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CHARACTERIZATION OF TRUNCATED TOEPLITZ OPERATORS BY CONJUGATIONS

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Abstract. Truncated Toeplitz operators are C-symmetric with respect to the canonical conjugation given on an appropriate model space. However, by considering only one conjugation one cannot characterize truncated Toeplitz operators. It will be proved, for some classes of inner functions and the model spaces connected with them, that if an operator on a model space is C–symmetric for a certain family of conjugations in the model space, then is has to be truncated Toeplitz. A characterization of classical Toeplitz operators is also presented in terms of conjugations.

1. Introduction

Let $\mathcal{H}$ denote a complex Hilbert space. Denote by $L(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. A *conjugation* is an antilinear involution $C\colon \mathcal{H} \to \mathcal{H}$ such that $\langle Cf, Cg\rangle = \langle g, f\rangle$ for all $f, g \in \mathcal{H}$. An operator $A \in L(\mathcal{H})$ is called C-*symmetric* if $CAC = A^*$.

Let $\mathbb{D}$ denote the open unit disk, let $\mathbb{T} = \partial\mathbb{D}$ denote the unit circle and let m be the normalized Lebesgue measure on $\mathbb{T}$. Denote by L^2 the space $L^2(\mathbb{T}, m)$ and by $L^\infty = L^\infty(\mathbb{T}, m)$. Recall that a classical *Toeplitz operator* T_φ with a symbol $\varphi \in L^\infty$ on the Hardy space H^2 is given by the formula

$$T_\varphi f = P(\varphi f) \text{ for } f \in H^2,$$

where $P\colon L^2 \to H^2$ is the orthogonal projection. Denote by $\mathcal{T}$ the set of all Toeplitz operators, i.e., $\mathcal{T} = \{T_\varphi : \varphi \in L^\infty\}$.

Let θ be a nonconstant inner function. Consider the so-called *model space* $K_\theta^2 = H^2 \ominus \theta H^2$ and the orthogonal projection $P_\theta\colon L^2 \to K_\theta^2$. A *truncated Toeplitz operator* A_φ^θ with a symbol $\varphi \in L^2$ is defined as

$$A_\varphi^\theta\colon D(A_\varphi^\theta) \subset K_\theta^2 \to K_\theta^2; \quad A_\varphi^\theta f = P_\theta(\varphi f)$$

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for $f \in D(A_\varphi^\theta) = \{f \in K_\theta^2 : \varphi f \in L^2\}$. Denote by $\mathscr{T}(\theta)$ the set of all bounded truncated Toeplitz operators on K_θ^2.

The conjugation C_θ defined for $f \in L^2$ by the formula

$$C_\theta f(z) = \theta(z)\overline{zf(z)}, \quad |z| = 1,$$

is a very useful tool in investigating Toeplitz operators. In fact, all truncated Toeplitz operators are C_θ-symmetric [7].

Truncated Toeplitz operators have been recently strongly investigated (see for instance [10, 1, 3, 4, 5, 6, 8]). However, usually only one (canonical) conjugation was involved in analysis on these operators. In this paper we suggest to consider a family of conjugations to study Toeplitz operators. In particular, we give a characterization of the classical Toeplitz operators as well as some special cases of truncated Toeplitz operators using conjugations.

It is easy to see that if $\theta = z^N$, then $K_\theta^2 = \mathbb{C}^N$. The natural conjugation $C_N = C_{z^N}$ in $\mathbb{C}^N$ can be expressed as $C_N(z_1, \ldots, z_N) = (\bar{z}_N, \ldots, \bar{z}_1)$. Note that a matrix $(a_{i,j})_{i,j=1,\ldots,N}$ is C_N-symmetric if and only if it is symmetric with respect to the second diagonal, i.e.,

$$a_{i,j} = a_{N-j+1,N-i+1} \quad \text{for} \quad i,j = 1,\ldots,N.$$

On the other hand, a finite matrix $(a_{i,j})_{i,j=1,\ldots,N}$ is a Toeplitz matrix if and only if it has constant diagonals, that is,

$$a_{i,j} = a_{k,l} \quad \text{if} \quad i-j = k-l.$$

Hence, as D. Sarason in [10] observed, each $N \times N$ Toeplitz matrix is C_N-symmetric but the reverse implication is true only if $N \leqslant 2$. However, one can notice that for a given matrix $(a_{i,j})_{i,j=1,\ldots,N}$, if the matrix is C_n-symmetric for every $n \leqslant N$, i.e.,

$$a_{i,j} = a_{n-j+1,n-i+1} \quad \text{for} \quad n \leqslant N \quad \text{and} \quad i,j = 1,\ldots n,$$

then the matrix $(a_{i,j})_{i,j=1,\ldots,N}$ has to be Toeplitz. Corollary 2.3 gives a precise proof of this fact. One can ask if a similar property can be obtained for other inner functions than $\theta = z^N$. Using known matrix descriptions [5, 6, 8] we obtained the positive answer: for a Blaschke product with a single zero in Section 3, for a finite Blaschke product with distinct zeros in Section 4 (the most demanding case), for an infinite Blaschke product with uniformly separated zeros in Section 5. For a general case we put the conjecture in Section 6. However, even for the simplest singular inner function $\theta(z) = \exp(\frac{z+1}{z-1})$ no similar description is known and to solve the conjecture probably a different approach is needed. In Section 2 we also give similar characterization of the classical Toeplitz operators on the Hardy space in terms of conjugations.

2. Characterization of Toeplitz operators by conjugations

Let α and θ be two nonconstant inner functions. We say that α divides θ ($\alpha \leqslant \theta$) if $\bar{\alpha}\theta$ is an inner function. It is easy to verify that $K^2_\alpha \subset K^2_\theta$ for every $\alpha \leqslant \theta$. It is known that truncated Toeplitz operators on K^2_θ are C_θ-symmetric but this property does not characterize them, i.e., there are C_θ-symmetric operators on K^2_θ, which are not truncated Toeplitz ([7], [10, Lemma 2.1, Corollary on p. 504]). Note however that A^θ_φ is C_α-symmetric for every $\alpha \leqslant \theta$. Namely:

LEMMA 2.1. *Let $A^\theta_\varphi\colon K^2_\theta \to K^2_\theta$ be a truncated Toeplitz operator. For every $\alpha \leqslant \theta$ the operator $P_\alpha A^\theta_{\varphi|K^2_\alpha}$ is C_α-symmetric.*

Proof. Note that $P_\alpha A^\theta_{\varphi|K^2_\alpha}$ belongs to $\mathscr{T}(\alpha)$. Actually, $P_\alpha A^\theta_{\varphi|K^2_\alpha} = A^\alpha_\varphi$, hence it is C_α-symmetric by [10, Lemma 2.1]. $\square$

A similar argument shows that if $A \in \mathscr{T}$, then $P_\alpha A_{|K^2_\alpha}$ is C_α-symmetric for all inner functions α. The latter can be used to characterize all Toeplitz operators on H^2:

THEOREM 2.2. *Let $A \in L(H^2)$. Then the following conditions are equivalent:*

(1) $A \in \mathscr{T}$;

(2) $C_\alpha A_\alpha C_\alpha = A^*_\alpha$ for all nonconstant inner functions α, where $A_\alpha = P_\alpha A_{|K^2_\alpha}$;

(3) $C_\alpha A_\alpha C_\alpha = A^*_\alpha$ for all $\alpha = z^n$, where $A_\alpha = P_\alpha A_{|K^2_\alpha}$.

Proof. The proof of the implication $(1) \Rightarrow (2)$ is similar to the proof of Lemma 2.1. Since $(2) \Rightarrow (3)$ is obvious, we will prove now that $(3) \Rightarrow (1)$.

The equivalent condition for a bounded operator on H^2 to be Toeplitz is that it has to annihilate all rank-two operators of the form

$$t = z^m \otimes z^r - z^{m+1} \otimes z^{r+1} \quad \text{with} \quad m, r \geqslant 0,$$

in the sense that $\operatorname{tr}(At) = 0$ (it follows form the well known Brown–Halmos characterization of Toeplitz operators given in [2]). Each such operator can be obtained from $1 \otimes z^k - z^l \otimes z^{k+l}$ or $z^k \otimes 1 - z^{k+l} \otimes z^l$, with $k, l \geqslant 0$. Hence our reasoning will be held only for such operators.

Fix $k, l \geqslant 0$ and let $\alpha = z^n$, $n = k + l + 1$. Since

$$C_\alpha z^k = z^{n-k-1} = z^l \quad \text{and} \quad C_\alpha 1 = z^{n-1} = z^{k+l},$$

the C_α-symmetry of A_α gives

$$\begin{aligned} \operatorname{tr}(A(1 \otimes z^k)) &= \langle A1, z^k \rangle = \langle A_\alpha 1, z^k \rangle = \langle C_\alpha z^k, C_\alpha A_\alpha 1 \rangle \\ &= \langle C_\alpha z^k, A^*_\alpha C_\alpha 1 \rangle = \langle z^l, A^*_\alpha z^{k+l} \rangle = \langle A_\alpha z^l, z^{k+l} \rangle = \operatorname{tr}(A(z^l \otimes z^{k+l})). \end{aligned}$$

Similarly,

$$\operatorname{tr}(A(z^k \otimes 1)) = \operatorname{tr}(A(z^{k+l} \otimes z^l)).$$

Therefore all operators of the form $1 \otimes z^k - z^l \otimes z^{k+l}$, $z^k \otimes 1 - z^{k+l} \otimes z^l$ for $k,l \geqslant 0$, are annihilated by A. Hence A is Toeplitz. □

From the previous proof we can obtain

COROLLARY 2.3. *Let $A \in L(K^2_{z^N})$, $N \in \mathbb{N}$. Then $A \in \mathscr{T}(z^N)$ if and only if for every $1 \leqslant n \leqslant N$ the operator A_n is C_{z^n}-symmetric, i.e., $C_{z^n} A_n C_{z^n} = A_n^*$, where $A_n = P_n A_{|K^2_{z^n}}$ and $P_n \colon K^2_{z^N} \to K^2_{z^n}$ is the orthogonal projection.*

3. The case of a Blaschke product with a single zero

Let α, θ be any nonconstant inner functions. We say that a unitary operator $U \colon K^2_\theta \to K^2_\alpha$ defines a spatial isomorphism between $\mathscr{T}(\theta)$ and $\mathscr{T}(\alpha)$ if $U\mathscr{T}(\theta)U^* = \mathscr{T}(\alpha)$, that is, $A \in \mathscr{T}(\theta)$ if and only if $UAU^* \in \mathscr{T}(\alpha)$. If such U exists, $\mathscr{T}(\theta)$ and $\mathscr{T}(\alpha)$ are said to be spatially isomorphic. The spatial isomorphism between spaces of truncated Toeplitz operators is discussed in [6, Chapter 13.7.4].

PROPOSITION 3.1. *Let α, θ be any nonconstant inner functions. Let $U \colon K^2_\theta \to K^2_\alpha$ be such that U defines a spatial isomorphism between $\mathscr{T}(\theta)$ and $\mathscr{T}(\alpha)$. Then $UC_\theta = C_\alpha U$.*

Proof. It is known [6, Chapter 13.7.4] that there are three basic types of unitary operators that define a spatial isomorphism between $\mathscr{T}(\theta)$ and $\mathscr{T}(\alpha)$. The requested intertwining property for one of those basic types is proved in [10, Lemma 13.1]. The proof for two other types is similar. Since every $U \colon K^2_\theta \to K^2_\alpha$ such that U defines a spatial isomorphism between $\mathscr{T}(\theta)$ and $\mathscr{T}(\alpha)$, is a composition of at most three of those basic types of operators, it follows that U also has this intertwining property. □

Let $a \in \mathbb{D}$ and $N \in \mathbb{N}$. Denote $b_a(z) = \frac{z-a}{1-\bar{a}z}$.

PROPOSITION 3.2. *Let $A \in L(K^2_{b_a^N})$. Then $A \in \mathscr{T}(b_a^N)$ if and only if for every $1 \leqslant n \leqslant N$ the operator A_n is $C_{b_a^n}$-symmetric, i.e., $C_{b_a^n} A_n C_{b_a^n} = A_n^*$, where $A_n = P_n A_{|K^2_{b_a^n}}$ and $P_n \colon K^2_{b_a^N} \to K^2_{b_a^n}$ is the orthogonal projection.*

Proof. The operator U_{b_a} given by

$$U_{b_a} f(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z} f \circ b_a(z)$$

defines a spatial isomorphism between $\mathbb{C}^n = K^2_{z^n}$ and $K^2_{b_a^n}$ for each $n = 1, \ldots, N$ (see [6, chapter 13.7.4(i)]). By Proposition 3.1, U_{b_a} intertwines the conjugations C_{z^n} and $C_{b_a^n}$. Application of Corollary 2.3 finishes the proof. □

4. The case of a finite Blaschke product with distinct zeros

Let B be a finite Blaschke product of degree N with distinct zeros $a_1,\dots,a_N$,

$$B(z) = e^{i\gamma}\prod_{j=1}^{N}\frac{z-a_j}{1-\bar{a}_j z}, \tag{4.1}$$

where $\gamma\in\mathbb{R}$. As usual, for $w\in\mathbb{D}$ by

$$k_w^B(z) = \frac{1-\overline{B(w)}B(z)}{1-\bar{w}z}$$

we denote the reproducing kernel for K_B^2, that is,

$$f(w) = \langle f, k_w^B\rangle$$

for $f\in K_B^2$. Note that for $j=1,\dots,N$ we have

$$k_j(z) := k_{a_j}^B(z) = \frac{1}{1-\bar{a}_j z}. \tag{4.2}$$

As it was observed in [5], the model space K_B^2 is N-dimensional and the functions $k_1,\dots,k_N$ form a (non–orthonormal) basis for K_B^2.

A simple computation gives the following.

LEMMA 4.1. ([5], p. 5)

(1) $(C_Bk_j)(z) = \frac{B(z)}{z-a_j}$ *for* $j=1,\dots,N$.

(2) $\langle C_Bk_j, k_i\rangle = \begin{cases} 0 & \text{for } i\neq j,\\ B'(a_j) & \text{for } i=j.\end{cases}$

(3) $\langle k_j, k_i\rangle = \frac{1}{1-\bar{a}_j a_i}$.

LEMMA 4.2. *Let B be a finite Blaschke product of degree N with distinct zeros $a_1,\dots,a_N$. Let C_B be the conjugation in K_B^2 given by $C_Bf(z) = B(z)\overline{zf(z)}$ for $f\in K_B^2$. Assume that an operator $A\in L(K_B^2)$ has a matrix representation $(b_{i,j})_{i,j=1,\dots,N}$ with respect to the basis $\{k_1,\dots,k_N\}$. Then the following are equivalent:*

(1) *A is C_B-symmetric;*

(2) $\langle Ak_i, C_Bk_j\rangle = \langle Ak_j, C_Bk_i\rangle$ *for all* $i,j=1,\dots,N$;

(3) $\overline{B'(a_j)}b_{j,i} = \overline{B'(a_i)}b_{i,j}$ *for all* $i,j=1,\dots,N$.

Proof. The implication $(1) \Rightarrow (2)$ follows from

$$\langle Ak_i, C_B k_j\rangle = \langle C_B^2 k_j, C_B A k_i\rangle = \langle k_j, A^* C_B k_i\rangle = \langle Ak_j, C_B k_i\rangle.$$

The reverse implication can be proved similarly.

To prove that $(2) \Leftrightarrow (3)$ note that $Ak_i = \sum_{m=1}^{N} b_{m,i} k_m$. Hence, by Lemma 4.1(2),

$$\langle Ak_i, C_B k_j\rangle = \sum_{m=1}^{N} b_{m,i} \langle k_m, C_B k_j\rangle = \overline{B'(a_j)} b_{j,i}.$$

Analogously,

$$\langle Ak_j, C_B k_i\rangle = \overline{B'(a_i)} b_{i,j}. \quad \square$$

Let $1 \leqslant n \leqslant N$. Denote by B_n the finite Blaschke product with n distinct zeros $a_1, \ldots, a_n$,

$$B_n(z) = \prod_{j=1}^{n} \frac{z - a_j}{1 - \bar{a}_j z}, \tag{4.3}$$

and by $C_n = C_{B_n}$ the conjugation in $K_{B_n}^2$ given by

$$(C_n f)(z) = B_n(z)\overline{z f(z)}, \quad |z| = 1.$$

THEOREM 4.3. *Let B be a finite Blaschke product of degree N with distinct zeros $a_1, \ldots, a_N$. Denote by B_n the Blaschke product of degree n with zeros $a_1, \ldots, a_n$ and by P_n the orthogonal projection from K_B^2 onto $K_{B_n}^2$ for $n = 1, \ldots, N$. Let $A \in L(K_B^2)$. The following conditions are equivalent:*

(1) $A \in \mathscr{T}(B)$;

(2) for every Blaschke product B_σ dividing B the operator $A_\sigma = P_{B_\sigma} A_{|K_{B_\sigma}^2}$ is C_{B_σ}-symmetric;

(3) for every $n = 1, \ldots, N$ the operator $A_n = P_n A_{|K_{B_n}^2}$ is C_n-symmetric.

To give the proof of Theorem 4.3 we need two technical lemmas. Firstly, let us observe by (4.2) that $k_{a_j}^{B_n} = k_{a_j}^{B} = k_j$ for $1 \leqslant n \leqslant N$, $j = 1, \ldots, n$. Hence $\{k_1, \ldots, k_n\}$ is a basis for $K_{B_n}^2 \subset K_B^2$.

LEMMA 4.4. *For $1 \leqslant m, n \leqslant N$ the following holds:*

(1) $$\langle C_n k_j, k_m\rangle = \begin{cases} 0 & \text{for} \quad m \leqslant n, m \neq j, \\ B_n'(a_j) & \text{for} \quad m \leqslant n, m = j, \\ \frac{B_n(a_m)}{a_m - a_j} & \text{for} \quad m > n, \end{cases} \qquad \text{for } j = 1, \ldots, n;$$

(2) $P_n k_m = \sum_{j=1}^{n} \frac{\overline{B_n(a_m)}}{\overline{B_n'(a_j)}(\bar{a}_m - \bar{a}_j)} k_j$ *for $n < m$;*

(3) $\frac{B_{n-1}(a_n)}{B_n'(a_n)} = 1-|a_n|^2$ *for* $n>1$*;*

(4) $\frac{B_{n-1}'(a_j)}{B_n'(a_j)} = \frac{1-\bar{a}_n a_j}{a_j-a_n}$ *for* $n>1,\ j=1,\dots,n-1$.

Proof. To show (1) note that $C_n k_j \in K^2_{B_n} \subset K^2_B$ for $1 \leqslant n \leqslant N,\ j=1,\dots,n$, and that

$$(C_n k_j)(z) = \frac{B_n(z)}{z-a_j}$$

by Lemma 4.1(1). If $m>n$, then the reproducing property of k_m yields

$$\langle C_n k_j, k_m\rangle = (C_n k_j)(a_m) = \frac{B_n(a_m)}{a_m-a_j}.$$

On the other hand, if $m \leqslant n$, then it follows from Lemma 4.1(2) that

$$\langle C_n k_j, k_m\rangle = \begin{cases} 0 & \text{for} \quad m \neq j, \\ B_n'(a_j) & \text{for} \quad m = j. \end{cases}$$

To show (2) assume that $m>n$ and $P_n k_m = \sum\limits_{l=1}^{n} d_l k_l$. Then, by part (1), for $j = 1,\dots,n$,

$$\frac{B_n(a_m)}{a_m-a_j} = \langle C_n k_j, k_m\rangle = \langle C_n k_j, P_n k_m\rangle = \sum_{l=1}^{n} \bar{d}_l \langle C_n k_j, k_l\rangle = B_n'(a_j)\bar{d}_j.$$

Hence

$$d_j = \frac{\overline{B_n(a_m)}}{\overline{B_n'(a_j)}(\bar{a}_m-\bar{a}_j)},$$

which proves (2). The statements (3) and (4) follow directly from

$$B_n'(z) = B_{n-1}'(z)\frac{z-a_n}{1-\bar{a}_n z} + B_{n-1}(z)\frac{1-|a_n|^2}{(1-\bar{a}_n z)^2}. \quad \square$$

LEMMA 4.5. *Let* $A \in L(K^2_{B_n})$ *have a matrix representation* $(b^{(n)}_{i,j})_{i,j=1,\dots,n}$ *with respect to the basis* $\{k_1,\dots,k_n\}$*. Then* $A_{n-1} = P_{n-1}A_{|K^2_{B_{n-1}}}$ *has a matrix representation* $(b^{(n-1)}_{i,j})_{i,j=1,\dots,n-1}$,

$$b^{(n-1)}_{i,j} = b^{(n)}_{i,j} + \frac{\overline{B_{n-1}(a_n)}\, b^{(n)}_{n,j}}{\overline{B_{n-1}'(a_i)}(\bar{a}_n-\bar{a}_i)},$$

with respect to the basis $\{k_1,\dots,k_{n-1}\}$.

Proof. Note that by Lemma 4.4(2),

$$P_{n-1}k_n = \sum_{m=1}^{n-1} \frac{\overline{B_{n-1}(a_n)}}{\overline{B_{n-1}'(a_m)}(\bar{a}_n-\bar{a}_m)}\, k_m.$$

Hence, for $j = 1,\dots,n-1$, we have

$$P_{n-1}(Ak_j) = P_{n-1}\left(\sum_{m=1}^{n} b_{m,j}^{(n)} k_m\right) = P_{n-1}\left(\sum_{m=1}^{n-1} b_{m,j}^{(n)} k_m\right) + P_{n-1} b_{n,j}^{(n)} k_n$$
$$= \sum_{m=1}^{n-1}\left(b_{m,j}^{(n)} + \frac{\overline{B_{n-1}(a_n)} b_{n,j}^{(n)}}{\overline{B'_{n-1}(a_m)}(\bar a_n - \bar a_m)}\right) k_m.$$

Since

$$b_{i,j}^{(n-1)} = \frac{1}{\overline{B'_{n-1}(a_i)}} \langle P_{n-1}(Ak_j), C_{n-1}k_i\rangle, \quad 1 \leqslant i,j \leqslant n-1,$$

we get

$$b_{i,j}^{(n-1)} = \frac{1}{\overline{B'_{n-1}(a_i)}} \sum_{m=1}^{n-1}\left(b_{m,j}^{(n)} + \frac{\overline{B_{n-1}(a_n)} b_{n,j}^{(n)}}{\overline{B'_{n-1}(a_i)}(\bar a_n - \bar a_m)}\right)\langle k_m, C_{n-1}k_i\rangle$$
$$= b_{i,j}^{(n)} + \frac{\overline{B_{n-1}(a_n)} b_{n,j}^{(n)}}{\overline{B'_{n-1}(a_i)}(\bar a_n - \bar a_i)}$$

by Lemma 4.4(1). $\square$

Proof of Theorem 4.3. Since multiplying B by a constant of modulus 1 does not change K_B^2, we can assume without any loss of generality that B is given by (4.1) with $\gamma = 0$, that is, $B = B_N$.

The implication $(1) \Rightarrow (2)$ follows from Lemma 2.1 and the implication $(2) \Rightarrow (3)$ is obvious. We only need to prove the implication $(3) \Rightarrow (1)$. This will be proved by induction. Note firstly that it is true for $N = 2$ by [10, p. 505].

Assume now that the assertion is true for $n-1 < N$, which means that $A_{n-1} = P_{n-1}A_{|K^2_{B_{n-1}}}$ is Toeplitz and has a matrix representation $(b_{i,j}^{(n-1)})_{i,j=1,\dots,n-1}$ with respect to the basis $\{k_1,\dots,k_{n-1}\}$ satisfying

$$b_{i,j}^{(n-1)} = \frac{\overline{B'_{n-1}(a_1)}}{\overline{B'_{n-1}(a_i)}}\left(\frac{b_{1,i}^{(n-1)}(\bar a_1 - \bar a_i) + b_{1,j}^{(n-1)}(\bar a_j - \bar a_1)}{\bar a_j - \bar a_i}\right), \tag{4.4}$$

for $1 \leqslant i,j \leqslant n-1$, $i \neq j$, by [5, Theorem 1.4]. Assume also that $A \in L(K^2_{B_n})$ is C_n-symmetric and has a matrix representation $(b_{i,j}^{(n)})_{i,j=1,\dots,n}$ with respect to the basis $\{k_1,\dots,k_n\}$. We will show that A is Toeplitz, i.e., $b_{i,j}^{(n)}$ satisfies [5, Theorem 1.4]:

$$b_{i,j}^{(n)} = \frac{\overline{B'_n(a_1)}}{\overline{B'_n(a_i)}}\left(\frac{b_{1,i}^{(n)}(\bar a_1 - \bar a_i) + b_{1,j}^{(n)}(\bar a_j - \bar a_1)}{\bar a_j - \bar a_i}\right),$$

for $1 \leqslant i,j \leqslant n$, $i \neq j$.

Since A_{n-1} is C_{n-1}-symmetric, for $i,j=1,\dots,n-1$ we have, by Lemma 4.2 and Lemma 4.5,

$$b_{j,i}^{(n-1)} = \frac{\overline{B'_{n-1}(a_i)}}{B'_{n-1}(a_j)} b_{i,j}^{(n-1)} = \frac{\overline{B'_{n-1}(a_i)}}{B'_{n-1}(a_j)} \left(b_{i,j}^{(n)} + \frac{\overline{B_{n-1}(a_n)} b_{n,j}^{(n)}}{B'_{n-1}(a_i)(\bar{a}_n - \bar{a}_i)} \right). \tag{4.5}$$

On the other hand, by Lemma 4.5 and using the C_n-symmetry of A,

$$\begin{aligned} b_{j,i}^{(n-1)} &= b_{j,i}^{(n)} + \frac{\overline{B_{n-1}(a_n)}}{B'_{n-1}(a_j)} \frac{b_{n,i}^{(n)}}{\bar{a}_n - \bar{a}_j} \\ &= \frac{\overline{B'_n(a_i)}}{B'_n(a_j)} b_{i,j}^{(n)} + \frac{\overline{B'_n(a_i)}}{B'_n(a_n)} \frac{\overline{B_{n-1}(a_n)}}{B'_{n-1}(a_j)} \frac{b_{i,n}^{(n)}}{\bar{a}_n - \bar{a}_j}. \end{aligned} \tag{4.6}$$

Comparing (4.5) with (4.6) and putting $i=1$ we obtain

$$\frac{\overline{B_{n-1}(a_n)}}{B'_{n-1}(a_j)} \frac{b_{n,j}^{(n)}}{\bar{a}_n - \bar{a}_1} = \frac{\overline{B'_n(a_1)}}{B'_n(a_n)} \frac{\overline{B_{n-1}(a_n)}}{B'_{n-1}(a_j)} \frac{b_{1,n}^{(n)}}{\bar{a}_n - \bar{a}_j} + \left(\frac{\overline{B'_n(a_1)}}{B'_n(a_j)} - \frac{\overline{B'_{n-1}(a_1)}}{B'_{n-1}(a_j)} \right) b_{1,j}^{(n)}.$$

Hence

$$b_{n,j}^{(n)} = \frac{\overline{B'_n(a_1)}}{B'_n(a_n)} \frac{b_{1,n}^{(n)}(\bar{a}_n - \bar{a}_1)}{\bar{a}_n - \bar{a}_j} + \left(\frac{\overline{B'_n(a_1)}}{\overline{B_{n-1}(a_n)}} \frac{B'_{n-1}(a_j)}{B'_n(a_j)} - \frac{\overline{B'_{n-1}(a_1)}}{\overline{B_{n-1}(a_n)}} \right) b_{1,j}^{(n)}(\bar{a}_n - \bar{a}_1). \tag{4.7}$$

Using Lemma 4.4 we can simplify

$$\begin{aligned} &\frac{\overline{B'_n(a_1)}}{\overline{B_{n-1}(a_n)}} \frac{B'_{n-1}(a_j)}{B'_n(a_j)} - \frac{\overline{B'_{n-1}(a_1)}}{\overline{B_{n-1}(a_n)}} \\ &\qquad = \frac{\overline{B'_n(a_1)}}{\overline{B'_n(a_n)}} \frac{1}{1-|a_n|^2} \left(\frac{1-a_n\bar{a}_j}{\bar{a}_j - \bar{a}_n} + \frac{1-a_n\bar{a}_1}{\bar{a}_n - \bar{a}_1} \right) \\ &\qquad = \frac{\overline{B'_n(a_1)}}{\overline{B'_n(a_n)}} \frac{(\bar{a}_j - \bar{a}_1)}{(\bar{a}_j - \bar{a}_n)(\bar{a}_n - \bar{a}_1)}, \end{aligned}$$

which together with (4.7) gives

$$b_{n,j}^{(n)} = \frac{\overline{B'_n(a_1)}}{\overline{B'_n(a_n)}} \left(\frac{b_{1,n}^{(n)}(\bar{a}_1 - \bar{a}_n) + b_{1,j}^{(n)}(\bar{a}_j - \bar{a}_1)}{\bar{a}_j - \bar{a}_n} \right) \tag{4.8}$$

for $1 \leqslant j \leqslant n-1$. From (4.8), the C_n-symmetry of A_n and Lemma 4.2 we also get

$$b_{i,n}^{(n)} = \frac{\overline{B'_n(a_n)}}{B'_n(a_i)} b_{n,i}^{(n)} = \frac{\overline{B'_n(a_1)}}{B'_n(a_i)} \left(\frac{b_{1,i}^{(n)}(\bar{a}_1 - \bar{a}_i) + b_{1,n}^{(n)}(\bar{a}_n - \bar{a}_1)}{\bar{a}_n - \bar{a}_i} \right) \tag{4.9}$$

for $1 \leqslant i \leqslant n-1$. By Lemma 4.2 and Lemma 4.5, we have for $i = 1, \dots, n-1$,

$$b_{1,i}^{(n-1)} = \frac{\overline{B'_{n-1}(a_i)}}{\overline{B'_{n-1}(a_1)}} b_{i,1}^{(n-1)} = \frac{\overline{B'_{n-1}(a_i)}}{\overline{B'_{n-1}(a_1)}} \left(b_{i,1}^{(n)} + \frac{\overline{B_{n-1}(a_n)}}{\overline{B'_{n-1}(a_i)}} \frac{b_{n,1}^{(n)}}{\bar{a}_n - \bar{a}_i} \right).$$

Using Lemma 4.2 again we obtain

$$b_{1,i}^{(n-1)} = \frac{\overline{B'_n(a_1)}}{\overline{B'_n(a_i)}} \frac{\overline{B'_{n-1}(a_i)}}{\overline{B'_{n-1}(a_1)}} b_{1,i}^{(n)} + \frac{\overline{B'_n(a_1)}}{\overline{B'_n(a_n)}} \frac{\overline{B_{n-1}(a_n)}}{\overline{B'_{n-1}(a_1)}} \frac{b_{1,n}^{(n)}}{\bar{a}_n - \bar{a}_i}. \tag{4.10}$$

Now applying Lemma 4.5 to the left–hand side of (4.4), and formula (4.10) to the right–hand side of (4.4) we can calculate for all $i, j = 1, \dots, n-1$,

$$\begin{aligned} b_{i,j}^{(n)} &+ \frac{\overline{B_{n-1}(a_n)}}{\overline{B'_{n-1}(a_i)}} \frac{b_{n,j}^{(n)}}{\bar{a}_n - \bar{a}_i} \\ &= \frac{\overline{B'_n(a_1)}}{\overline{B'_n(a_i)}} \frac{\bar{a}_1 - \bar{a}_i}{\bar{a}_j - \bar{a}_i} b_{1,i}^{(n)} + \frac{\overline{B'_n(a_1)}}{\overline{B'_n(a_n)}} \frac{\overline{B_{n-1}(a_n)}}{\overline{B'_{n-1}(a_i)}} \frac{(\bar{a}_1 - \bar{a}_i) b_{1,n}^{(n)}}{(\bar{a}_n - \bar{a}_i)(\bar{a}_j - \bar{a}_i)} \\ &\quad + \frac{\overline{B'_n(a_1)}}{\overline{B'_n(a_j)}} \frac{\overline{B'_{n-1}(a_j)}}{\overline{B'_{n-1}(a_i)}} \frac{\bar{a}_j - \bar{a}_1}{\bar{a}_j - \bar{a}_i} b_{1,j}^{(n)} + \frac{\overline{B'_n(a_1)}}{\overline{B'_n(a_n)}} \frac{\overline{B_{n-1}(a_n)}}{\overline{B'_{n-1}(a_i)}} \frac{\bar{a}_j - \bar{a}_1}{(\bar{a}_n - \bar{a}_j)(\bar{a}_j - \bar{a}_i)} b_{1,n}^{(n)}. \end{aligned} \tag{4.11}$$

Note that

$$\begin{aligned} &\frac{\overline{B'_n(a_1)}}{\overline{B'_n(a_j)}} \frac{\overline{B'_{n-1}(a_j)}}{\overline{B'_{n-1}(a_i)}} \frac{\bar{a}_j - \bar{a}_1}{\bar{a}_j - \bar{a}_i} b_{1,j}^{(n)} \\ &\qquad = \frac{\overline{B'_n(a_1)}}{\overline{B'_n(a_i)}} \frac{\bar{a}_j - \bar{a}_1}{\bar{a}_j - \bar{a}_i} b_{1,j}^{(n)} + \frac{\overline{B_{n-1}(a_n)}}{\overline{B'_{n-1}(a_i)}} \frac{\overline{B'_n(a_1)}}{\overline{B'_n(a_n)}} \frac{\bar{a}_j - \bar{a}_1}{(\bar{a}_n - \bar{a}_i)(\bar{a}_j - \bar{a}_n)} b_{1,j}^{(n)} \end{aligned}$$

by Lemma 4.4. Moreover,

$$\frac{1}{\bar{a}_j - \bar{a}_i} \left(\frac{\bar{a}_1 - \bar{a}_i}{\bar{a}_n - \bar{a}_i} + \frac{\bar{a}_j - \bar{a}_1}{\bar{a}_n - \bar{a}_j} \right) = \frac{\bar{a}_n - \bar{a}_1}{(\bar{a}_n - \bar{a}_i)(\bar{a}_n - \bar{a}_j)}. \tag{4.12}$$

Hence, (4.11) and (4.12) give

$$\begin{aligned} b_{i,j}^{(n)} + \frac{\overline{B_{n-1}(a_n)}}{\overline{B'_{n-1}(a_i)}} \frac{b_{n,j}^{(n)}}{\bar{a}_n - \bar{a}_i} &= \frac{\overline{B'_n(a_1)}}{\overline{B'_n(a_i)}} \frac{b_{1,i}^{(n)}(\bar{a}_1 - \bar{a}_i) + b_{1,j}^{(n)}(\bar{a}_j - \bar{a}_1)}{\bar{a}_j - \bar{a}_i} \\ &+ \frac{\overline{B_{n-1}(a_n)}}{\overline{B'_{n-1}(a_i)}} \frac{1}{\bar{a}_n - \bar{a}_i} \frac{\overline{B'_n(a_1)}}{\overline{B'_n(a_n)}} \frac{b_{1,n}^{(n)}(\bar{a}_1 - \bar{a}_n) + b_{1,j}^{(n)}(\bar{a}_j - \bar{a}_1)}{\bar{a}_j - \bar{a}_n}. \end{aligned}$$

Taking into account (4.8) and (4.9), the above equation implies that

$$b_{i,j}^{(n)} = \frac{\overline{B'_n(a_1)}}{\overline{B'_n(a_i)}} \left(\frac{b_{1,i}^{(n)}(\bar{a}_1 - \bar{a}_i) + b_{1,j}^{(n)}(\bar{a}_j - \bar{a}_1)}{\bar{a}_j - \bar{a}_i} \right)$$

for all $1 \leqslant i, j \leqslant n$, $i \neq j$, which completes the proof. $\square$

5. An infinite Blaschke product with uniformly separated zeros

Let B be an infinite Blaschke product,

$$B(z) = e^{i\gamma} \prod_{j=1}^{\infty} \frac{\bar{a}_j}{|a_j|} \frac{a_j - z}{1 - \bar{a}_j z}, \quad \gamma \in \mathbb{R}, \tag{5.1}$$

(if $a_j = 0$, then $\bar{a}_j/|a_j|$ is interpreted as -1) with uniformly separated zeros $a_1, a_2, \ldots$, i.e.,

$$\inf_n \prod_{j \neq n} \left| \frac{a_j - a_n}{1 - \bar{a}_j a_n} \right| \geqslant \delta \tag{5.2}$$

for some $\delta > 0$. In particular, the zeros $\{a_j\}_{j=1}^{\infty}$ are distinct. As before, B_n, $n \in \mathbb{N}$, denotes the finite Blaschke product with zeros $a_1, \ldots, a_n$, given by (4.3).

THEOREM 5.1. *Let B be an infinite Blaschke product with uniformly separated zeros $\{a_j\}_{j=1}^{\infty}$. Denote by B_n the Blaschke product of degree n with distinct zeros $\{a_1, \ldots, a_n\}$ and by P_n the orthogonal projection form K_B^2 onto $K_{B_n}^2$ for $n \in \mathbb{N}$. Let $A \in L(K_B^2)$. The following conditions are equivalent:*

(1) $A \in \mathscr{T}(B)$;

(2) for every Blaschke product B_σ dividing B the operator $A_\sigma = P_{B_\sigma} A_{|K_{B_\sigma}^2}$ is C_{B_σ}-symmetric;

(3) for every $n \in \mathbb{N}$ the operator $A_n = P_n A_{|K_{B_n}^2}$ is C_n-symmetric.

Again, before we give the proof some preparations are necessary. Clearly, $K_{B_n}^2 \subset K_B^2$ for all $n \in \mathbb{N}$ and $k_{a_j}^B = k_j$ for all $j \in \mathbb{N}$. Condition (5.2) implies that the reproducing kernels k_j, $j \in \mathbb{N}$, form a basis for K_B^2 (for more details see [6, Chapter 12], [7] or [9]). In particular, every $f \in K_B^2$ can be written as

$$f = \sum_{j=1}^{\infty} \frac{\langle f, C_B k_j \rangle}{\overline{B'(a_j)}} k_j,$$

where the series converges in the norm.

LEMMA 5.2. *Let $A \in L(K_B^2)$ have a matrix representation $(b_{i,j})_{i,j=1}^{\infty}$ with respect to the basis $\{k_i \colon i \in \mathbb{N}\}$. Then $A_n = P_n A_{|K_{B_n}^2}$ has a matrix representation $(b_{i,j}^{(n)})_{i,j=1,\ldots,n}$,*

$$b_{i,j}^{(n)} = b_{i,j} + \sum_{m=n+1}^{\infty} \frac{\overline{B_n(a_m)} b_{m,j}}{\overline{B_n'(a_i)} (\bar{a}_m - \bar{a}_i)},$$

with respect to the basis $\{k_1, \ldots, k_n\}$.

Proof. Let $n \in \mathbb{N}$ and $1 \leqslant i, j \leqslant n$. Since

$$A_n k_j = \sum_{m=1}^{n} b_{m,j}^{(n)} k_m,$$

Lemma 4.4(1) gives

$$b_{i,j}^{(n)} = \frac{1}{B_n'(a_i)} \langle A_n k_j, C_n k_i \rangle.$$

Since

$$A k_j = \sum_{m=1}^{\infty} b_{m,j} k_m,$$

and the series converges in norm, we get

$$\begin{aligned} b_{i,j}^{(n)} &= \frac{1}{B_n'(a_i)} \langle A k_j, C_n k_i \rangle = \frac{1}{B_n'(a_i)} \sum_{m=1}^{\infty} b_{m,j} \langle k_m, C_n k_i \rangle \\ &= b_{i,j} + \frac{1}{B_n'(a_i)} \sum_{m=n+1}^{\infty} \frac{\overline{B_n(a_m)}}{\bar{a}_m - \bar{a}_i} b_{m,j} \end{aligned}$$

by Lemma 4.4(1). $\square$

COROLLARY 5.3. *For all $i, j \in \mathbb{N}$,*

$$b_{i,j} = \lim_{n \to \infty} b_{i,j}^{(n)}.$$

Proof. It is known that the infinite Blaschke product B converges uniformly on compact subsets of $\mathbb{D}$. It follows that if

$$\lambda_n = (-1)^n \prod_{j=1}^{n} \frac{\bar{a}_j}{|a_j|}, \quad n \in \mathbb{N},$$

then $\lambda_n B_n \to B$ and $\lambda_n B_n' \to B'$ as $n \to \infty$ (uniformly on compact subsets of $\mathbb{D}$). In particular,

$$\lambda_n B_n'(a_i) \to B'(a_i) \quad \text{as} \quad n \to \infty$$

for each $i \in \mathbb{N}$. Fix $i, j \in \mathbb{N}$. Let $n \geqslant \max\{i, j\}$ and write

$$A k_j = \sum_{m=1}^{n} b_{m,j} k_m + r_n, \qquad \text{where} \qquad r_n = \sum_{m=n+1}^{\infty} b_{m,j} k_m.$$

As in the proof of Lemma 5.2,

$$b_{i,j}^{(n)} = b_{i,j} + \tfrac{1}{B_n'(a_i)} \langle r_n, C_n k_i \rangle = b_{i,j} + \tfrac{1}{\lambda_n B_n'(a_i)} \langle r_n, \lambda_n C_n k_i \rangle,$$

where the last equality follows form the fact that $\lambda_n \in \mathbb{T}$. Since r_n tends to zero in the norm, the sequence $(\lambda_n C_n k_i)_{n\geqslant i}$ is bounded and $\lambda_n B_n'(a_i) \to B'(a_i)$, we get

$$\lim_{n\to\infty} b_{i,j}^{(n)} = b_{i,j} + \lim_{n\to\infty}\left(\frac{1}{\lambda_n B_n'(a_i)}\langle r_n, \lambda_n C_n k_i\rangle\right) = b_{i,j}. \quad \square$$

Proof of Theorem 5.1. As in the proof of Theorem 4.3, without loss of generality, assume that B is given by (5.1) with $\gamma = 0$. The implication $(1) \Rightarrow (2)$ follows from Lemma 2.1 and the implication $(2) \Rightarrow (3)$ is obvious. We only need to prove $(3) \Rightarrow (1)$.

Let $A \in L(K_B^2)$ and assume that $A_n = P_n A_{|K_{B_n}^2}$ is C_n-symmetric for every $n \in \mathbb{N}$. By [8, Remark 2.4], to prove that $A \in \mathscr{T}(B)$ it is enough to show that

$$b_{i,j} = \frac{\overline{B'(a_1)}}{\overline{B'(a_i)}}\left(\frac{b_{1,i}(\bar a_1 - \bar a_i) + b_{1,j}(\bar a_j - \bar a_1)}{\bar a_j - \bar a_i}\right) \tag{5.3}$$

for all $i \neq j$, where $(b_{i,j})_{i,j=1}^{\infty}$ is the matrix representation of A with respect to the basis $\{k_i \colon i \in \mathbb{N}\}$. Fix $i, j \in \mathbb{N}$, $i \neq j$, and take an arbitrary $N \geqslant \max\{i,j\}$. By (3), $P_n A_{N|K_{B_n}^2} = A_n$ is C_n-symmetric for all $n = 1,\dots,N$. Hence Theorem 4.3 implies that $A_N \in \mathscr{T}(B_N)$. By [5, Theorem 4.1],

$$b_{i,j}^{(N)} = \frac{\overline{B_N'(a_1)}}{\overline{B_N'(a_i)}}\left(\frac{b_{1,i}^{(N)}(\bar a_1 - \bar a_i) + b_{1,j}^{(N)}(\bar a_j - \bar a_1)}{\bar a_j - \bar a_i}\right), \tag{5.4}$$

where $(b_{i,j}^{(N)})_{i,j=1,\dots,N}$ is the matrix representation of the operator A_N with respect to the basis $\{k_1,\dots,k_N\}$. Taking the limit in (5.4) as N tends to infinity we get (5.3) because $b_{i,j}^{(N)} \to b_{i,j}$ and $B_N'(a_i) \to B'(a_i)$ by Corollary 5.3 and its proof. $\square$

6. Conjecture

Theorems 2.2, 4.3, 5.1 and Proposition 3.2 suggest that the following conjecture can be true:

CONJECTURE 6.1. *Let θ be a nonconstant inner function, and let $A \in L(K_\theta^2)$. Then $A \in \mathscr{T}(\theta)$ if and only if for every nonconstant inner function α dividing θ the operator $A_\alpha = P_\alpha A_{|K_\alpha^2}$ is C_α-symmetric.*

The following example supports the conjecture.

EXAMPLE 6.2. Consider

$$B(z) = z^2\frac{w - z}{1 - \overline{w}z}, \quad \text{where} \quad w \neq 0.$$

Then the space K_B^2 has dimension 3 and the set $\{1, z, \frac{z^2}{\|k_w\|}k_w\}$, $k_w(z) = (1-\overline{w}z)^{-1}$, is an orthonormal basis for K_B^2.

We first describe the operators form $\mathscr{T}(B)$ in terms of their matrix representations with respect to the basis $\{1, z, \frac{z^2}{\|k_w\|}k_w\}$. Let A^B_φ, $\varphi \in L^2$, be an operator from $\mathscr{T}(B)$, and let $M_{A^B_\varphi} = (b_{i,j})$ be its matrix representation. By [10, Theorem 3.1] we can assume that $\varphi \in \overline{BH^2} + BH^2$, namely, that

$$\varphi = c_{-2}\frac{\overline{z}^2}{\|k_w\|}\overline{k}_w + c_{-1}\overline{z} + c_0 + c_1 z + c_2\frac{z^2}{\|k_w\|}k_w.$$

It is now a matter of a simple computation to see that the matrix $M_{A^B_\varphi} = (b_{i,j})$ is given by

$$\begin{pmatrix} c_0 & c_{-1} & c_{-2} \\ c_1 & c_0 & c_{-2}\overline{w} + \frac{c_{-1}}{\|k_w\|} \\ c_2 & \frac{c_1}{\|k_w\|} + c_2 w & c_{-2}\overline{w}^2\|k_w\| + c_{-1}\overline{w} + c_0 + c_1 w + c_2 w^2\|k_w\| \end{pmatrix}.$$

From this, the elements $b_{i,j}$ are described by the following system of equations

$$b_{2,2} = b_{1,1} \tag{6.1}$$
$$b_{2,3} = \overline{w}b_{1,3} + \|k_w\|^{-1}b_{1,2} \tag{6.2}$$
$$b_{3,2} = \|k_w\|^{-1}b_{2,1} + wb_{3,1} \tag{6.3}$$
$$\begin{aligned} b_{3,3} &= b_{1,1} + \overline{w}\|k_w\|b_{2,3} + w\|k_w\|b_{3,2} \\ &= b_{1,1} + \overline{w}^2\|k_w\|b_{1,3} + \overline{w}b_{1,2} + wb_{2,1} + w^2\|k_w\|b_{3,1}. \end{aligned} \tag{6.4}$$

Clearly, each 3×3 matrix $(b_{i,j})$ satisfying (6.1)–(6.4) is determined by five elements (the first row and the first column) and the space $\mathscr{M}_B$ of all such matrices has dimension 5. As matrices representing operators from $\mathscr{T}(B)$ have to belong to $\mathscr{M}_B$ and the dimension of $\mathscr{T}(B)$ in this case is also 5, we conclude that a linear operator A form K^2_B into K^2_B belongs to $\mathscr{T}(B)$ if and only if its matrix representation with respect to $\{1, z, \frac{z^2}{\|k_w\|}k_w\}$ satisfies (6.1)–(6.4).

Now let A be an operator from K^2_B into K^2_B such that for every $B_\sigma \leqslant B$ the compression $A_\sigma = P_{B_\sigma}A_{|B_\sigma}$ is C_{B_σ}-symmetric. Using the above characterization we show that A must belong to $\mathscr{T}(B)$. Let $M_A = (b_{i,j})$ be the matrix representation of A with respect to the basis $\{1, z, \frac{z^2}{\|k_w\|}k_w\}$. Our goal is to show that $(b_{i,j})$ satisfies (6.1)–(6.4).

Let $B_1(z) = z^2$ and $A_1 = P_{B_1}A_{|B_1}$. Then the space $K^2_{B_1}$ is spanned by $\{1, z\}$,

$$C_{B_1}1 = z, \quad C_{B_1}z = 1,$$

and the C_{B_1}-symmetry of A_1 gives (6.1).

Let $B_2(z) = z\frac{w-z}{1-\overline{w}z}$ and $A_2 = P_{B_2}A_{|B_2}$. Then the space $K^2_{B_2}$ is spanned by $\{1, k_w\}$,

$$C_{B_2}1 = \tfrac{w-z}{1-\overline{w}z}, \quad \text{and} \quad C_{B_2}k_w = -zk_w.$$

Moreover, we have

$$C_B z = \tfrac{w-z}{1-\overline{w}z} = C_{B_2}1, \quad C_B(z^2k_w) = -k_w,$$

and

$$\overline{w}b_{1,3}+\|k_w\|^{-1}b_{1,2}=\|k_w\|^{-1}\langle A\left(\overline{w}z^2k_w+z\right),1\rangle.$$

Since

$$\overline{w}z^2k_w+z=zk_w$$

and A, A_2 are symmetric with respect to C_B and C_{B_2}, respectively, we obtain (6.2). Namely,

$$\begin{aligned}\overline{w}b_{1,3}+\|k_w\|^{-1}b_{1,2}&=\|k_w\|^{-1}\langle A\left(zk_w\right),1\rangle=-\|k_w\|^{-1}\langle A_2C_{B_2}k_w,1\rangle\\&=-\|k_w\|^{-1}\langle C_{B_2}A_2^*k_w,1\rangle=-\|k_w\|^{-1}\langle A_2C_{B_2}1,k_w\rangle\\&=\|k_w\|^{-1}\langle AC_Bz,C_B(z^2k_w)\rangle=\|k_w\|^{-1}\langle C_BA^*z,C_B(z^2k_w)\rangle\\&=\|k_w\|^{-1}\langle A(z^2k_w),z\rangle=b_{2,3}.\end{aligned}$$

Similarly we can obtain (6.3).

To get (6.4) firstly, by using C_B-symmetry of A, we have

$$\begin{aligned}b_{3,3}&=\|k_w\|^{-2}\langle A(z^2k_w),z^2k_w\rangle\\&=\|k_w\|^{-2}\langle AC_Bk_w,C_Bk_w\rangle\\&=\|k_w\|^{-2}\langle Ak_w,k_w\rangle.\end{aligned}$$

From this

$$\begin{aligned}b_{3,3}-b_{1,1}&=(1-|w|^2)\langle Ak_w,k_w\rangle-\langle A1,1\rangle=\langle A\left(1-\overline{w}\tfrac{w-z}{1-\overline{w}z}\right),k_w\rangle-\langle A1,1\rangle\\&=\langle A1,k_w-1\rangle-\overline{w}\langle A\left(\tfrac{w-z}{1-\overline{w}z}\right),k_w\rangle=w\langle A1,zk_w\rangle+\overline{w}\langle AC_Bz,C_B(z^2k_w)\rangle\\&=-w\langle A_21,C_{B_2}k_w\rangle+\overline{w}\langle A(z^2k_w),z\rangle=-w\langle A_2k_w,C_{B_2}1\rangle+\overline{w}\|k_w\|b_{2,3}\\&=w\langle AC_B(z^2k_w),C_Bz\rangle+\overline{w}\|k_w\|b_{2,3}=w\langle Az,z^2k_w\rangle+\overline{w}\|k_w\|b_{2,3}\\&=w\|k_w\|b_{3,2}+\overline{w}\|k_w\|b_{2,3},\end{aligned}$$

which completes the proof.

REFERENCES

[1] A. BARANOV, I. CHALENDAR, E. FRICAIN, J. MASHREGHI AND D. TIMOTIN, *Bounded symbols and reproducing kernels thesis for truncated Toeplitz operators*, J. Funct. Anal. **259** (2010), 2673–2701.

[2] A. BROWN, P. R. HALMOS, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963/64), 89–102.

[3] C. CÂMARA, J. JURASIK, K. KLIŚ-GARLICKA AND M. PTAK, *Characterizations of asymmetric truncated Toeplitz operators*, Banach J. Math. Anal. (to appear).

[4] M. C. CÂMARA AND J. R. PARTINGTON, *Spectral properties of truncated Toeplitz operators by equivalence after extension*, J. Math. Anal. Appl. **433** (2016), 762–784.

[5] J. A. CIMA, W. T. ROSS AND W. R. WOGEN *Truncated Toeplitz operators on finite dimensional spaces*, Oper. Matrices **2** (2008), 357–369.

[6] S. R. GARCIA, J. MASHREGHI AND W. T. ROSS, *Introduction to model spaces and their operators*, Cambridge University Press 2016.

[7] S. R. GARCIA, M. PUTINAR, *Complex symmetric operators and applications*, Trans. Amer. Math.Soc. **358** (2006), 1285–1315.

[8] B. ŁANUCHA, *Matrix representations of truncated Toeplitz operators*, J. Math. Anal. Appl. **413** (2014), 430–437.

[9] N. NIKOL'SKIĬ, *Treatise on the shift operator*, Springer-Verlag, Berlin, 1986.

[10] D. SARASON *Algebraic properties of truncated Toeplitz operators*, Oper. Matrices **1** (2007), 491–526.

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