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MODULE AND GENERALIZED MODULE LEFT (m,n)–DERIVATIONS

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Abstract. Fošner [3, 4] defined a module left (m,n)-derivation and a generalized module left (m,n)-derivation and proved the Hyers-Ulam stability of module left (m,n)-derivations and generalized module left (m,n)-derivations.

In this note, we investigate the properties of module left (m,n)-derivation and generalized module left (m,n)-derivation. Furthermore, we prove that every module left (m,n)-derivation in C^*-algebras is a zero mapping and that every generalized module left (m,n)-derivation in C^*-algebras is a zero mapping.

1. Introduction and preliminaries

Let A be an algebra and M be a left A-module. An additive mapping $d: A \to M$ is called a *module left derivation* if $d(xy) = x \cdot d(y) + y \cdot d(x)$ for all $x, y \in A$.

DEFINITION 1.1. [3] Let A be an algebra and M be a left A-module. An additive mapping $d: A \to M$ is called a *module left* (m,n)*-derivation* if

$$(m+n)d(xy) = 2mx \cdot d(y) + 2ny \cdot d(x) \tag{1.1}$$

for all $x, y \in A$. Here m and n are nonnegative integers with $m+n \neq 0$.

When $m = n$ in Definition 1.1, the additive mapping $d: A \to M$ is just a module left derivation.

Let A be an algebra and M be a left A-module. An additive mapping $g: A \to M$ is called a *generalized module left derivation* if there exists a module left derivation $d: A \to M$ such that $g(xy) = x \cdot g(y) + y \cdot d(x)$ for all $x, y \in A$.

DEFINITION 1.2. [4] Let A be an algebra and M be a left A-module. An additive mapping $g: A \to M$ is called a *generalized module left* (m,n)*-derivation* if there exists a module left (m,n)-derivation $d: A \to M$ such that

$$(m+n)g(xy) = 2mx \cdot g(y) + 2ny \cdot d(x) \tag{1.2}$$

for all $x, y \in A$. Here m and n are nonnegative integers with $m+n \neq 0$.

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When $m = n$ in Definition 1.2, the additive mapping $d : A \to M$ is just a generalized module left derivation.

From now on, assume that $e \cdot x = x$ for all $x \in M$, where e is the unit in a unital algebra A.

PROPOSITION 1.3. [1] *Let A be a unital algebra with unit e and M be a left A-module. Assume that m and n are nonnegative integers with $m+n \neq 0$ and $m \neq n$. Then each module left (m,n)-derivation $d : A \to M$ is a zero mapping.*

Proof. Letting $x = y = e$ in (1.1), we get $(m+n)d(e) = 2(m+n)d(e)$ and so $d(e) = 0$.

Letting $y = e$ in (1.1), we get

$$(m+n)d(x) = 2mx \cdot d(e) + 2nd(x) = 2nd(x)$$

for all $x \in A$.

Letting $x = e$ and replacing y by x in (1.1), we get

$$(m+n)d(x) = 2m \cdot d(x) + 2nx \cdot d(e) = 2md(x)$$

for all $x \in A$. So $2nd(x) = 2md(x)$ for all $x \in A$. Since $m \neq n$, $d(x) = 0$ for all $x \in A$, as desired. □

PROPOSITION 1.4. [6] *Let A be a unital algebra with unit e and M be a left A-module. Assume that m and n are nonnegative integers with $m+n \neq 0$ and $m \neq n$. Then each generalized module left (m,n)-derivation $g : A \to M$ is a zero mapping.*

Proof. By Proposition 1.3, $d(e) = 0$.

Letting $x = y = e$ in (1.2), we get $(m+n)g(e) = 2mg(e) + 2nd(e) = 2mg(e)$ and so $g(e) = 0$, since $m \neq n$.

Letting $y = e$ in (1.2), we get

$$(m+n)g(x) = 2mx \cdot g(e) + 2nd(x) = 2nd(x) = 0$$

for all $x \in A$, since $d(x) = 0$ and $g(e) = 0$. Since $m+n \neq 0$, $g(x) = 0$ for all $x \in A$, as desired. □

In the proofs of Propositions 1.3 and 1.4, we did not use the fact that m and n are nonnegative integers. Propositions 1.3 and 1.4 hold true when m and n are nonzero real numbers with $m+n \neq 0$ and $m \neq n$. From now on, assume that m and n are nonzero real numbers with $m+n \neq 0$ and $m \neq n$.

In Section 2, we investigate module left (m,n)-derivations and generalized module left (m,n)-derivations for (non-unital) algebras.

In Section 3, we investigate module left (m,n)-derivations and generalized module left (m,n)-derivations for (non-unital) C^*-algebras.

2. Generalized module left (m,n)-derivations

We investigate module left (m,n)-derivations for non-unital algebras.

THEOREM 2.1. *Let A be an algebra and M be a left A-module. Then each module left (m,n)-derivation $d: A \to M$ satisfies the following:*
(1) $x^k \cdot d(x) = 0$ *for all* $x \in A$ *and* $k \geqslant 2$.
(2) $x^k \cdot d(x^2) = 0$ *for all* $x \in A$ *and* $k \geqslant 1$.
(3) $d(x^k) = 0$ *for all* $x \in A$ *and* $k \geqslant 3$.

Proof. (1) Letting $y = x^2$ in (1.1), we get

$$\begin{aligned}(m+n)d(x^3) &= 2mx \cdot d(x^2) + 2nx^2 \cdot d(x) \\ &= 2mx \cdot \frac{2mx \cdot d(x) + 2nx \cdot d(x)}{m+n} + 2nx^2 \cdot d(x) \\ &= \frac{4m^2x^2 + 4mnx^2 + 2mnx^2 + 2n^2x^2}{m+n} \cdot d(x)\end{aligned} \tag{2.1}$$

for all $x \in A$.

Replacing x by x^2 and letting $y = x$ in (1.1), we get

$$\begin{aligned}(m+n)d(x^3) &= 2mx^2 \cdot d(x) + 2nx \cdot d(x^2) \\ &= 2mx^2 \cdot d(x) + 2nx \cdot \frac{2mx \cdot d(x) + 2nx \cdot d(x)}{m+n} \\ &= \frac{2m^2x^2 + 2mnx^2 + 4mnx^2 + 4n^2x^2}{m+n} \cdot d(x)\end{aligned} \tag{2.2}$$

for all $x \in A$.

It follows from (2.1) and (2.2) that $2m^2x^2 \cdot d(x) = 2n^2x^2 \cdot d(x)$ for all $x \in A$. So

$$x^2 \cdot d(x) = 0 \tag{2.3}$$

for all $x \in A$. Thus

$$x^k \cdot d(x) = x^{k-2} \cdot (x^2 \cdot d(x)) = 0 \tag{2.4}$$

for all $x \in A$ and $k \geqslant 2$.

(2) It follows from (2.1), (2.2) and (2.3) that

$$(m+n)d(x^3) = 2mx \cdot d(x^2) = 2nx \cdot d(x^2)$$

for all $x \in A$. So

$$x \cdot d(x^2) = 0 \tag{2.5}$$

for all $x \in A$. Thus $x^k \cdot d(x^2) = x^{k-1} \cdot (x \cdot d(x^2)) = 0$ for all $x \in A$ and $k \geqslant 1$.

(3) It follows from (2.1), (2.3) and (2.5) that

$$(m+n)d(x^3) = 2mx\cdot d(x^2) + 2nx^2\cdot d(x) = 0$$

for all $x \in A$. So

$$d(x^3) = 0 \tag{2.6}$$

for all $x \in A$.

Now we prove $d(x^k) =$ for all $k \geqslant 3$ by using the mathematical induction on k.

(i) For $k = 3$, we proved $d(x^3) = 0$ in (2.6).

(ii) For $k = l-1$, assume that $d(x^{l-1}) = 0$ holds for a fixed integer l with $l \geqslant 4$. Then for $k = l$, it follows from (2.3) and the inductive hypothesis that

$$(m+n)d(x^l) = 2mx^{l-1}\cdot d(x) + 2nx\cdot d(x^{l-1}) = 2mx^{l-3}\cdot(x^2\cdot d(x)) + 2nx\cdot d(x^{l-1}) = 0$$

for all $x \in A$. Thus $d(x^l) = 0$ for all $x \in A$. By the mathematical induction, we obtain that $d(x^k) = 0$ for all $x \in A$ and $k \geqslant 3$. □

PROPOSITION 2.2. *Let A be an algebra and M be a left A-module. Then $d(x^2) = 0$ for all $x \in A$ if and only if $x\cdot d(x) = 0$ for all $x \in A$.*

Proof. Letting $y = x$ in (1.1), we get

$$(m+n)d(x^2) = 2mx\cdot d(x) + 2nx\cdot d(x) = 2(m+n)x\cdot d(x) \tag{2.7}$$

for all $x \in A$. So $d(x^2) = 0$ for all $x \in A$ if and only if $x\cdot d(x) = 0$ for all $x \in A$. □

PROBLEM 2.3. *Let A be a non-unital algebra and M be a left A-module.*

(1) *Does $d(x^2) = 0$ hold for all $x \in A$?*

(2) *Does $d(x) = 0$ hold for all $x \in A$?*

PROBLEM 2.4. [1] *Let A be a non-unital algebra and M be a left A-module. Assume that m and n are nonnegative integers with $m+n \neq 0$ and $m \neq n$.*

(1) *Is there a non-trivial module left (m,n)-derivation $d : A \to M$?*

(2) *Construct a non-trival module left (m,n)-derivation $d : A \to M$.*

Now we investigate generalized module left (m,n)-derivations for non-unital algebras.

THEOREM 2.5. *Let A be an algebra and M be a left A-module. Then each generalized module left (m,n)-derivation $g : A \to M$ satisfies the following:*

(1) *$x^k\cdot g(x) = 0$ for all $x \in A$ and $k \geqslant 2$.*

(2) *$x^k\cdot g(x^2) = 0$ for all $x \in A$ and $k \geqslant 1$.*

(3) *$g(x^k) = 0$ for all $x \in A$ and $k \geqslant 3$.*

Proof. (1) Letting $y = x^2$ in (1.2), we get

$$\begin{aligned}(m+n)g(x^3) &= 2mx \cdot g(x^2) + 2nx^2 \cdot d(x) \\ &= 2mx \cdot \frac{2mx \cdot g(x) + 2nx \cdot d(x)}{m+n} \\ &= \frac{4m^2}{m+n} x^2 \cdot g(x)\end{aligned} \tag{2.8}$$

for all $x \in A$ by Theorem 2.1.

Replacing x by x^2 and letting $y = x$ in (1.1), we get

$$(m+n)g(x^3) = 2mx^2 \cdot g(x) + 2nx \cdot d(x^2) = 2mx^2 \cdot g(x) \tag{2.9}$$

for all $x \in A$ by Theorem 2.1.

It follows from (2.8) and (2.9) that $2m^2x^2 \cdot g(x) = 2mnx^2 \cdot g(x)$ for all $x \in A$. So

$$x^2 \cdot g(x) = 0 \tag{2.10}$$

for all $x \in A$. Thus

$$x^k \cdot g(x) = x^{k-2} \cdot (x^2 \cdot g(x)) = 0 \tag{2.11}$$

for all $x \in A$ and $k \geqslant 2$.

(2) It follows from (2.8), (2.9), (2.10) and Theorem 2.1 that

$$(m+n)g(x^3) = 2mx \cdot g(x^2) = 2mx^2 \cdot g(x) = 0$$

for all $x \in A$. So

$$x \cdot g(x^2) = 0 \tag{2.12}$$

for all $x \in A$. Thus $x^k \cdot g(x^2) = x^{k-1} \cdot (x \cdot g(x^2)) = 0$ for all $x \in A$ and $k \geqslant 1$.

(3) It follows from (2.9) and (2.10) that

$$(m+n)g(x^3) = 2mx \cdot g(x^2) = 0$$

for all $x \in A$. So

$$g(x^3) = 0 \tag{2.13}$$

for all $x \in A$.

Now we prove $g(x^k) =$ for all $k \geqslant 3$ by using the mathematical induction on k.

(i) For $k = 3$, we proved $g(x^3) = 0$ in (2.13).

(ii) For $k = l - 1$, assume that $g(x^{l-1}) = 0$ holds for a fixed integer l with $l \geqslant 4$. Then for $k = l$, it follows from Theorem 2.1 and the inductive hypothesis that

$$(m+n)g(x^l) = 2mx \cdot g(x^{l-1}) + 2nx^{l-1} \cdot d(x) = 2mx \cdot g(x^{l-1}) + 2nx^{l-3} \cdot (x^2 \cdot d(x)) = 0$$

for all $x \in A$. Thus $g(x^l) = 0$ for all $x \in A$. By the mathematical induction, we obtain that $g(x^k) = 0$ for all $x \in A$ and $k \geqslant 3$. $\square$

PROPOSITION 2.6. *Let A be an algebra and M be a left A-module. Assume that $d(x^2) = 0$ for all $x \in A$. Then $g(x^2) = 0$ for all $x \in A$ if and only if $x \cdot g(x) = 0$ for all $x \in A$.*

Proof. Letting $y = x$ in (1.2), we get

$$(m+n)g(x^2) = 2mx \cdot g(x) + 2nx \cdot d(x) = 2mx \cdot g(x)$$

for all $x \in A$ by Proposition 2.2. So $g(x^2) = 0$ for all $x \in A$ if and only if $x \cdot g(x) = 0$ for all $x \in A$. □

PROBLEM 2.7. *Let A be a non-unital algebra and M be a left A-module.*
(1) *Does $g(x^2) = 0$ hold for all $x \in A$?*
(2) *Does $g(x) = 0$ hold for all $x \in A$?*

PROBLEM 2.8. [6] *Let A be a non-unital algebra and M be a left A-module. Assume that m and n are nonnegative integers with $m+n \neq 0$ and $m \neq n$.*
(1) *Is there a non-trivial generalized module left (m,n)-derivation $d : A \to M$?*
(2) *Construct a non-trival generalized module left (m,n)-derivation $d : A \to M$.*

3. Generalized module left (m,n)-derivations in C^*-algebras

In this section, we investigate module left (m,n)-derivations and generalized module left (m,n)-derivations for C^*-algebras.

If an additive mapping $d : A \to M$ satisfies $d(\lambda x) = \lambda d(x)$ for all $\lambda \in \mathbb{C}$, then we say that d is $\mathbb{C}$-linear.

DEFINITION 3.1. Let A be a C^*-algebra and M be a left A-module. A $\mathbb{C}$-linear mapping $d : A \to M$ is called a *module left (m,n)-derivation* if

$$(m+n)d(xy) = 2mx \cdot d(y) + 2ny \cdot d(x) \tag{3.1}$$

for all $x, y \in A$.

DEFINITION 3.2. Let A be a C^*-algebra and M be a left A-module. A $\mathbb{C}$-linear mapping $g : A \to M$ is called a *generalized module left (m,n)-derivation* if there exists a module left (m,n)-derivation $d : A \to M$ such that

$$(m+n)g(xy) = 2mx \cdot g(y) + 2ny \cdot d(x) \tag{3.2}$$

for all $x, y \in A$.

DEFINITION 3.3. [2] Let A be a C^*-algebra and $x \in A$ a self-adjoint element, i.e., $x^* = x$. Then x is said to be *positive* if it is of the form yy^* for some $y \in A$.

The set of positive elements of A is denoted by A^+.

Note that A^+ is a closed convex cone and each self-adjoint element $x = x^+ - x^-$ for some positive elements x^+ and x^- (see [2]).

It is well-known that for a positive element x and a positive integer n there exists a unique positive element $y \in A^+$ such that $x = y^n$. We denote y by $x^{\frac{1}{n}}$ (see [5]).

Now we investigate module left (m,n)-derivations for C^*-algebras.

THEOREM 3.4. *Let A be a C^*-algebra and M be a left A-module. Then each module left (m,n)-derivation $d : A \to M$ is a zero mapping.*

Proof. For each $x \in A$, $x = x_1 + ix_2$, where $x_1 = \frac{x+x^*}{2}$ and $x_2 = \frac{x-x^2}{2i}$ are self-adjoint elements. So $x_1 = x_1^+ - x_1^-$ and $x_2 = x_2^+ - x_2^-$, where $x_1^+, x_1^-, x_2^+, x_2^-$ are positive elements.

For each positive element $p \in A^+$, there exists a positive element $q \in A^+$ such that $p = q^3$. By Theorem 2.1, $d(p) = d(q^3) = 0$ for all $p \in A^+$. So

$$\begin{aligned} d(x) &= d(x_1 + ix_2) \\ &= d(x_1^+ - x_1^- + ix_2^+ - ix_2^-) \\ &= d(x_1^+) - d(x_1^-) + id(x_2^+) - id(x_2^-) = 0 \end{aligned}$$

for all $x \in A$, as desired. □

Finally, we investigate generalized module left (m,n)-derivations for C^*-algebras.

THEOREM 3.5. *Let A be a C^*-algebra and M be a left A-module. Then each generalized module left (m,n)-derivation $g : A \to M$ is a zero mapping.*

Proof. For each $x \in A$, $x = x_1 + ix_2$, where $x_1 = \frac{x+x^*}{2}$ and $x_2 = \frac{x-x^2}{2i}$ are self-adjoint elements. So $x_1 = x_1^+ - x_1^-$ and $x_2 = x_2^+ - x_2^-$, where $x_1^+, x_1^-, x_2^+, x_2^-$ are positive elements.

For each positive element $p \in A^+$, there exists a positive element $q \in A^+$ such that $p = q^3$. By Theorem 2.5, $g(p) = g(q^3) = 0$ for all $p \in A^+$. So

$$\begin{aligned} g(x) &= g(x_1 + ix_2) \\ &= g(x_1^+ - x_1^- + ix_2^+ - ix_2^-) \\ &= g(x_1^+) - g(x_1^-) + ig(x_2^+) - ig(x_2^-) = 0 \end{aligned}$$

for all $x \in A$, as desired. □

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