Operators
and
Matrices
Volume 11, Number 3 (2017), 831–844 doi:10.7153/oam-11-58

SPECTRUM OF (n,k)–QUASIPARANORMAL OPERATORS

JIANGTAO YUAN AND CAIHONG WANG

(*Communicated by H. Radjavi*)

Abstract. In this work, some spectral properties of (n,k)-quasiparanormal operators are considered. Let T be a (n,k)-quasiparanormal operator, $\mathscr{M}$ a nontrivial closed invariant subspace of T and $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$ on $\mathscr{M} \oplus \mathscr{M}^{\perp}$. (i) Isolated spectral points and poles. Every nonzero isolated spectral point of T is a pole of order one. (ii) Point spectrum and finite ascent. If $\lambda \neq 0$ and $\mathscr{M} = \ker(T-\lambda) \neq \{0\}$, then $\ker(T_{22}-\lambda) = \{0\}$. Thus $\ker(T-\lambda) = \ker(T-\lambda)^2$. In particular, if λ is nonzero isolated spectral point, then $T_{22}-\lambda$ is invertible. (iii) Riesz idempotent. The Riesz idempotent $E_{\lambda}(T)$ associated with a nonzero isolated spectral point λ is self-adjoint under some assumptions. (iv) Approximate point spectrum and orthogonal eigen-spaces. T has the spectral property (II-1). Meanwhile some examples are given: (i) There exists an operator T such that T is $(n+1)$-paranormal, T is not n-paranormal, T^{-1} is not normaloid and T^* is not m-paranormal for every positive integer m. (ii) There exists an operator T such that T is $(n,1)$-quasiparanormal, T is not n-paranormal, $\mathscr{M} = \ker T \neq \{0\}$ and $\ker T_{22} \neq \{0\}$.

1. Introduction

In this paper, an operator T means a bounded linear operator on a complex Hilbert space $\mathscr{H}$. Let $\mathscr{M}$ be a nontrivial closed invariant subspace of an operator T and $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$ on $\mathscr{M} \oplus \mathscr{M}^{\perp}$. It is well known that paranormal operators have many interesting properties (see [3, 11]), for example:

(i) Isolated spectral points and poles. If T is invertible, then T^{-1} is also paranormal (Thus T^{-1} is normaloid, that is, $\|T^{-1}\| = r(T^{-1})$). In particular, every isolated spectral point of T is a pole of order one;

(ii) Point spectrum. If $\mathscr{M} = \ker(T-\lambda) \neq \{0\}$, then $\ker(T_{22}-\lambda) = \{0\}$;

(iii) Riesz idempotent. The Riesz idempotent $E_{\lambda}(T)$ associated with a nonzero isolated spectral point is self-adjoint;

(iv) Orthogonal eigen-spaces. Two different eigen-spaces are orthogonal to each other.

Mathematics subject classification (2010): 47B20, 47A10, 47A63.

Keywords and phrases: Paranormal operator, isolated spectral point, point spectrum, Riesz idempotent, approximate point spectrum.

The first author was supported in part by National Natural Science Foundation of China (11301155). The second author was supported in part by Doctoral Foundation of Henan Polytechnic University (B2011-061).

The preceding properties are important to other properties of operators, such as Weyl type theorems (see [2, 1, 11, 9, 16]).

In this paper, we will consider the following nonnormal operators related to class A and paranormal operators. Let n be a positive integer and k a nonnegative integer.

(1) T belongs to k-quasiclass $A(n)$ (we denote this class by k-$QA(n)$) if

$$T^{*^k}|T^{1+n}|^{\frac{2}{1+n}}T^k \geqslant T^{*^k}|T|^2T^k$$

where T^* and $|T|$ mean the adjoint and polar factor $(T^*T)^{\frac{1}{2}}$ of T respectively. The class k-$QA(1)$ is equal to k-QA and 0-quasiclass $A(n)$ means class $A(n)$ [13]. It is well-known that, for each n, class $A(n)$ includes every p-hyponormal operators [5, 4].

(2) T is called (n,k)-quasiparanormal (we denote this class by (n,k)-QP) if

$$\|T^{1+n}(T^kx)\|^{\frac{1}{1+n}}\|T^kx\|^{\frac{n}{1+n}} \geqslant \|T(T^kx)\|$$

for $x \in \mathscr{H}$ [14]. A $(n,0)$-quasiparanormal operator means a n-paranormal operator [13], the class of n-paranormal operators includes all class $A(n)$ operators, and every n-paranormal operator is normaloid [6, Theorem 1]. A k-$QA(n)$ operator is (n,k)-quasiparanormal [7, Theorem 2.2].

Recently, the properties (i)–(iv) of paranormal operators are extended to n-paranormal or (n,k)-quasiparanormal operators.

Let $\rho(T)$, $p_0(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_a(T)$, $r(T)$ and iso $\sigma(T)$ mean the resolvent set, the set of poles of the resolvent function, spectrum, point spectrum, approximate point spectrum, spectral radius and the set of all isolated points of the spectrum of an operator T respectively.

An operator T is called isoloid if iso $\sigma(T) \subset \sigma_p(T)$, and polaroid if iso $\sigma(T) \subset p_0(T)$.

THEOREM 1.1. ([10]) *Let T be n-paranormal.*

(1) *If T is invertible, then*

$$\|T^{-1}\| \leqslant r(T^{-1})^{\frac{n(n+1)}{2}} r(T)^{\frac{(n-1)(n+2)}{2}}.$$

(2) *If $\sigma(T) \subseteq \partial\mathscr{D} := \{z|\ |z| = 1\}$, then T is unitary.*

(3) *If $\sigma(T) = \{\lambda\}$, then $T = \lambda$.*

(4) *If $\lambda \in$ iso $\sigma(T)$, then the Riesz idempotent $E_\lambda(T)$ satisfies* $R(E_\lambda(T)) = \ker(T-\lambda)$.

THEOREM 1.2. ([13, 10]) *Let T be n-paranormal, $\lambda \in \sigma_p(T)$ and $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$ on* $\ker(T-\lambda) \oplus (\ker(T-\lambda))^\perp$.

(1) *If* $\lambda \neq 0$, *then*

$$T_{12}\left(\lambda^{n-1}T_{22}+\cdots+T_{22}^{n}\right)=n\lambda^{n}T_{12}, \tag{1.1}$$

$$\left\|T_{22}^{1+n}x\right\|^{\frac{2}{1+n}}\cdot\|x\|^{\frac{2n}{1+n}}\geqslant\left\|T_{12}x\right\|^{2}+\left\|T_{22}x\right\|^{2}. \tag{1.2}$$

In particular, T_{22} *is also* n*-paranormal.*

(2) $\ker(T_{22}-\lambda)=\{0\}$.

(3) *If* $\lambda \in iso\,\sigma(T)$, *then* $\lambda \in \rho(T_{22})$.

Let $F_{n,\lambda}(z)$ denote the polynomial $F_{n,\lambda}(z) := -n\lambda^{n}+\lambda^{n-1}z+\cdots+z^{n}$.

THEOREM 1.3. ([14, 10]) *Let* $0 \neq \lambda \in iso\,\sigma(T)$.

(1) *If* T *is* k*-quasiparanormal and* $\ker(T_{22})^{*k}=0(=\ker(T_{22})^{*})$, *then* $E_{\lambda}(T)=(E_{\lambda}(T))^{*}$.

(2) *If* T *is* n*-paranormal and* $\sigma(T)\cap\{z|F_{n,\lambda}(z)=0\}=\{\lambda\}$, *then* $E_{\lambda}(T)=(E_{\lambda}(T))^{*}$.

An operator T is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc D of λ_0, the only analytic function $f : D \longrightarrow \mathscr{H}$ which satisfies the equation $(\lambda I-T)f(\lambda)=0$ for all $\lambda \in D$ is the function $f \equiv 0$. An operator T is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Uchiyama-Tanahashi [12] introduced the spectral properties (I)–(II) which imply SVEP (single valued extension property).

(I) For each $\lambda \in \sigma_a(T)$ and sequence of bounded vectors $\{x_n\}$, if $\lim_{n\to 0}\|(T-\lambda)x_n\|=0$, then $\lim_{n\to 0}\|(T-\lambda)^{*}x_n\|=0$.

(I-1) For each $0 \neq \lambda \in \sigma_a(T)$ and sequence of bounded vectors $\{x_n\}$, if $\lim_{n\to 0}\|(T-\lambda)x_n\|=0$, then $\lim_{n\to 0}\|(T-\lambda)^{*}x_n\|=0$.

(II) For $\lambda,\mu \in \sigma_a(T)$ $(\lambda \neq \mu)$ and sequences of bounded vectors $\{x_n\}$ and $\{y_n\}$, if $\lim_{n\to 0}\|(T-\lambda)x_n\|=\lim_{n\to 0}\|(T-\mu)y_n\|=0$, then $\lim_{n\to 0}\langle x_n,y_n\rangle=0$ where $\langle\cdot,\cdot\rangle$ means the inner product.

It is known that (I)$\Rightarrow$(I-1)$\Rightarrow$(II).

THEOREM 1.4. ([12, 10]) *If* T *is* n*-paranormal, then* T *satisfies the spectral property* (II). *In particular, any two different eigen-spaces are orthogonal to each other.*

This work will further consider the spectral properties of (n,k)-quasiparanormal operators.

In Section 2, it is to discuss the isolated spectral points and poles. Theorem 1.1 is extended to (n,k)-quasiparanormal operators. In particular, we show an operator T such that T is $(n+1)$-paranormal, T is not n-paranormal, T^{-1} is not normaloid and T^* is not m-paranormal for every positive integer m.

Section 3 is devoted to point spectrum and finite ascent. Among others, we give an operator T such that T is $(n,1)$-quasiparanormal, T is not n-paranormal, $\mathscr{M} = \ker T \neq \{0\}$ and $\ker T_{22} \neq \{0\}$. This implies that the case $\lambda = 0$ of Theorem 1.2 does not holds for (n,k)-quasiparanormal operators.

In Section 4, the Riesz idempotent $E_\lambda(T)$ is considered and Theorem 1.3 is generalized.

Section 5 is devoted to approximate point spectrum and orthogonal eigenspaces. The spectral properties (II-1)–(III-1) are introduced, and it is shown that (n,k)-quasiparanormal operators satisfy (II-1).

2. Isolated spectral points and poles

Let $[R(T)]$ be the closure of range $R(T)$ of an operator T, $\mathscr{M}$ a nontrivial closed invariant subspace of T, $P_{\mathscr{M}}$ the (orthogonal) projection on $\mathscr{M}$, and $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$ on $\mathscr{M} \oplus \mathscr{M}^\perp$.

THEOREM 2.1. *Let $T \in (n,k)$-QP.*

(1) *If T is invertible, then*

$$\|T^{-1}\| \leqslant r(T^{-1})^{\frac{n(n+1)}{2}} r(T)^{\frac{(n-1)(n+2)}{2}}.$$

(2) *If $\sigma(T) \subseteq \partial\mathscr{D} := \{z \mid |z| = 1\}$, then T is unitary.*

(3) *If $\sigma(T) = \{\lambda\}$, then $T = \lambda$ if $\lambda \neq 0$ and $T^{1+k} = 0$ if $\lambda = 0$.*

(4) *If $\lambda \in \operatorname{iso}\sigma(T)$, then $R(E_\lambda(T)) = \ker(T-\lambda)$ with $\lambda \neq 0$, and $R(E_0(T)) = \ker(T^{1+k})$.*

A part of an operator T is its restriction to a closed invariant subspace. Let $\mathscr{C}$ be a class of operators, $\mathscr{C}$ is called a hereditary class if each part of $T \in \mathscr{C}$ belongs to $\mathscr{C}$.

LEMMA 2.2. *Let $\mathscr{C}$ be a hereditary class, then the following assertion (1) implies (2).*

(1) *If $T \in \mathscr{C}$ and $\sigma(T) = \{\lambda\}$, then $T = \lambda$ if $\lambda \neq 0$ and $T^{1+k} = 0$ if $\lambda = 0$.*

(2) *If $T \in \mathscr{C}$ and $\lambda \in \operatorname{iso}\sigma(T)$, then $R(E_\lambda(T)) = \ker(T-\lambda)$ when $\lambda \neq 0$, and $R(E_0(T)) = \ker(T^{1+k})$.*

Lemma 2.2 implies that, if T belongs to a hereditary class and satisfies (1) of Lemma 2.2, then T is polaroid and isoloid.

Proof. Let $T \in \mathscr{C}$ and $\lambda \in \operatorname{iso} \sigma(T)$, then $T|_{E_\lambda(T)\mathscr{H}} \in \mathscr{C}$ and $\sigma(T|_{E_\lambda(T)\mathscr{H}}) = \{\lambda\}$. By (1), $(T|_{E_0(T)\mathscr{H}})^{1+k} = 0$, and $T|_{E_\lambda(T)\mathscr{H}} = \lambda$ when $\lambda \neq 0$. That is, $E_0(T)\mathscr{H} = \ker T^{1+k}$, and $E_\lambda(T)\mathscr{H} = \ker(T-\lambda)$ when $\lambda \neq 0$. $\square$

Proof of Theorem 2.1. (1) If T is invertible, then T is n-paranormal and the result holds by Theorem 1.1 directly. Here, we give a simplified proof.

By definition, $S := T^{-1}$ satisfies

$$\|T^{1+n}(S^k x)\|^{\frac{1}{1+n}} \|S^k x\|^{\frac{n}{1+n}} \geqslant \|T(S^k x)\|$$

for nonnegative integer k and $x \in \mathscr{H}$. So

$$\begin{aligned} \frac{\|S^k x\|}{\|S^{n+k} x\|} &\geqslant \frac{\|S^{n+k} x\|^n}{\|S^{n+k+1} x\|^n}, \\ \prod_{k=0}^{l} \frac{\|S^k x\|}{\|S^{n+k} x\|} &\geqslant \prod_{k=0}^{l} \frac{\|S^{n+k} x\|^n}{\|S^{n+k+1} x\|^n}. \end{aligned} \tag{2.1}$$

Hence

$$\frac{\|x\| \cdots \|S^l x\|}{\|S^n x\| \cdots \|S^{n+l} x\|} \geqslant \frac{\|S^n x\|^n}{\|S^{n+l+1} x\|^n}. \tag{2.2}$$

It is easy to check that (2.2) is equivalent to

$$\frac{\|x\| \cdots \|S^{n-1} x\|}{\|S^{l+1} x\| \cdots \|S^{n+l} x\|} \geqslant \frac{\|S^n x\|^n}{\|S^{n+l+1} x\|^n}. \tag{2.3}$$

Then, since $\|T\| = r(T)$,

$$\begin{aligned} \|S \frac{S^{n-1}x}{\|S^{n-1}x\|}\| &\leqslant \frac{\|x\|}{\|S^n x\|} \cdots \frac{\|S^{n-2}x\|}{\|S^n x\|} \frac{\|S^{n+l+1}x\|^n}{\|S^{l+1}x\| \cdots \|S^{n+l}x\|} \\ &= \|T^n \frac{S^n x}{\|S^n x\|}\| \cdots \|T^2 \frac{S^n x}{\|S^n x\|}\| \frac{\|S^{n+l+1}x\|^n}{\|S^{l+1}x\| \cdots \|S^{n+l}x\|} \\ &\leqslant \|T^n\| \cdots \|T^2\| \frac{\|S^{n+l+1}x\|^n}{\|S^{l+1}x\| \cdots \|S^{n+l}x\|} \\ &= r(T)^{\frac{(n-1)(n+2)}{2}} \frac{\|S^{n+l+1}x\|^n}{\|S^{l+1}x\| \cdots \|S^{n+l}x\|}. \end{aligned} \tag{2.4}$$

Thus

$$\|S\| \leqslant d(n) \frac{\|S^{n+l+1}x\|^n}{\|S^{l+1}x\| \cdots \|S^{n+l}x\|} \tag{2.5}$$

where $d(n)=r(T)^{\frac{(n-1)(n+2)}{2}}$. For sufficiently large m,

$$\begin{aligned}\|S\|^{m+1} &\leqslant d(n)^{m+1}\prod_{l=0}^{m}\frac{\|S^{n+l+1}x\|^n}{\|S^{l+1}x\|\cdots\|S^{n+l}x\|}\\ &= d(n)^{m+1}\frac{\|S^{n+1}x\|^n\cdots\|S^{n+m+1}x\|^n}{\|Sx\|\cdots\|S^{n-1}x\|^{n-1}\|S^nx\|^n\cdots\|S^{m+1}x\|^n\|S^{m+2}x\|^{n-1}\cdots\|S^{n+m}x\|}\\ &= d(n)^{m+1}\frac{\|S^{m+2}x\|\|S^{m+3}x\|^2\cdots\|S^{n+m+1}x\|^n}{\|Sx\|\cdots\|S^{n-1}x\|^{n-1}\|S^nx\|^n}\\ &\leqslant d(n)^{m+1}\frac{\|S^{m+2}\|\|S^{m+3}\|^2\cdots\|S^{n+m+1}\|^n\|x\|^{\frac{n(n+1)}{2}}}{\|Sx\|\cdots\|S^{n-1}x\|^{n-1}\|S^nx\|^n}.\end{aligned}$$

Therefore

$$\|S\|\leqslant d(n)\frac{\|S^{m+2}\|^{\frac{1}{m+1}}\|S^{m+3}\|^{\frac{2}{m+1}}\cdots\|S^{n+m+1}\|^{\frac{n}{m+1}}\|x\|^{\frac{n(n+1)}{2(m+1)}}}{\left(\|Sx\|\cdots\|S^{n-1}x\|^{n-1}\|S^nx\|^n\right)^{\frac{1}{m+1}}}.$$

By letting $m\to\infty$, $\|S\|\leqslant d(n)r(S)^{\frac{n(n+1)}{2}}$ follows.

(2) By (1), $\|T^{-1}\|\leqslant r(T^{-1})=1$ and $\|x\|=\|T^{-1}Tx\|\leqslant\|T^{-1}\|\|Tx\|=\|Tx\|\leqslant\|x\|$. So T is an invertible isometry, that is, T is unitary.

(3) If $\lambda\neq 0$, then T is n-paranormal and the assertion holds by Theorem 1.1. If $\lambda=0$, let $T=\begin{pmatrix}T_{11} & T_{12}\\ 0 & T_{22}\end{pmatrix}$ on $[R(T^k)]\oplus\ker(T^{*k})$. By [14, Theorem 2.1], T_{11} is n-paranormal, $T_{22}^k=0$ and $\sigma(T_{11})=\{0\}$. Hence $T_{11}=0$ and $T^{1+k}=\begin{pmatrix}0 & T_{12}T_{22}^k\\ 0 & T_{22}^{1+k}\end{pmatrix}=0$.

(4) follows by (3) and Lemma 2.2. $\square$

Theorem 1.1 and 2.1 imply that, if T is invertible, n-paranormal and $\sigma(T)\subseteq\partial\mathscr{D}$, then T^{-1} is unitary and normaloid.

Now we show an example such that T is invertible and $(n+1)$-paranormal, but T^{-1} is not normaloid.

Let $\{e_n\}_{n=-\infty}^{\infty}$ be a canonical orthogonal basis of $l^2(\mathbb{Z})$ and $x=(x_n)_{n\in\mathbb{Z}}\in l^2(\mathbb{Z})$.

EXAMPLE 2.3. Let n be a positive integer, k a nonnegative integer, $w=(w_n)_{n\in\mathbb{Z}}$ be a bounded sequence of positive numbers, and $Tx=\sum_{n=-\infty}^{\infty}w_nx_ne_{n+1}$.

(1) $T\in k$-$QA(n)$ if and only if $w_{i+k}^n\leqslant w_{i+k+1}\cdots w_{i+k+n}$ for each $i\in\mathbb{Z}$.

(2) $T\in(n,k)$-QP if and only if $T\in k$-$QA(n)$.

(3) If $a>1$ and $w_i=\begin{cases}a^{\frac{1}{n+1}}, & i=-1,-2,\cdots\\ a, & i=0\\ 1, & i=1,\cdots,n\\ a^{n+1}, & i=n+1,\cdots\end{cases}$, then T is $(n+1)$-paranormal, but not n-paranormal, and T^{-1} is not normaloid. Meanwhile, T^* is normaloid, but not m-paranormal for every positive integer m.

LEMMA 2.4. ([15]) *Let $a>0$, $b>0$, $c>0$, $p>0$ and $r>0$.*

(1) $a^pc^r \geqslant b^{p+r}$ *if and only if* $ap\mu^{p+r}-b(p+r)\mu^p+cr \geqslant 0$ *holds for every* $\mu>0$.

(2) *If n is a positive integer, then* $a^nc \geqslant b^{1+n}$ *if and only if* $na\mu^{1+n}-b(1+n)\mu^n+c \geqslant 0$ *holds for every* $\mu>0$.

Let $u=(u_n)_{n\in\mathbb{Z}}$ and $w=(w_n)_{n\in\mathbb{Z}}$ be bounded sequences of real numbers. It is easy to check that the following lemma holds for each diagonal S (defined by $Sx=\sum_{n=-\infty}^{\infty}u_nx_ne_n$) and weighted unilateral right shift operator T (defined by $Tx=\sum_{n=-\infty}^{\infty}w_nx_ne_{n+1}$).

LEMMA 2.5. *If $Sx=\sum_{n=-\infty}^{\infty}u_nx_ne_n$ and $Tx=\sum_{n=-\infty}^{\infty}w_nx_ne_{n+1}$, then for positive integer m,*

$$T^{*^m}ST^mx=\sum_{n=-\infty}^{\infty}w_n^2\cdots w_{n+m-1}^2u_{n+m}x_ne_n.$$

$$T^mST^{*^m}x=\sum_{n=-\infty}^{\infty}w_n^2\cdots w_{n+m-1}^2u_nx_{n+m}e_{n+m}.$$

Proof of Example 2.3. By Lemmas 2.4–2.5, (1)–(2) hold in a similar manner to (1) and (4) of [15, Example 4.5].

(3) By (1)–(2), T is n-paranormal if and only if $w_i^n \leqslant w_{i+1}\cdots w_{i+n}$ for each $i\in\mathbb{Z}$. Since $w_0^n=a^n>1=w_1\cdots w_n$, T is not n-paranormal. It is easy to check that T is $(n+1)$-paranormal. So $\|T\|=r(T)=a^{n+1}$. By assumption, $\|T^{-1}\|=1$ and $r(T^{-1})=\lim_{m\to\infty}\|T^{-m}\|^{\frac{1}{m}}=\lim_{m\to\infty}a^{-\frac{m-1}{n+1}\frac{1}{m}}=a^{-\frac{1}{n+1}}$. Hence T^{-1} is not normaloid.

Meanwhile, T is normaloid ensures that T^* is also normaloid. By assumption, T^* is m-paranormal if and only if $w_i^m \leqslant w_{i-1}\cdots w_{i-m}$ for each $i\in\mathbb{Z}$. Since $w_0^m=a^m>a^{\frac{m}{n+1}}=w_{-1}\cdots w_{-m}$, T^* is not m-paranormal. $\square$

3. Point spectrum and finite ascent

THEOREM 3.1. *Let $T\in(n,k)$-QP, $0\neq\lambda\in\sigma_p(T)$ and $T=\begin{pmatrix}\lambda & T_{12}\\ 0 & T_{22}\end{pmatrix}$ on* $\ker(T-\lambda)\oplus(\ker(T-\lambda))^{\perp}$.

(1) *The following assertions hold.*

$$T_{12}\big(\lambda^{n-1}T_{22}+\cdots+T_{22}^n\big)T_{22}^k=n\lambda^nT_{12}T_{22}^k, \tag{3.1}$$

$$\big\|T_{22}^{1+n}T_{22}^kx\big\|^{\frac{2}{1+n}}\cdot\|T_{22}^kx\|^{\frac{2n}{1+n}} \geqslant \big\|T_{12}T_{22}^kx\big\|^2+\big\|T_{22}T_{22}^kx\big\|^2. \tag{3.2}$$

In particular, T_{22} is also (n,k)-quasiparanormal.

(2) $\ker(T_{22}-\lambda)=\{0\}$ *and* $\ker(T-\lambda)=\ker(T-\lambda)^2$.

(3) *If $\lambda\in iso\,\sigma(T)$, then $\lambda\in\rho(T_{22})$.*

Theorem 3.1 is known (see [14, Theorem 3.1–3.2], [10, Lemma 6]). Here we give alternate proofs of (2)–(3) of Theorem 3.1, and an example which implies that Theorem 3.1 (2) does not holds for $\lambda = 0$.

LEMMA 3.2. *Let T be an operator, $\lambda \in \sigma_p(T)$ and $T = \begin{pmatrix} \lambda & T_{12} \\ 0 & T_{22} \end{pmatrix}$ on $\ker(T-\lambda) \oplus (\ker(T-\lambda))^{\perp}$. If $\ker(T_{22}-\lambda) = \{0\}$, then $\ker(T-\lambda) = \ker(T-\lambda)^2$.*

Lemma 3.2 says that the matrix representation associated with point spectrum relates to finite ascent closely.

Proof. Let $S := T - \lambda = \begin{pmatrix} 0 & T_{12} \\ 0 & T_{22}-\lambda \end{pmatrix}$ and $x = x_1 \oplus x_2 \in \ker(S^2)$, then $0 = S^2x = S(Sx) = S(T_{12}x_2 \oplus (T_{22}-\lambda)x_2) = T_{12}(T_{22}-\lambda)x_2 \oplus (T_{22}-\lambda)^2x_2$. So $(T_{22}-\lambda)^2x_2 = 0$ and $x_2 = 0$ follows by $\ker(T_{22}-\lambda) = \{0\}$. Hence $x = x_1 \in \ker(T-\lambda)$. □

Proof of Theorem 3.1. (2) It is sufficient to prove $\ker(T_{22}-\lambda) = \{0\}$ because of Lemma 3.2. The assertion $\ker(T_{22}-\lambda) = \{0\}$ can be proved by (3.2) see ([14, Corollary 3.2]). Here we prove $\ker(T_{22}-\lambda) = \{0\}$ by using Theorem 2.1. It is easy to check that $\mathscr{M} := \ker(T-\lambda) \oplus \ker(T_{22}-\lambda)$ is an invariant subspace of T and $\mathscr{M} \subseteq \ker(T-\lambda)^2$. By [14, Theorem 2.1], $T|_{\mathscr{M}} \in (n,k)\text{-}QP$, so $T|_{\mathscr{M}} = \lambda$ follows by $\sigma(T|_{\mathscr{M}}) = \{\lambda\}$ and Theorem 2.1. Hence $\mathscr{M} \subseteq \ker(T-\lambda)$ and $\ker(T_{22}-\lambda) \subseteq \ker(T-\lambda) \oplus (\ker(T-\lambda))^{\perp} = \{0\}$.

(3) If $\lambda \in \operatorname{iso} \sigma(T)$, then $\lambda \in \rho(T_{22}) \cup \operatorname{iso} \sigma(T_{22})$. By (1), $T_{22} \in ((n,k)\text{-}QP)$. If $\lambda \in \operatorname{iso} \sigma(T_{22})$, then $\lambda \in \sigma_p(T_{22})$ by (4) of Theorem 2.1. So $\lambda \in \rho(T_{22})$ holds by (2). □

LEMMA 3.3. ([14]) *Let $T \in (n,k)\text{-}QP$, then $\ker T^{1+k} = \ker T^{2+k}$ and $\ker(T-\lambda) = \ker(T-\lambda)^2$ for $\lambda \neq 0$. In particular, T has SVEP.*

Let $\sigma_w(T)$, $\sigma_{uw}(T)$, $\sigma_{bw}(T)$ and $\sigma_{ubw}(T)$ mean the Weyl spectrum, upper semi-Weyl spectrum, B-Weyl spectrum and upper semi-B-Weyl spectrum of an operator T respectively (see [2]).

Denote $\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \dim\ker(T-\lambda) < \infty\}$, $\pi_0(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \dim\ker(T-\lambda)\}$, $\pi_{00}^a(T) := \{\lambda \in \operatorname{iso} \sigma_a(T) : 0 < \dim\ker(T-\lambda) < \infty\}$, and $\pi_0^a(T) := \{\lambda \in \operatorname{iso} \sigma_a(T) : 0 < \dim\ker(T-\lambda)\}$.

$T \in (W)$ means Weyl theorem holds for T, that is, $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$.

$T \in (aW)$ means a-Weyl theorem holds for T, that is, $\sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}^a(T)$.

$T \in (gW)$ means generalized Weyl theorem holds for T, that is,

$$\sigma(T) \setminus \sigma_{bw}(T) = \pi_0(T).$$

$T \in (gaW)$ means generalized a-Weyl theorem holds for T, that is,

$$\sigma_a(T) \setminus \sigma_{ubw}(T) = \pi_0^a(T).$$

$T \in (w)$ means the property (w) holds for T, that is, $\sigma_a(T) \setminus \sigma_{uw}(T) = \pi_{00}(T)$.

$T\in(gw)$ means the property (gw) holds for T, that is, $\sigma_a(T)\setminus\sigma_{ubw}(T)=\pi_0(T)$.

It is well-known that $(gaW)\Rightarrow(aW)\Rightarrow(W),(gaW)\Rightarrow(gW)\Rightarrow(W)$, and $(gw)\Rightarrow(w)\Rightarrow(W)$.

Let $H(\sigma(T))$ be the set of all functions analytic on some open neighborhood $\mathscr{U}$ of $\sigma(T)$, and $f\in\mathscr{H}_{nc}(\sigma(T))$ means f is holomorphic and locally nonconstant on an open set $\mathscr{U}$ containing $\sigma(T)$. The following result follows by (4) of Theorem 2.1, Lemma 3.3 and [2, Theorem 3.12 and 3.14].

COROLLARY 3.4. *Let $T\in(n,k)$-QP and $f\in H_{nc}(\sigma(T))$, then*

$$f(T)\in(gW)\cap(gaW)\cap(gw).$$

Let $\{e_n\}_{n=0}^{\infty}$ be a canonical orthogonal basis of $l^2(\mathbb{N})$ and $x=(x_n)_{n\in\mathbb{N}}\in l^2(\mathbb{N})$.

EXAMPLE 3.5. Let $w=(w_n)_{n\in\mathbb{N}}$ be a bounded sequence of nonnegative numbers, and $Tx=\sum_{n=0}^{\infty}w_nx_ne_{n+1}$. If $w_0>0,\ w_1=0,w_i=\frac{i}{i+1},\ i\geqslant 2$, then the following assertions hold.

(1) $T\in(n,1)$-QP, but T is not n-paranormal. If $w_0>1$, then T is not normaloid.

(2) $\ker(T)=(0,x_1,0,\cdots)$, $\ker(T^2)=(x_0,x_1,0,\cdots)$ and $\ker(T_{22})\neq\{0\}$.

Example 3.5 implies that, for $\lambda=0$, Theorem 3.1 is not true.

Let $u=(u_n)_{n\in\mathbb{N}}$ and $w=(w_n)_{n\in\mathbb{N}}$ be bounded sequences of real numbers.

LEMMA 3.6. ([15]) *If $Sx=\sum_{n=0}^{\infty}u_nx_ne_n$ and $Tx=\sum_{n=0}^{\infty}w_nx_ne_{n+1}$, then, for positive integer m,*

$$\begin{aligned}T^{*^m}ST^m=&(w_0^2\cdots w_{m-1}^2u_m)\oplus(w_1^2\cdots w_m^2u_{m+1})\oplus(w_2^2\cdots w_{m+1}^2u_{m+2})\oplus\cdots.\\ T^mST^{*^m}=&\overbrace{0\oplus\cdots\oplus 0}^{m\ items}\oplus(w_0^2\cdots w_{m-1}^2u_0)\oplus(w_1^2\cdots w_m^2u_1)\oplus\\ &\oplus(w_2^2\cdots w_{m+1}^2u_2)\oplus\cdots.\end{aligned}$$

Proof of Example 3.5. By [14, Lemma 2.2], $T\in(n,k)$-QP if and only if, for any $\mu>0$,

$$T^{*^k}T^{*^{1+n}}T^{1+n}T^k-(1+n)\mu^nT^{*^k}T^*TT^k+n\mu^{1+n}T^{*^k}T^k\geqslant 0.$$

By Lemma 3.6, $T\in(n,k)$-QP if and only if, for any $\mu>0$ and $i\in\mathbb{N}$,

$$w_{k+i}^2\cdots w_{k+i+n}^2-(1+n)\mu^nw_{k+i}^2+n\mu^{1+n}\geqslant 0.$$

(1) By Lemma 2.4, $T\in(n,1)$-QP if and only if, for any $i\in\mathbb{N}$,

$$w_{1+i}^2\cdots w_{1+i+n}^2\geqslant w_{1+i}^{2(1+n)}.$$

That is, $w_{1+i}^n \leqslant w_{1+i+1}\cdots w_{1+i+n}$ for $i\in\mathbb{N}$. So $T\in(n,1)$-QP follows by the assumption that $w_{1+i}^n=(\frac{i+1}{i+2})^n$ for $i\geqslant 1$.

Similarly, T is n-paranormal if and only if, for any $\mu>0$ and $i\in\mathbb{N}$,

$$w_i^2\cdots w_{i+n}^2-(1+n)\mu^n w_i^2+n\mu^{1+n}\geqslant 0. \tag{3.3}$$

It is clear that w does not satisfy the case $i=0$ of (3.3). Hence T is not n-paranormal.

If $w_0>1$, then $\|T\|=w_0$, $r(T)=\lim_{m\to\infty}\|T^m\|^{\frac{1}{m}}=\lim_{m\to\infty}\sup_{i\geqslant 2}\{\frac{i}{i+m}\}^{\frac{1}{m}}=1$. So T is not normaloid.

(2) If $x\in\ker(T)$, then $0=Tx=(0,w_0x_0,w_1x_1,\cdots)$ and $\ker(T)=(0,x_1,0,\cdots)$. If $x\in\ker(T^2)$, then $0=Tx=(0,0,w_0w_1x_0,w_1w_2x_1,\cdots)$ and $\ker(T^2)=(x_0,x_1,0,\cdots)$. Lemma 3.2 ensures that $\ker(T_{22})\neq\{0\}$. $\square$

4. Riesz idempotent

Let

$$G_{n,\lambda}(z):=\sum_{i=1}^{n} i\lambda^{i-1}z^{n-i}=n\lambda^{n-1}+(n-1)\lambda^{n-2}z+\cdots+z^{n-1}.$$

It is easy to check that $F_{n,\lambda}(z)=(z-\lambda)G_{n,\lambda}(z)$ and $G_{n,\lambda}(\lambda)\neq 0$.

THEOREM 4.1. *Let $T\in(n,k)$-QP, $0\neq\lambda\in iso\,\sigma(T)$ and $T=\begin{pmatrix}\lambda & T_{12}\\ 0 & T_{22}\end{pmatrix}$ on* $\ker(T-\lambda)\oplus(\ker(T-\lambda))^{\perp}$.

(1) *If $\sigma(T)\cap\{z|G_{n,\lambda}(z)=0\}=\phi$ and $\ker(T_{22})^{*k}=0(=\ker(T_{22})^*)$, then $E_\lambda(T)=(E_\lambda(T))^*$.*

(2) *If $\sigma(T)\cap\{z|z^kG_{n,\lambda}(z)=0\}=\phi$, then $E_\lambda(T)=(E_\lambda(T))^*$.*

REMARK 4.2. It is obvious that Theorem 4.1 (1) implies (2). If $n=1$, then $G_{1,\lambda}(z)\equiv 1$ and $\sigma(T)\cap\{z|G_{1,\lambda}(z)=0\}=\phi$ always holds. So the case $n=1$ of Theorem 4.1 (1) is just (1) of Theorem 1.3 ([14, Theorem 5.1]), and case $n=1$ and $k=0$ of Theorem 4.1 (1) is just Uchiyama's result [11, Theorem 3.7].

REMARK 4.3. By (3) of Theorem 3.1, the condition $\sigma(T)\cap\{z|G_{n,\lambda}(z)=0\}=\phi$ is equivalent to $\sigma(T)\cap\{z|F_{n,\lambda}(z)=0\}=\{\lambda\}$. Therefore the case $k=0$ of Theorem 4.1 (2) is just (2) of Theorem 1.3 ([10, Theorem 7]).

LEMMA 4.4. ([14]) *Let m be a positive integer and $\lambda\in iso\,\sigma(T)$.*

(1) *The following assertions are equivalent to each other.*

(a) $E\mathscr{H}=\ker(T-\lambda)^m$.

(b) $\ker E=(T-\lambda)^m\mathscr{H}$.

(2) *If $\lambda \in p_0(T)$ and the order of λ is m, the following assertions are equivalent to each other.*

(a) *E is self-adjoint.*

(b) $\ker(T-\lambda)^m = \ker(T-\lambda)^{*m}$.

(c) $\ker(T-\lambda)^m \subseteq \ker(T-\lambda)^{*m}$.

Proof of Theorem 4.1. It is sufficient to prove (1) because of Remark 4.2. By (3) of Theorem 3.1,

$$\begin{aligned}(3.1) &\Longleftrightarrow T_{12}\big(T_{22}^n + \lambda T_{22}^{n-1} + \cdots + \lambda^{n-1}T_{22} - n\lambda^n\big)T_{22}^k = 0\\ &\Longleftrightarrow T_{12}\big(T_{22}^n - \lambda^n + \lambda(T_{22}^{n-1} - \lambda^{n-1}) + \cdots + \lambda^{n-1}(T_{22}-\lambda)\big)T_{22}^k = 0\\ &\Longleftrightarrow T_{12}(T_{22}-\lambda)G_{n,\lambda}(T_{22})T_{22}^k = 0\\ &\Longleftrightarrow T_{12}G_{n,\lambda}(T_{22})T_{22}^k = 0.\end{aligned}$$

(1) Since $\sigma(T)\cap\{z|G_{n,\lambda}(z)=0\} = \phi$, $G_{n,\lambda}(T_{22})$ is invertible and (3.1) is equivalent to $T_{12}T_{22}^k = 0$. Hence $T_{12} = 0$ follows by $\ker(T_{22})^{*k} = 0 (= \ker(T_{22})^*)$. By Lemma 4.4, the assertion follows. $\square$

LEMMA 4.5. ([15]) *If $T \in k$-$QA(n)$, then $T \in R_3$.*

Lemma 4.5 implies that, if $\lambda \neq 0$, then $\ker(T-\lambda)$ reduces T. The following result follows by Theorem 2.1 (4), Lemma 4.5 and Lemma 4.4.

COROLLARY 4.6. *If $T \in k$-$QA(n)$, $0 \neq \lambda \in iso\,\sigma(T)$, then*

$$E_\lambda(T) = \ker(T-\lambda) = \ker(T-\lambda)^*.$$

5. Approximate point spectrum and orthogonal eigen-spaces

Let us introduce spectral properties (II-1)–(III-1) which imply SVEP.

(II-1) For $\lambda,\mu \in \sigma_a(T)$ $(\lambda \neq \mu)$, $\lambda\mu \neq 0$ and sequences of bounded vectors $\{x_n\}$ and $\{y_n\}$, if $\lim_{n\to 0}\|(T-\lambda)x_n\| = \lim_{n\to 0}\|(T-\mu)y_n\| = 0$, then $\lim_{n\to 0}\langle x_n, y_n\rangle = 0$.

(III) For $\lambda,\mu \in \sigma_p(T)$ $(\lambda \neq \mu)$, $\ker(T-\lambda) \perp \ker(T-\mu)$ holds.

(III-1) For $\lambda,\mu \in \sigma_p(T)$ $(\lambda \neq \mu)$ and $\lambda\mu \neq 0$, $\ker(T-\lambda) \perp \ker(T-\mu)$ holds.

It is obvious that (II) $\Rightarrow$(III)$\Rightarrow$(III-1) and (II)$\Rightarrow$(II-1)$\Rightarrow$(III-1)$\Rightarrow$SVEP ([14, Lemma 3.5]). It is known that n-paranormal operators satisfy the spectral property (II) (Theorem 1.4) and (n,k)-QP operators satisfy (III-1) ([14, Corollary 3.3]).

THEOREM 5.1. *If $T \in (n,k)$-QP, then T satisfies the spectral property (II-1). In particular, T satisfies (III-1) and has SVEP.*

It should be pointed out that Theorem 3.1 (1) also ensures that the assertion "T satisfies (III-1)" above (see [14, Corollary 3.3]).

LEMMA 5.2. *Let* $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$ *on* $\mathscr{M} \oplus \mathscr{M}^{\perp}$. *If* T_{11} *satisfies (II-1) (or (III-1)) and* $\sigma(T_{22}) = \{0\}$, *then* T *satisfies (II-1) (or (III-1))*

Proof. Let $\lambda, \mu \in \sigma_a(T)$ $(\lambda \neq \mu)$, $\lambda\mu \neq 0$, $\{x_n\}$ and $\{y_n\}$ be sequences of unit vectors such that $x_n = x_{n1} \oplus x_{n2}$, $y_n = y_{n1} \oplus y_{n2}$ and $\lim_{n\to 0} \|(T-\lambda)x_n\| = \lim_{n\to 0} \|(T-\mu)y_n\| = 0$.

Since $\sigma(T_{22}) = \{0\}$ and $\lambda\mu \neq 0$,

$$\begin{aligned} \lim_{n\to 0} \|(T-\lambda)x_n\| = 0 &\Longleftrightarrow \lim_{n\to 0} \|(T_{11}-\lambda)x_{n1} + T_{12}x_{n2}\| = 0 = \lim_{n\to 0} \|(T_{22}-\lambda)x_{n2}\| \\ &\Longleftrightarrow \lim_{n\to 0} \|(T_{11}-\lambda)x_{n1} + T_{12}x_{n2}\| = 0 = \lim_{n\to 0} \|x_{n2}\| \\ &\Longleftrightarrow \lim_{n\to 0} \|(T_{11}-\lambda)x_{n1}\| = 0 = \lim_{n\to 0} \|x_{n2}\|. \end{aligned}$$

Similarly,

$$\lim_{n\to 0} \|(T-\mu)y_n\| = 0 \Longleftrightarrow \lim_{n\to 0} \|(T_{11}-\mu)y_{n1}\| = 0 = \lim_{n\to 0} \|y_{n2}\|.$$

Since T_{11} satisfies (II-1), $\lim_{n\to 0}\langle x_n, y_n\rangle = \lim_{n\to 0}(\langle x_{n1}, y_{n1}\rangle + \langle x_{n2}, y_{n2}\rangle) = 0$ holds. □

Proof of Theorem 5.1. Let $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$ on $[R(T^k)] \oplus \ker(T^{*k})$. If $T \in (n,k)$-QP, then T_{11} is n-paranormal, $T_{22}^k = 0$ and $\sigma(T) = \sigma(T_{11}) \cup \{0\}$ ([14, Theorem 2.1]). Therefore the assertion follows by Lemma 5.2 and Theorem 1.4. □

At the end, for convenience, we provide a simplified proof of Theorem 1.4.

Proof of Theorem 1.4. Let $\lambda, \mu \in \sigma_a(T)$ $(\lambda \neq \mu)$, $\{x_m\}$ and $\{y_m\}$ be sequences of unit vectors such that $\lim_{m\to 0} \|(T-\lambda)x_m\| = \lim_{m\to 0} \|(T-\mu)y_m\| = 0$.

Without loss of generality, we may assume that $\mu = 1$, $|\lambda| \leqslant 1$. Assume to the contrary that $\lim_{m\to 0}\langle x_m, y_m\rangle \neq 0$, by considering subsequence and replacing a with $e^{i\theta}a$, we may assume that $\lim_{m\to 0}\langle x_m, y_m\rangle = a > 0$.

For $\varepsilon > 0$ and every complex number c such that $|c| = 1$,

$$\|T(\varepsilon c x_m + y_m)\|^{1+n} \leqslant \|T^{1+n}(\varepsilon c x_m + y_m)\| \|\varepsilon c x_m + y_m\|^n.$$

By letting $m \to \infty$ and $\lim_{m\to 0}(\langle Tx_m, Ty_m\rangle - \langle \lambda x_m, \mu y_m\rangle) = 0$,

$$\begin{aligned} &(\varepsilon^2 c^2 |\lambda|^2 + 1 + 2Re(\varepsilon c \lambda a))^{1+n} \\ &\leqslant (\varepsilon^2 c^2 |\lambda|^{2(1+n)} + 1 + 2Re(\varepsilon c \lambda^{1+n} a))(\varepsilon^2 c^2 + 1 + 2Re(\varepsilon c a))^n \end{aligned}$$

Then

$$\begin{aligned}
&1+(1+n)2Re(\varepsilon c\lambda a)+o(\varepsilon)\\
&\leqslant (1+2Re(\varepsilon c\lambda^{1+n}a)+o(\varepsilon))(1+n2Re(\varepsilon ca)+o(\varepsilon))\\
&=1+2Re(\varepsilon c\lambda^{1+n}a)+2Re(n\varepsilon ca)+o(\varepsilon).
\end{aligned}$$

So that $Re((1+n)c\lambda a) \leqslant Re(c\lambda^{1+n}a)+Re(nca)+o(1)$.
By letting $\varepsilon \to \infty$,

$$Re(ac(n+\lambda^{1+n}-(1+n)\lambda)) \geqslant 0.$$

Noting that c is an arbitrary complex number such that $|c|=1$,

$$n+\lambda^{1+n}-(1+n)\lambda = 0.$$

Since

$$n+\lambda^{1+n}-(1+n)\lambda = n(1-\lambda)+\lambda(\lambda^n-1) = (1-\lambda)(n-\lambda(1+\cdots+\lambda^{n-1})),$$

we have

$$n = \lambda+\cdots+\lambda^n \leqslant |\lambda|+\cdots+|\lambda^n| \leqslant n.$$

Therefore $\lambda = |\lambda| = 1$, this is a contradiction. $\square$

REFERENCES

[1] P. AIENA, *Fredholm and local spectral theory, with application to multipliers*, Kluwer Acad. Publishers, 2004.

[2] P. AIENA, E. APONTE AND E. BALZAN, *Weyl type theorems for left and right polaroid operators*, Integral Equations Operator Theory **66** (2010) 1–20.

[3] N. N. CHOURASIA AND P. B. RAMANUJAN, *Paranormal operators on Banach spaces*, Bull. Austral. Math. Soc. **21** (1980) 161–168.

[4] T. FURUTA, *Invitation to Linear Operators*, Taylor & Francis, London, 2001.

[5] T. FURUTA, M. ITO AND T. YAMAZAKI, *A subclass of paranormal operators including class of* log*-hyponormal and several classes*, Sci. Math. **1** (1998) 389–403.

[6] I. ISTRATESCU AND V. ISTRATESCU, *On some classes of operators*, Proc. Japan Acad. Ser. A Math. Sci. **43** (1967) 605–606.

[7] X. LI AND F. GAO, *On properties of k-quasiclass $A(n)$ operators*, J. Inequal. Appl. **2014**, 2014:91.

[8] K. TANAHASHI, I. H. JEON, I. H. KIM AND A. UCHIYAMA, *Quasinilpotent part of class A or (p,k)-quasihyponormal operators*, Oper. Theory Adv. Appl. **187** (2008) 199–210.

[9] K. TANAHASHI, S. M. PATEL AND A. UCHIYAMA, *On extensions of some Fuglede-Putnam type theorems involving (p,k)-quasihyponormal, spectral, and dominant operators*, Math. Nachr. **282** (2009) 1022–1032.

[10] K. TANAHASHI AND A. UCHIYAMA, *A note on $*$-paranormal operators and related classes of operators*, Bull. Korean Math. Soc. **51** (2014) 357–371.

[11] A. UCHIYAMA, *On the isolated points of the spectrum of paranormal operators*, Integral Equations Operator Theory **55** (2006), 291–298.

[12] A. UCHIYAMA AND K. TANAHASHI, *Bishop's property β for paranormal operator*, Oper. Matrices **3** (2009), 517–524.

[13] J. T. YUAN AND Z. S. GAO, *Weyl spectrum of class $A(n)$ and n-paranormal operators*, Integral Equations Operator Theory **60** (2008) 289–298.
[14] J. T. YUAN AND G. X. JI, *On (n,k)-quasiparanormal operators*, Studia Math. **209** (2012) 289–301.
[15] J. T. YUAN AND C. H. WANG, *Reducibility of invariant subspaces of operators related to k-quasiclass-$A(n)$ operators*, Complex Anal Oper. Theory **10** (2016), 153–169.
[16] Q. P. ZENG AND H. J. ZHONG, *On (n,k)-quasi-$*$-paranormal operators*, Bull. Malays. Math. Sci. Soc. (2) (2015), doi:10.1007/s40840-015-0119-z.

(Received September 28, 2016)

Jiangtao Yuan
School of Mathematics and Information Science
Henan Polytechnic University
Jiaozuo 454000, Henan Province, China
e-mail: jtyuan@hpu.edu.cn

Caihong Wang
School of Mathematics and Information Science
Henan Polytechnic University
Jiaozuo 454000, Henan Province, China
e-mail: chwang@hpu.edu.cn