\limsup_{m\to\infty}\|\Lambda_m(\Lambda_k(T))\|^{\frac{1}{m}}\leqslant r(\Lambda_k(T)^2-I)$ by Theorem 2.3(i)

(ii) If $T^2=I$, then T is complex symmetric with a conjugation C from [9]. Thus (8) implies that $\Lambda_m(T)=0$ and so T is an (m,C)-isometric operator. On the other hand, if $T^2=I+Q$ where Q is quasinilpotent, then $r(T^2-I)=0$ and therefore T is an (∞,C)-isometric operator. □

REMARK 2.5. We observe from Theorem 2.3(iii) that if S is an isometry, then γS is not an (∞,C)-isometric operator where γ is a constant for $0<|\gamma|<1$. Moreover, if $T\in\mathscr{L}(\mathscr{H})$ and $x\in\ker(T-\lambda)$ where $\lambda\neq 1$ and x is not isotropic, then we know from Theorem 2.3(ii) that T is not an (∞,C)-isometric operator.

We investigate the quasinilpotent perturbations of an (∞, C)-isometric operator and show that their class is norm closed.

LEMMA 2.6. *If T and Q are in $\mathscr{L}(\mathscr{H})$ with $TQ = QT$ and $T^*CQC = CQCT^*$, then, for $m \geqslant 2$,*

$$\|\Lambda_m(T+Q)\| \leqslant K^m \big(\max_{l \leqslant n \leqslant m} \|\Lambda_n(T)\| + \max_{l \leqslant n \leqslant m} \|Q^n\| \big)$$

where $K = 2((\|T\| + \|Q\|)^2 + 2\|T\| + 1)$ and $l = [\frac{m}{3}]$ is the integer part of $\frac{m}{3}$.

Proof. Since

$$\begin{aligned}[(a+b)(c+d)-1]^m &= [(ac-1)+(a+b)d+bc]^m \\ &= \sum_{m_1+m_2+m_3=m} \binom{m}{m_1, m_2, m_3} (a+b)^{m_1} b^{m_2} (ac-1)^{m_3} c^{m_2} d^{m_1},\end{aligned}$$

it follows that

$$\Lambda_m(T+Q) = \sum_{m_1+m_2+m_3=m} \binom{m}{m_1, m_2, m_3} (T^*+Q^*)^{m_1} Q^{*m_2} \Lambda_{m_3}(T) C T^{m_2} C C Q^{m_1} C. \tag{10}$$

Assume that $l = [\frac{m}{3}]$ is the integer part of $\frac{m}{3}$. Put

$$M_i = \sum_{m_1+m_2+m_3=m \text{ and } m_i \geqslant l} \binom{m}{m_1, m_2, m_3} \|(T^*+Q^*)^{m_1} Q^{*m_2} \Lambda_{m_3}(T) C T^{m_2} Q^{m_1} C\|$$

for $i = 1,2,3$. Since $m_1 + m_2 + m_3 = m$, it follows that $m_j \geqslant l$ for some $j = 1,2,3$. Therefore, we get that

$$\begin{aligned}&\|\Lambda_m(T+Q)\| \\ &\leqslant \sum_{m_1+m_2+m_3=m} \binom{m}{m_1, m_2, m_3} \|(T^*+Q^*)^{m_1} Q^{*m_2} \Lambda_{m_3}(T) C T^{m_2} Q^{m_1} C\| \\ &\leqslant M_1 + M_2 + M_3.\end{aligned} \tag{11}$$

On the other hand, since $\|C\| = 1$, we get that

$$\begin{aligned} M_3 &= \sum_{m_1+m_2+m_3=m,\ m_1 \geqslant l} \binom{m}{m_1, m_2, m_3} \|(T^*+Q^*)^{m_1} Q^{*m_2} \Lambda_{m_3}(T) C T^{m_2} Q^{m_1} C\| \\ * &\leqslant \sum_{m_1+m_2+m_3=m,\ m_1 \geqslant l} \binom{m}{m_1, m_2, m_3} (\|T^*\| + \|Q^*\|)^{m_1} \|Q^*\|^{m_2} \|\Lambda_{m_3}(T)\| \|T\|^{m_2} \|Q\|^{m_1} \\ * &\leqslant \max_{l \leqslant n \leqslant m} \|\Lambda_n(T)\| \cdot \sum_{\substack{m_1+m_2+m_3=m, \\ m_1 \geqslant l}} \binom{m}{m_1, m_2, m_3} (\|T\| + \|Q\|)^{m_1} \|Q\|^{m_2} \|T\|^{m_2} \|Q\|^{m_1} \\ * &= \max_{l \leqslant n \leqslant m} \|\Lambda_n(T)\| \cdot \big((\|T\| + \|Q\|)\|Q\| + \|T\|\|Q\| + 1\big)^m \\ * &\leqslant \max_{l \leqslant n \leqslant m} \|\Lambda_n(T)\| \cdot \Big(\frac{K}{2}\Big)^m. \end{aligned} \tag{12}$$

Since $\|\Lambda_k(T)\| \leqslant (\|T\|+1)^k$ for all $k \in \mathbb{N}$, it follows from a similar method of (12) that

$$\begin{aligned} M_1 &\leqslant \max_{l \leqslant n \leqslant m} \|CQ^nC\| \cdot \big((\|T^*\|+\|Q^*\|)+\|Q^*\|\|T\|+(\|T\|+1)\big)^m \\ &\leqslant \max_{l \leqslant n \leqslant m} \|Q^n\| \cdot \Big(\frac{K}{2}\Big)^m \end{aligned}$$

and

$$\begin{aligned} M_2 &\leqslant \max_{l \leqslant n \leqslant m} \|Q^{*n}\| \cdot \big((\|T^*\|+\|Q^*\|)\|Q\|+\|T\|+(\|T\|+1)\big)^m \\ &\leqslant \max_{l \leqslant n \leqslant m} \|Q^n\| \cdot \Big(\frac{K}{2}\Big)^m. \end{aligned}$$

Hence (11) implies that

$$\begin{aligned} \|\Lambda_m(T+Q)\| &\leqslant \Big(\frac{K}{2}\Big)^m \max_{l \leqslant n \leqslant m} \|\Lambda_n(T)\| + 2\Big(\frac{K}{2}\Big)^m \max_{l \leqslant n \leqslant m} \|Q^n\| \\ &\leqslant K^m \big(\max_{l \leqslant n \leqslant m} \|\Lambda_n(T)\| + \max_{l \leqslant n \leqslant m} \|Q^n\|\big), \end{aligned}$$

because $m \geqslant 2$. Hence this completes the proof. $\square$

THEOREM 2.7. *Let $T \in \mathcal{L}(\mathcal{H})$ and let C be a conjugation on $\mathcal{H}$. Then the following statements hold:*

(i) *If T is an (∞, C)-isometric operator and Q is a quasinilpotent operator where $TQ = QT$ and $T^*CQC = CQCT^*$, then $T+Q$ is an (∞, C)-isometric operator with conjugation C.*

(ii) *If $\{T_n\}$ is a sequence of commuting (∞, C)-isometric operators with conjugation C such that $\lim_{n\to\infty} \|T_n - T\| = 0$, then T is an (∞, C)-isometric operator.*

Proof. (i) Since T is an (∞, C)-isometric operator and Q is a quasinilpotent operator, it follows that for given $0 < \varepsilon < 1$, there exists N such that

$$\|\Lambda_n(T)\| \leqslant \varepsilon^n \text{ and } \|Q^n\| \leqslant \varepsilon^n$$

for all $n \geqslant N$. By Lemma 2.6, for $m \geqslant 3N$ and $l = [\frac{m}{3}] \geqslant N$, we get that

$$\begin{aligned} \|\Lambda_m(T+Q)\|^{\frac{1}{m}} &\leqslant K\big(\max_{l \leqslant n \leqslant m} \|\Lambda_n(T)\| + \max_{l \leqslant n \leqslant m} \|Q^n\|\big)^{\frac{1}{m}} \leqslant K(2\varepsilon^n)^{\frac{1}{m}} \leqslant K(2\varepsilon^l)^{\frac{1}{m}} \\ &= 2^{\frac{1}{m}} K \varepsilon^{\frac{l}{m}} (= 2^{\frac{1}{m}} K \varepsilon^{\frac{1}{m}[\frac{m}{3}]}) \text{ since } \varepsilon < 1. \end{aligned}$$

Since ε is arbitrary, $\limsup_{m\to\infty} \|\Lambda_m(T+Q)\|^{\frac{1}{m}} = 0$. Hence $T+Q$ is an (∞, C)-isometric operator.

(ii) If $T_nT_k = T_kT_n$ for all $k, n \in \mathbb{N}$, then $TT_n = T_nT$ for all $n \geqslant 1$. For a given $0 < \varepsilon < 1$, there exists n_0 such that

$$\|T - T_{n_0}\| \leqslant \varepsilon \text{ and } \|\Lambda_n(T_{n_0})\| \leqslant \varepsilon^n$$

for all $n \geqslant n_0$. By Lemma 2.6, for $m \geqslant 3n_0$ and $l = [\frac{m}{3}] \geqslant n_0$, we obtain that

$$\begin{aligned} \|\Lambda_m(T)\|^{\frac{1}{m}} &= \|\Lambda_m(T_{n_0} + T - T_{n_0})\|^{\frac{1}{m}} \\ &\leqslant K\big(\max_{l \leqslant n \leqslant m} \|\Lambda_n(T_{n_0})\| + \max_{l \leqslant n \leqslant m} \|T - T_{n_0}\|^n \big)^{\frac{1}{m}} \\ &\leqslant 2^{\frac{1}{m}} K \varepsilon^{\frac{l}{m}} = 2^{\frac{1}{m}} K \varepsilon^{\frac{1}{m}[\frac{m}{3}]}. \end{aligned}$$

Since ε is arbitrary, it follows that $\limsup_{m \to \infty} \|\Lambda_m(T)\|^{\frac{1}{m}} = 0$. Hence T is an (∞, C)-isometric operator. □

Let us recall that a closed subspace $\mathscr{M}$ is *hyperinvariant* for T if it is invariant for every operator in $\{T\}'$ where $\{T\}' = \{R \in \mathscr{L}(\mathscr{H}) : TR = RT\}$.

COROLLARY 2.8. *Let C be a conjugation on $\mathscr{H}$ and Q be a nonzero quasinilpotent operator on $\mathscr{H}$. Then $\mu I + Q$ is an (∞, C)-isometric operator with $|\mu| = 1$. Moreover, $\mu I + Q$ has a nontrivial hyperinvariant subspace.*

Proof. If $T = \mu I$ for $|\mu| = 1$, then T is clearly an (∞, C)-isometric operator. Hence the proof follows from Theorem 2.7. For the second statement, we know from [6, Theorem 2.18] that Q has a nontrivial hyperinvariant subspace. Hence $\mu I + Q$ has a nontrivial hyperinvariant subspace. □

COROLLARY 2.9. *Let C be the canonical conjugation on $\mathscr{H}$ given by*

$$C\Big(\sum_{n=0}^{\infty} x_n e_n\Big) = \sum_{n=0}^{\infty} \overline{x_n} e_n$$

where $\{e_n\}$ is an orthonormal basis of $\mathscr{H}$ with $Ce_n = e_n$. If W is the weighted shift on $\mathscr{H}$ defined by $We_n = \alpha_n e_{n+1}$ $(n = 0, 1, 2, \ldots)$ where $\{\alpha_n\}_{n=0}^{\infty}$ is a weight sequence which is decreasing to 0, then $T = I + W$ is an (∞, C)-isometric operator.

Proof. For any $\varepsilon > 0$, since W is a quasinilpotent operator, $\sigma(W) = \{0\}$, $WC = CW$, and $\Lambda_m(T) = \Lambda_m(W)$, it follows from [5] that

$$\limsup_{m \to \infty} \|\Lambda_m(T)\|^{\frac{1}{m}} = \limsup_{m \to \infty} \|\Lambda_m(W)\|^{\frac{1}{m}} \leqslant \varepsilon.$$

Since ε is arbitrary, it follows that T is an (∞, C)-isometric operator. □

EXAMPLE 2.10. Under the same conjugation C as in Corollary 2.9, if W is the weighted shift on $\mathscr{H}$ defined by $We_n = \frac{1}{n+1} e_{n+1}$ $(n = 0, 1, 2, \ldots)$, then $T = I + W$ is an (∞, C)-isometric operator from Corollary 2.9.

Finally , we study properties of products of (∞, C)-isometric operators.

LEMMA 2.11. *Let $T,S \in \mathcal{L}(\mathcal{H})$ satisfy $TS = ST$ and $S^*(CTC) = (CTC)S^*$. Then*

$$\Lambda_m(TS) = \sum_{j=0}^{m} \binom{m}{j} T^{*j}\Lambda_{m-j}(T)CT^jC\Lambda_j(S) \tag{13}$$

where $\Lambda_0(T) = I$ and $\Lambda_0(S) = I$.

Proof. Assume that $TS = ST$ and $S^*(CTC) = (CTC)S^*$. Since $S^{*j}(CT^kC) = (CT^kC)S^{*j}$ holds for all $j,k \in \mathbb{N}$ and

$$\begin{aligned}(abcd-1)^m &= [(ab-1)+a(cd-1)b]^m \\ &= \sum_{j=0}^{m} \binom{m}{j} a^j(ab-1)^{m-j}b^j(cd-1)^j,\end{aligned}$$

it follows that

$$\begin{aligned}\Lambda_m(TS) &= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(TS)^{*m-j}C(TS)^{m-j}C \\ &= \sum_{j=0}^{m}\binom{m}{j}T^{*j}\Lambda_{m-j}(T)CT^jC\Lambda_j(S)\end{aligned}$$

where $\Lambda_0(T) = I$ and $\Lambda_0(S) = I$. □

THEOREM 2.12. *Let T and S be (∞,C)-isometric operators with conjugation C. Assume that $TS = ST$ and $S^*(CTC) = (CTC)S^*$. Then TS is an (∞,C)-isometric operator.*

Proof. Assume that T and S are (∞,C)-isometric operators. Then for a given $0 < \varepsilon < 1$, there exist N_1 and N_2 such that

$$\|\Lambda_{n_1}(T)\| \leqslant \varepsilon^n \text{ and } \|\Lambda_{n_2}(S)\| \leqslant \varepsilon^n$$

for $n_1 \geqslant N_1$ and $n_2 \geqslant N_2$. Put $N = \max\{N_1,N_2\}$. Then it suffices to show that there is a constant $K > 0$ such that for $m \geqslant 2N$,

$$\|\Lambda_m(TS)\| \leqslant K^m\varepsilon^{\frac{m}{2}}.$$

Let $l = [\frac{m}{2}]$ denote the integer part of $\frac{m}{2}$. Then by (13), we have

$$\begin{aligned}\Lambda_m(TS) &= \sum_{j=0}^{l}\binom{m}{j}T^{*j}\Lambda_{m-j}(T)CT^jC\Lambda_j(S) \\ &\quad + \sum_{j=l+1}^{m}\binom{m}{j}T^{*j}\Lambda_{m-j}(T)CT^jC\Lambda_j(S).\end{aligned} \tag{14}$$

If $j \leqslant l = [\frac{m}{2}]$, then $m - j \geqslant [\frac{m}{2}] = l \geqslant N$, and so $\|\Lambda_{m-j}(T)\| \leqslant \varepsilon^{m-j} \leqslant \varepsilon^l$. Since $\|C\| = 1$, it follows that $\|\Lambda_j(S)\| \leqslant (\|S\|+1)^j$ for all $j \geqslant 1$. Thus by (14) we get that

$$\|\sum_{j=0}^{l}\binom{m}{j}T^{*j}\Lambda_{m-j}(T)CT^jC\Lambda_j(S)\|$$

$$
\begin{aligned}
&\leqslant \sum_{j=0}^{l}\binom{m}{j}\|\Lambda_{m-j}(T)\|\|T^{*j}\|\|CT^jC\|\|\Lambda_j(S)\| \\
&\leqslant \sum_{j=0}^{l}\binom{m}{j}\varepsilon^{m-j}\|T\|^j\|T\|^j\|(\|S\|+1)^j \\
&\leqslant \varepsilon^l \sum_{j=0}^{m}\binom{m}{j}\|T\|^{2j}(\|S\|+1)^j = \varepsilon^l(1+\|T\|^2(\|S\|+1))^m. \qquad (15)
\end{aligned}
$$

Similarly, if $j \geqslant l+1 \geqslant N$, then $\|\Lambda_j(S)\| \leqslant \varepsilon^l$ and hence we have

$$
\|\sum_{j=l+1}^{m}\binom{m}{j}T^{*j}\Lambda_{m-j}(T)CT^jC\Lambda_j(S)\| \leqslant \varepsilon^l(\|T\|^2+(\|T\|+1))^m. \qquad (16)
$$

From (15) and (16), we know that for $n \geqslant 2N$

$$
\|\Lambda_m(TS)\| \leqslant \varepsilon^{[\frac{m}{2}]}\big((1+\|T\|^2(\|S\|+1))^m+(\|T\|^2+(\|T\|+1))^m\big).
$$

Thus $\limsup_{m\to\infty}\|\Lambda_m(TS)\|^{\frac{1}{m}} = 0$. Hence TS is an (∞, C)-isometric operator. □

We illustrate the following example by Theorem 2.12.

EXAMPLE 2.13. Let $C : \mathscr{H} \to \mathscr{H}$ be the conjugation given by

$$
C(\sum_{n=1}^{\infty} x_n e_n) = \sum_{n=1}^{\infty} \overline{x_n} e_n
$$

where $\{x_n\}$ is a sequence in $\mathbb{C}$ with $\sum_{n=1}^{\infty}|x_n|^2 < \infty$. Suppose that $A, B \in \mathscr{L}(\mathscr{H})$ are the weighted shifts given by $Ae_n = \alpha_n e_{n+1}$ and $Be_n = \beta_n e_{n+1}$ with $\beta_n = \frac{1}{n}$ for all $n \geqslant 1$. If $|\alpha_n|^2 = 1$, $\frac{\alpha_{n-1}}{\alpha_n} = \frac{n-1}{n}$, and $\frac{\alpha_{n+1}}{\alpha_n} = \frac{n}{n+1}$ for $n \geqslant 2$, then A is a $(1, C)$-isometry and it is easy to compute

$$
ACB^*Ce_n = ACB^*e_n = AC(\overline{\beta_{n-1}}e_{n-1}) = A\beta_{n-1}e_{n-1} = \alpha_{n-1}\beta_{n-1}e_n
$$

and

$$
CB^*CAe_n = CB^*C(\alpha_n e_{n+1}) = CB^*(\overline{\alpha_n}e_{n+1}) = C(\overline{\alpha_n\beta_n e_n}) = \alpha_n\beta_n e_n.
$$

Moreover, $ABe_n = A\beta_n e_{n+1} = \beta_n\alpha_{n+1}e_{n+1}$ and $BAe_n = B\alpha_n e_n = \alpha_n\beta_{n+1}e_{n+1}$. Therefore, A and $B+I$ are (∞, C)-isometric operators. Hence $A(I+B)$ is an (∞, C)-isometric operator from Theorem 2.12.

COROLLARY 2.14. *Let T and S be (∞, C)-isometric operators with conjugation C. Suppose that $T^*(CTC) = (CTC)T^*$. Then the following arguments hold.*

(i) *If $TS = ST$ and $S^*(CTC) = (CTC)S^*$, then T^kS^j and S^jT^k are (∞, C)-isometric operators for any $k, j \in \mathbb{N}$.*

(ii) *T^n is an (∞, C)-isometric operator for any $n \in \mathbb{N}$.*

Proof. (i) By Theorem 2.12, TS is an (∞,C)-isometric operator. It suffices to show that T^kS is an (∞,C)-isometric operator. Since $TS = ST$, $S^*(CTC) = (CTC)S^*$, and $T^*(CTC) = (CTC)T^*$, it follows that $T^{k-1}(TS) = (TS)T^{k-1}$ and

$$(TS)^*CT^{k-1}C = S^*T^*(CTC)^{k-1} = (CTC)^{k-1}S^*T^* = CT^{k-1}C(TS)^*.$$

By Theorem 2.12, $T^{k-1}TS = T^kS$ is an (∞,C)-isometric operator. Similarly, T^kS^j is an (∞,C)-isometric operator. Also, we can show that S^jT^k is an (∞,C)-isometric operator by a similar method.

(ii) If $n = 2$, then it is clear. Assume that the above statement holds for $n = k$. Put $S = T^k$. Then $TS = T^{k+1}$ is an (∞,C)-isometric operator from Theorem 2.12. □

Let us recall that $\mathscr{H}_1 \otimes \mathscr{H}_2$ denotes the completion (endowed with a sensible uniform cross-norm) of the algebraic tensor product $\mathscr{H}_1 \otimes \mathscr{H}_2$ of $\mathscr{H}_1$ and $\mathscr{H}_2$ where $\mathscr{H}_1$ and $\mathscr{H}_2$ are separable complex Hilbert spaces. For operators $T \in \mathscr{L}(\mathscr{H}_1)$ and $S \in \mathscr{L}(\mathscr{H}_2)$, we define the *tensor product* operator $T \otimes S$ on $\mathscr{L}(\mathscr{H}_1 \otimes \mathscr{H}_2)$ by

$$(T \otimes S)(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j) = \sum_{j=1}^{n} \alpha_j Tx_j \otimes Sy_j.$$

Then it is well known that $T \otimes S \in \mathscr{L}(\mathscr{H}_1 \otimes \mathscr{H}_2)$.

The definition of $T \otimes S$ is extended from these finite linear combinations of simple tensors to the whole space.

Since $T \otimes S = (T \otimes I)(I \otimes S) = (I \otimes S)(T \otimes I)$ and $T \otimes I = \oplus_{n=1}^{\infty} T$, it is clear that an operator T is an (m,C)-isometric operator with conjugation C if and only if $T \otimes I$ and $I \otimes T$ are (m,C)-isometric operators with conjugation C. If C and D are conjugations on $\mathscr{H}$, we define $C \otimes D$ on $\mathscr{H} \otimes \mathscr{H}$ by

$$(C \otimes D)(\sum_{j=1}^{n} \alpha_j x_j \otimes y_j) = \sum_{j=1}^{n} \overline{\alpha_j} Cx_j \otimes Dy_j.$$

Then $C \otimes D$ is a conjugation on $\mathscr{H} \otimes \mathscr{H}$ (see [4]).

COROLLARY 2.15. *If T is an (∞,C)-isometric operator and S is an (∞,D)-isometric operator, then $T \otimes S$ is an $(\infty, C \otimes D)$-isometric operator.*

Proof. It is clear that $T \otimes I$ is (∞,C)-isometric operator and $I \otimes S$ is an (∞,D)-isometric operator, respectively. Since $C \otimes D$ is a conjugation on $\mathscr{H} \otimes \mathscr{H}$ by [4] and $(T \otimes I, I \otimes S)$ is a commuting pair and satisfies

$$(I \otimes S)^*\big((C \otimes D)(T \otimes I)(C \otimes D)\big) = \big((C \otimes D)(T \otimes I)(C \otimes D)\big)(I \otimes S)^*,$$

it follows from Theorem 2.12 that $(T \otimes I)(I \otimes S) = T \otimes S$ is an $(\infty, C \otimes D)$-isometric operator.

PROPOSITION 2.16. *If $T \in \mathcal{L}(\mathcal{H})$ satisfies $T^*CTC = CTCT^*$, then the following statements hold.*

(i) *T is an (∞, C)-isometric operator if and only if T^* is an (∞, C)-isometric operator.*

(ii) *If T is an invertible and (∞, C)-isometric operator, then T^{-1} is an (∞, C)-isometric operator.*

Proof. (i) Suppose that T is an (∞, C)-isometric operator and $T^*CTC = CTCT^*$. Since $\Lambda_m(T^*) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{m-j}CT^{*m-j}C$, it follows that

$$\begin{aligned} C\Lambda_m(T^*)C &= \sum_{j=0}^{m}(-1)^j\binom{m}{j}CT^{m-j}CT^{*m-j} \\ &= \sum_{j=0}^{m}(-1)^j\binom{m}{j}T^{*m-j}CT^{m-j}C = \Lambda_m(T), \end{aligned}$$

and $\Lambda_m(T^*) = C\Lambda_m(T)C$. Therefore, we have

$$\begin{aligned} \limsup_{m\to\infty}\|\Lambda_m(T^*)\|^{\frac{1}{m}} &= \limsup_{m\to\infty}\|C\Lambda_m(T)C\|^{\frac{1}{m}} \\ &= \limsup_{m\to\infty}\|\Lambda_m(T)\|^{\frac{1}{m}} = 0. \end{aligned}$$

Hence T^* is an (∞, C)-isometric operator. The converse implication holds by a similar method.

(ii) Note for any $a, b \in \mathbb{C}$,

$$a^m(a^{-1}b^{-1} - 1)^m b^m = (1 - ab)^m = \sum_{j=0}^{m}(-1)^j\binom{m}{j}a^{m-j}b^{m-j}.$$

Take $a = T^*$ and $b = CTC$. Then we get $\Lambda_m(T) = (-1)^m(T^*)^m\Lambda_m(T^{-1})(CTC)^m$ and so $(-1)^m(T^*)^{-m}\Lambda_m(T) = \Lambda_m(T^{-1})(CTC)^m$. Therefore,

$$(-1)^m(T^*)^{-m}\Lambda_m(T)CT^{-m}C = \Lambda_m(T^{-1}).$$

Hence

$$\limsup_{m\to\infty}\|\Lambda_m(T^{-1})\|^{\frac{1}{m}} \leqslant \limsup_{m\to\infty}\|T^{*-1}\|\|\Lambda_m(T)\|^{\frac{1}{m}}\|T^{-1}\| = 0.$$

So T^{-1} is an (∞, C)-isometric operator. □

COROLLARY 2.17. *Under the same hypothesis as in Proposition 2.16, if T is an invertible and (∞, C)-isometric operator, then T^{-n} and T^{*-n} are (∞, C)-isometric operators for any $n \in \mathbb{N}$.*

Proof. The proof follows from Proposition 2.16 and Corollary 2.14. □

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REFERENCES

[1] J. AGLER AND M. STANKUS, *m-isometric transformations of Hilbert space I*, Int. Eq. Op. Th. **21** (1995), 383–429.

[2] M. CHŌ, S. OTA, K. TANAHASHI, AND M. UCHIYAMA, *Spectral properties of m-isometric operators*, Functional Anal. Appl. Com. **4:2** (2012), 33–39.

[3] M. CHŌ, E. KO, AND J. LEE, *On (m,C)-isometric operators*, Comp. Anal. Oper. Th. **10** (2016), 1679–1694.

[4] M. CHŌ, J. LEE AND H. MOTOYOSHI, *On $[m,C]$-isometric operators*, Filomat, **31**, 7 (2017), 2073–2080.

[5] M. CHŌ, C. GU AND W. Y. LEE, *Elementary properties of ∞-isometries on a Hilbert space*, Linear Alg. Appl. **511** (2016), 378–402.

[6] R. ESKANDARI AND F. MIRZAPOUR, *Hyperinvariant subspaces and quaisnilpotent operators*, Bull. Iranian Math. Soc. **41** (2015), 805–813.

[7] S. R. GARCIA, E. PRODAN, AND M. PUTINAR, *Mathematical and physical aspects of complex symmetric operators*, J. Phys. A: Math. Theory **47** (2014) 1–51.

[8] S. R. GARCIA AND M. PUTINAR, *Complex symmetric operators and applications*, Trans. Amer. Math. Soc. **358** (2006), 1285–1315.

[9] S. R. GARCIA, *Conjugation and Clark operators*, Contemp. Math., **393** (2006), 67–112.

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