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A SPECTRAL CHARACTERIZATION OF ABSOLUTELY NORMING OPERATORS ON S. N. IDEALS

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Abstract. The class of absolutely norming operators on complex Hilbert spaces of arbitrary dimensions was introduced in [6] and a spectral characterization theorem for these operators was established in [11]. In this paper we extend the concept of absolutely norming operators to various symmetric norms. We establish a few spectral characterization theorems for operators on complex Hilbert spaces that are absolutely norming with respect to various symmetric norms. It is also shown that for many symmetric norms the absolutely norming operators have the same spectral characterization as proven earlier for the class of operators that are absolutely norming with respect to the usual operator norm. Finally, we prove the existence of a symmetric norm on the algebra $\mathscr{B}(\mathscr{H})$ with respect to which even the identity operator does not attain its norm.

1. Introduction

Throughout this paper $\mathscr{H}$ and $\mathscr{K}$ will denote complex Hilbert spaces and we write $\mathscr{B}(\mathscr{H},\mathscr{K})$ (respectively $\mathscr{B}(\mathscr{H})$) for the set of all bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$ (respectively from $\mathscr{H}$ to $\mathscr{H}$). We recall that $\mathscr{B}(\mathscr{H},\mathscr{K})$ is a complex Banach space with respect to the operator norm $\|T\| = \sup\{\|Tx\|_{\mathscr{K}} : x \in \mathscr{H}, \|x\|_{\mathscr{H}} \leqslant 1\}$. We recall the following definition.

DEFINITION 1.1. [11, Definitions 1.1,1.2] An operator $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$ is said to be *norming* or *norm attaining* if there is an element $x \in \mathscr{H}$ with $\|x\| = 1$ such that $\|T\| = \|Tx\|$. We say that $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$ is *absolutely norming* if for every nontrivial closed subspace $\mathscr{M}$ of $\mathscr{H}$, $T|_{\mathscr{M}}$ is norming.

We let $\mathscr{N}(\mathscr{H},\mathscr{K})$ (or $\mathscr{N}$) and $\mathscr{A}\mathscr{N}(\mathscr{H},\mathscr{K})$ (or $\mathscr{A}\mathscr{N}$) respectively denote the set of norming and absolutely norming operators in $\mathscr{B}(\mathscr{H},\mathscr{K})$.

There is a wealth of information on norm attaining operators; see, for instance, [4, 3, 2, 1, 5, 13, 15, 16, 17] and references therein. The class of absolutely norming operators, however, was introduced recently in [6] and studied in [6], [14]. Carvajal and Neves [6] proved a partial structure theorem [6, Theorem 3.25] for the class of positive operators on complex Hilbert spaces that included an uncharacterized "remainder" operator. The result [11, Theorem 5.1] established a spectral characterization for positive

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operators which asserts that a positive operator is absolutely norming if and only if it is the sum of a positive compact operator, a self-adjoint finite-rank operator, and a non-negative scalar multiple of the identity operator. This theorem was then carried over to bounded operators which we recall here.

THEOREM 1.2. (Spectral Theorem for operators in $\mathscr{AN}(\mathscr{H},\mathscr{K})$) [11, Theorem 6.4] *Let $\mathscr{H}$ and $\mathscr{K}$ be complex Hilbert spaces of arbitrary dimensions, let $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$ and let $T = U|T|$ be its polar decomposition. Then $T \in \mathscr{AN}$ if and only if $|T|$ is of the form $|T| = \alpha I + F + K$, where $\alpha \geqslant 0$, K is a positive compact operator and F is self-adjoint finite-rank operator.*

The above theorem opened up new territories to explore. Can the concept of "absolutely norming" be carried over to norms other than the operator norm? If yes, then can we characterize such class of operators?

In this paper we extend the concept of absolutely norming operators to several particular symmetric norms that are equivalent to the operator norm. We do this with an eye towards the objective of characterizing these classes. We single out three of these symmetric norms for more detailed study: the Ky Fan k-norm, the weighted Ky Fan π, k-norm, and the (p,k)-singular norm.

In section 3 and 4 we develop results for $[k]$-norming and absolutely $[k]$-norming operators (see Definitions 3.2 and 3.3) on $\mathscr{B}(\mathscr{H},\mathscr{K})$. These results parallel those for norming and absolutely norming operators on $\mathscr{B}(\mathscr{H},\mathscr{K})$. In subsections 4.1 and 4.2 we establish necessary and sufficient conditions for a positive operator to belong to the class $\mathscr{AN}_{[k]}(\mathscr{H},\mathscr{K})$ (see Definition 3.3) and present a spectral characterization theorem (see Theorem 4.20) for such operators. This leads us to extend the result to bounded operators in subsection 4.3 and we establish a spectral characterization theorem for the family $\{\mathscr{AN}_{[k]}(\mathscr{H},\mathscr{K}) : k \in \mathbb{N}\}$ (see Theorem 4.22). The operators that belong to this family has the same form as that of those which belong to the class $\mathscr{AN}(\mathscr{H},\mathscr{K})$ or to the class $\mathscr{AN}_{[k]}(\mathscr{H},\mathscr{K})$ for any $k \in \mathbb{N}$.

Subsection 5.1 introduces weighted Ky Fan π, k-norm and study the family $\{\mathscr{AN}_{[\pi,k]}(\mathscr{H},\mathscr{K}) : \pi \in \Pi, k \in \mathbb{N}\}$ of operators (see Definition 5.3), where we use Π to denote the set of all nonincreasing sequences of positive numbers with their first term equal to 1. The family of these classes is large. For instance, it contains $\mathscr{AN}_{[k]}(\mathscr{H},\mathscr{K})$ for every $k \in \mathbb{N}$. In subsection 5.2 we introduce (p,k)-singular norm and work through the class of absolutely (p,k)-norming operators (see Definition 5.7) in $\mathscr{B}(\mathscr{H},\mathscr{K})$, where $p \in [1,\infty)$ and $k \in \mathbb{N}$. This family $\{\mathscr{AN}_{(p,k)}(\mathscr{H},\mathscr{K}) : p \in [1,\infty), k \in \mathbb{N}\}$ too contains $\mathscr{AN}_{[k]}(\mathscr{H},\mathscr{K})$ for every $k \in \mathbb{N}$. We develop the ground results and establish spectral characterization theorems for each of these families separately (see Theorem 5.12 and Theorem 5.13). These results along with the spectral characterization theorems, for each of the two families, can be thought of as a generalization of those we prove in section 4. The operators T that belong to either of these families have property that $|T|$ is also also of the form $\alpha I + K + F$, the notations being as before.

As a corollary to the several spectral characterization theorems we prove, we see that every positive operator of the form $\alpha I + K + F$ belongs to each of the families

$\mathscr{A}\mathscr{N}_{[k]}\mathscr{B}(\mathscr{H}), \mathscr{A}\mathscr{N}_{[\pi,k]}\mathscr{B}(\mathscr{H})$ and $\mathscr{A}\mathscr{N}_{(p,k)}\mathscr{B}(\mathscr{H})$. So, it might appear at this stage that with respect to *every* symmetric norm $\|\cdot\|_s$ on $\mathscr{B}(\mathscr{H})$, the positive operators on $\mathscr{B}(\mathscr{H})$ that are of the above form, are "absolutely s-norming". In section 6, we prove the following proposition that violates our intuition and renders the identity operator nonnorming. From section 6 onwards all Hilbert spaces are considered to be separable.

PROPOSITION 1.3. *There exists a symmetric norm $\|\cdot\|_{\Phi_\pi^*}$ on $\mathscr{B}(\ell^2(\mathbb{N}))$ such that $I \notin \mathscr{N}_{\Phi_\pi^*}(\ell^2(\mathbb{N}))$.*

We collect some facts about symmetrically-normed ideals (s.n. ideals) from [9] and then establish the definition of "s-norming" and "absolutely s-norming" operators on the s.n. ideal $(\mathscr{B}(\mathscr{H}), \|\cdot\|_s)$. We also prove that every compact operator in the s.n. ideal $(\mathscr{B}(\mathscr{H}), \|\cdot\|_s)$ is "absolutely s-norming", where $\mathscr{H}$ is a separable Hilbert space and $\|\cdot\|_s$ is an arbitrary symmetric norm on $\mathscr{B}(\mathscr{H})$ (see Theorem 6.17).

2. Preliminaries

Let $\mathscr{H}$ and $\mathscr{K}$ be complex Hilbert spaces and $T \in \mathscr{B}(\mathscr{H}, \mathscr{K})$. We define $|T| := \sqrt{T^*T}$ — this is conventionally known as the absolute value (or modulus) of the operator T — such that $|T|^2 = T^*T$. If T is compact, then $|T|$ is a positive compact operator on $\mathscr{H}$. We use $\mathscr{B}_0(\mathscr{H}, \mathscr{K})$ (respectively $\mathscr{B}_0(\mathscr{H})$) to denote the set of all compact operators in $\mathscr{B}(\mathscr{H}, \mathscr{K})$ (respectively from $\mathscr{H}$ to $\mathscr{H}$).

DEFINITION 2.1. (s-numbers of compact operators) Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces and $T \in \mathscr{B}_0(\mathscr{H}, \mathscr{K})$. The *singular values* or *s-numbers* of T are the eigenvalues of $|T|$. Needless to say that we can enumerate the nonzero eigenvalues $\lambda_1(|T|)$, $\lambda_2(|T|), \dots$ of $|T|$ in decreasing order, taking account of their multiplicities, that is, $\lambda_1(|T|) \geqslant \lambda_2(|T|) \geqslant \dots$; and hence can enumerate the nonzero s-numbers $s_1(T)$, $s_2(T), \dots$ of T in decreasing order, taking account of their multiplicities as well, so that $s_j(T) = \lambda_j(|T|)$ for each $j = 1, 2, \dots, rank(|T|)$. If $rank(|T|) < \infty$ we define $s_j(T) = 0$ for $j > rank(|T|)$.

We now generalize this concept from compact operators to bounded linear operators. This requires us to define the numbers $\lambda_j(|T|)$ for $T \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ which parallel the definition in the case when $T \in \mathscr{B}_0(\mathscr{H}, \mathscr{K})$. So, our next task is to define the numbers $\lambda_j(A)$ for a positive operator $A \in \mathscr{B}(\mathscr{H})$. For this we need the following definition.

DEFINITION 2.2. (essential spectrum of a self-adjoint operator) Let $T \in \mathscr{B}(\mathscr{H})$ be a self-adjoint operator. A point λ in the spectrum $\sigma(T)$ of T is said to be in the essential spectrum $\sigma_e(T)$ of T if it is either an accumulation point of $\sigma(T)$ or an eigenvalue of T with infinite multiplicity.

DEFINITION 2.3. Let $A \in \mathscr{B}(\mathscr{H})$ be a positive operator and let $\mu = \sup\{\nu : \nu \in \sigma(A)\}$. If $\mu \in \sigma_e(A)$ we define $\lambda_j(A) := \mu \ \ (j = 1, 2, \ldots, rank(A))$. If $\mu \notin \sigma_e(A)$ then it is an eigenvalue of A with finite multiplicity, say m. In this case, we define

$$\lambda_j(A) := \mu \qquad (j = 1, 2, \ldots, m).$$
$$\lambda_{m+j}(A) := \lambda_j(A_1) \qquad (j = 1, 2, \ldots, rank(A_1)).$$

where $A_1 = A - \mu P_{E_\mu}$ with P_{E_μ} being the orthogonal projection of $\mathscr{H}$ onto the eigenspace E_μ corresponding to the eigenvalue μ. If $rank(A) < \infty$ we define $\lambda_j(T) = 0$ for $j > rank(A)$.

This notion agrees with the original definition if A is compact. In the light of the above definition, the following definition makes sense.

DEFINITION 2.4. (s-numbers of arbitrary bounded linear operator) The s-numbers of an arbitrary operator $T \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ are defined as $s_j(T) = \lambda_j(|T|)$ for each $j = 1, 2, \ldots, rank(|T|)$. If $rank(|T|) < \infty$ we define $s_j(T) = 0$ for $j > rank(|T|)$.

We now define the notion of a symmetric norm on a two-sided ideal of $\mathscr{B}(\mathscr{H})$. An ideal of $\mathscr{B}(\mathscr{H})$ always means a two-sided ideal.

DEFINITION 2.5. (Symmetric norm) Let $\mathscr{I}$ be an ideal of the algebra $\mathscr{B}(\mathscr{H})$ of operators on a complex Hilbert space. A *symmetric norm* on $\mathscr{I}$ is a function $\|.\|_s : \mathscr{I} \longrightarrow [0, \infty)$ which satisfies the following six conditions:

(1) $\|X\|_s \geqslant 0$ for each $X \in \mathscr{I}$.
(2) $\|X\|_s = 0$ if and only if $X = 0$.
(3) $\|\lambda X\|_s = |\lambda|\|X\|_s$ for every $X \in \mathscr{I}$ and $\lambda \in \mathbb{C}$.
(4) $\|X + Y\|_s \leqslant \|X\|_s + \|Y\|_s$ for every $X, Y \in \mathscr{I}$.
(5) $\|AXB\|_s \leqslant \|A\|\|X\|_s\|B\|$ for every $A, B \in \mathscr{B}(\mathscr{H})$ and $X \in \mathscr{I}$.
(6) $\|X\|_s = \|X\| = s_1(X)$ for every rank-one operator $X \in \mathscr{I}$.

REMARK 2.6. In the definition of symmetric norm, if we consider the ideal $\mathscr{I}$ to be $\mathscr{B}(\mathscr{H})$, then it is said to be a *symmetric norm on* $\mathscr{B}(\mathscr{H})$. That is, this definition can be extended to the trivial ideals as well. Moreover, the following observations are obvious: (1) the usual operator norm on any ideal $\mathscr{I}$ of $\mathscr{B}(\mathscr{H})$, including the trivial ideals, is a symmetric norm; and (2) every symmetric norm on $\mathscr{B}(\mathscr{H})$ is topologically equivalent to the ordinary operator norm.

3. The classes $\mathscr{N}_{[k]}$ and $\mathscr{A}\mathscr{N}_{[k]}$

DEFINITION 3.1. (Ky Fan k-norm) [8] For a given natural number k, the Ky Fan k-norm $\|\cdot\|_{[k]}$ of an operator $T \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ is defined to be the sum of the k largest singular values of T, that is,

$$\|T\|_{[k]} = \sum_{j=1}^{k} s_j(T).$$

The Ky Fan k-norm on $\mathscr{B}(\mathscr{H},\mathscr{K})$ is, indeed, a norm. It is not difficult to see that it is, in fact, a symmetric norm on $\mathscr{B}(\mathscr{H})$. Note that the smallest of Ky Fan norms, the Ky Fan 1-norm, is equal to the operator norm.

DEFINITION 3.2. For any $k \in \mathbb{N}$, an operator $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$ is said to be $[k]$-*norming* if there are orthonormal elements $x_1,\dots,x_k \in \mathscr{H}$ such that $\|T\|_{[k]} = \|Tx_1\| + \dots + \|Tx_k\|$. If $\dim(\mathscr{H}) = r < k$, we define T to be $[k]$-*norming* if there exist orthonormal elements $x_1,\dots,x_r \in \mathscr{H}$ such that $\|T\|_{[k]} = \|Tx_1\| + \dots + \|Tx_r\|$. We let $\mathscr{N}_{[k]}(\mathscr{H},\mathscr{K})$ denote the set of $[k]$-norming operators in $\mathscr{B}(\mathscr{H},\mathscr{K})$.

A generalization of the above property leads to a new class of operators in $\mathscr{B}(\mathscr{H},\mathscr{K})$.

DEFINITION 3.3. For any $k \in \mathbb{N}$, an operator $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$ is said to be *absolutely* $[k]$-*norming* if for every nontrivial closed subspace $\mathscr{M}$ of $\mathscr{H}$, $T|_{\mathscr{M}}$ is $[k]$-*norming*. We let $\mathscr{A}\mathscr{N}_{[k]}(\mathscr{H},\mathscr{K})$ denote the set of absolutely $[k]$-norming operators in $\mathscr{B}(\mathscr{H},\mathscr{K})$.

Needless to say, every absolutely $[k]$-norming operator is $[k]$-norming, that is, $\mathscr{A}\mathscr{N}_{[k]}(\mathscr{H},\mathscr{K}) \subseteq \mathscr{N}_{[k]}(\mathscr{H},\mathscr{K})$.

REMARK 3.4. Since, in the finite-dimensional setting, the geometric multiplicity of an eigenvalue of a diagonalizable operator is the same as its algebraic multiplicity and the singular values of an operator T are precisely the eigenvalues of the positive operator $|T|$, it immediately follows that *every operator on a finite-dimensional Hilbert space is* $[k]$*-norming for any* $k \in \mathbb{N}$. This is not true when the Hilbert space in question is not finite-dimensional (see Example 4.5).

There is an important and useful criterion for an operator $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$ to be absolutely $[k]$-norming, which is stated in the following lemma.

LEMMA 3.5. *For a closed linear subspace $\mathscr{M}$ of $\mathscr{H}$ let $V_{\mathscr{M}} : \mathscr{M} \longrightarrow \mathscr{H}$ be the inclusion map from $\mathscr{M}$ to $\mathscr{H}$ defined as $V_{\mathscr{M}}(x) = x$ for each $x \in \mathscr{M}$ and let $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$. For any $k \in \mathbb{N}$, $T \in \mathscr{A}\mathscr{N}_{[k]}(\mathscr{H},\mathscr{K})$ if and only if for every nontrivial closed linear subspace $\mathscr{M}$ of $\mathscr{H}$, $TV_{\mathscr{M}} \in \mathscr{N}_{[k]}(\mathscr{M},\mathscr{K})$.*

PROPOSITION 3.6. *Let $\mathscr{H}$ and $\mathscr{K}$ be complex Hilbert spaces and let $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$. Then for every $k \in \mathbb{N}$, $T \in \mathscr{A}\mathscr{N}_{[k]}(\mathscr{H},\mathscr{K})$ if and only if $|T| \in \mathscr{A}\mathscr{N}_{[k]}(\mathscr{H})$.*

Proof. Let $\mathscr{M}$ be an arbitrary nontrivial closed subspace of $\mathscr{H}$ and let $V_{\mathscr{M}} : \mathscr{M} \longrightarrow \mathscr{H}$ be the inclusion map from $\mathscr{M}$ to $\mathscr{H}$ defined as $V_{\mathscr{M}}(x) = x$ for each $x \in \mathscr{M}$. Notice that $|TV_{\mathscr{M}}| = |\,|T|V_{\mathscr{M}}|$; for $|TV_{\mathscr{M}}|^2 = V_{\mathscr{M}}^*|T|^2V_{\mathscr{M}} = |\,|T|V_{\mathscr{M}}|^2$. Consequently, for every j, $s_j(TV_{\mathscr{M}}) = s_j(|T|V_{\mathscr{M}})$. This implies that for each $k \in \mathbb{N}$, we have $\|TV_{\mathscr{M}}\|_{[k]} = \|\,|T|V_{\mathscr{M}}\|_{[k]}$. That for each $x \in \mathscr{H}$, $\|TV_{\mathscr{M}}x\| = \|\,|T|V_{\mathscr{M}}x\|$ is a trivial observation. Since $\mathscr{M}$ is arbitrary, by Lemma 3.5 the assertion follows. $\square$

REMARK 3.7. For the remaining part of this article, we use $\mathcal{N}_{[k]}$ and $\mathcal{AN}_{[k]}$ for $\mathcal{N}_{[k]}(\mathcal{H},\mathcal{K})$ and $\mathcal{AN}_{[k]}(\mathcal{H},\mathcal{K})$ respectively, as long as the domain and codomain spaces are obvious from the context.

4. Spectral characterization of operators in $\mathcal{AN}_{[k]}$

PROPOSITION 4.1. *Suppose $A \in \mathcal{B}(\mathcal{H})$ be a positive operator, $\mu = \sup\{\nu : \nu \in \sigma(A)\}$, and $\mu \notin \sigma_e(A)$, in which case, it is an eigenvalue of A with finite multiplicity, say m, so that for every $j \in \{1,\dots,m\}$, $s_j(A) = \mu$. Then the following statements are equivalent.*

1. *$s_{m+1}(A)$ is an eigenvalue of A.*
2. *$s_{m+1}(A)$ is an eigenvalue of $A - \mu P_{E_\mu}$, where P_{E_μ} is the orthogonal projection of $\mathcal{H}$ onto the eigenspace E_μ corresponding to the eigenvalue μ.*
3. *$(A-\mu P_{E_\mu})|_{E_\mu^\perp} : E_\mu^\perp \longrightarrow E_\mu^\perp$ is norming, that is, $(A-\mu P_{E_\mu})|_{E_\mu^\perp} \in \mathcal{N}$.*
4. *$A|_{E_\mu^\perp} : E_\mu^\perp \longrightarrow E_\mu^\perp$ is norming, that is, $A|_{E_\mu^\perp} \in \mathcal{N}$.*
5. *$A \in \mathcal{N}_{[m+1]}$.*

Proof. $(1) \iff (2)$: The backward implication is trivial. For the forward implication, let $\lambda = s_{m+1}(A) := \sup\{\nu : \nu \in \sigma(A-\mu P_{E_\mu})\}$ be an eigenvalue of A. Then $Ax = \lambda x$ for some $0 \neq x \in \mathcal{H}$. It suffices to prove that $x \perp E_\mu$, for then $(A-\mu P_{E_\mu})x = Ax = \lambda x$. But $A \geqslant 0$ and $\lambda \neq \mu$ which implies that $x \perp E_\mu$.

$(2) \iff (3)$: Since

$$A - \mu P_{E_\mu}(x) = \begin{cases} 0 & \text{if } x \in E_\mu \\ Ax & \text{if } x \in E_\mu^\perp, \end{cases}$$

$A-\mu P_{E_\mu}$ is a positive operator on $\mathcal{B}(\mathcal{H})$ and $E_\mu^\perp$ is a closed subspace of $\mathcal{H}$ which is invariant under $A-\mu P_{E_\mu}$. This implies that $(A-\mu P)_{E_\mu}|_{E_\mu^\perp} : E_\mu^\perp \longrightarrow E_\mu^\perp$, viewed as an operator on $E_\mu^\perp$, is positive and $\|(A-\mu P_{E_\mu})|_{E_\mu^\perp}\| = s_{m+1}(A)$. By [11, Theorem 2.3] we know that a positive operator T belongs to $\mathcal{N}$ if and only if $\|T\|$ is an eigenvalue of T. Thus $(A-\mu P_{E_\mu})|_{E_\mu^\perp} \in \mathcal{N}$ if and only if $s_{m+1}(A)$ is an eigenvalue of $(A-\mu P_{E_\mu})|_{E_\mu^\perp}$ if and only if $s_{m+1}(A)$ is an eigenvalue of $A-\mu P_{E_\mu}$.

$(3) \iff (4)$: This equivalence follows trivially from the fact that the maps $(A - \mu P_{E_\mu})|_{E_\mu^\perp}$, and $A|_{E_\mu^\perp}$ are identical on $E_\mu^\perp$.

$(3) \iff (5)$: Notice that $A \in \mathcal{N}_{[m]}$; for $\|A\|_{[m]} = m\mu$ and since the geometric multiplicity of μ is m, we can find a set $\{v_1,\dots,v_m\}$ of m orthonormal vectors in $E_\mu \subseteq \mathcal{H}$ such that $\sum_{i=1}^m \|Av_i\| = m\mu = \|A\|_{[m]}$. Also, it is not very difficult to observe that if there exists any set $\{w_1,\dots,w_m\}$ of m orthonormal vectors in $\mathcal{H}$ such that $\sum_{i=1}^m \|Aw_i\| = m\mu$, then this set has to be contained in E_μ. This observation implies that $A \in \mathcal{N}_{[m+1]}$ if and only if there exists a unit vector $x \in E_\mu^\perp$ such that $\|Ax\| = s_{m+1}(A)$ which is possible if and only if $A - \mu P_{E_\mu}|_{E_\mu^\perp} : E_\mu^\perp \longrightarrow E_\mu^\perp$ is norming because $\|A - \mu P_{E_\mu}\| = \|(A - \mu P_{E_\mu})|_{E_\mu^\perp}\| = s_{m+1}(A)$. □

REMARK 4.2. The above proposition holds even if $\mu \in \sigma_e(A)$, μ is an accumulation point but not an eigenvalue; for we can consider it to be an eigenvalue with multiplicity 0. If $\mu \in \sigma_e(A)$ is an accumulation point as well as an eigenvalue with finite multiplicity, say m, then one can still prove $(2) \iff (3) \iff (4) \iff (5)$; the condition (1) no longer remains equivalent to other conditions.

PROPOSITION 4.3. *If $A \in \mathcal{B}(\mathcal{H})$ be a positive operator and $s_{m+1}(A) \neq s_m(A)$ for some $m \in \mathbb{N}$, then $A \in \mathcal{N}_{[m]}$. Moreover, in this case, $A \in \mathcal{N}_{[m+1]}$ if and only if $s_{m+1}(A)$ is an eigenvalue of A.*

Proof. It is easy to see that for every $j \in \{1, \dots, m\}$, $s_j(A) \notin \sigma_{\mathrm{e}}(A)$. Then the set $\{s_1(A), \dots, s_m(A)\}$ consists of eigenvalues (not necessarily distinct) of A, each having finite multiplicity. This guarantees the existence of an orthonormal set $\{v_1, \dots, v_m\} \subseteq K \subseteq \mathcal{H}$ such that $Av_j = s_j(A)v_j$ which yields $\|A\|_{[m]} = \|Av_1\| + \ldots + \|Av_m\|$, where K is the joint span of the eigenspaces corresponding to the eigenvalues $\{s_1(A), \dots, s_m(A)\}$, which implies that $A \in \mathcal{N}_{[m]}$. Furthermore, we observe that if there exists any orthonormal set $\{w_1, \dots, w_m\}$ of m vectors in $\mathcal{H}$ such that $\sum_{i=1}^m \|Aw_i\| = \sum_{j=1}^m s_j(A)$, then this set has to be contained in K. Note that $K^\perp$ is invariant under A and hence $A|_{K^\perp} : K^\perp \longrightarrow K^\perp$, viewed as an operator on $K^\perp$, is positive. It follows then that $A \in \mathcal{N}_{m+1}$ if and only if there exists a unit vector $x \in K^\perp$ such that $\|Ax\| = s_{m+1}(A)$, which is possible if and only if $A|_{K^\perp} : K^\perp \longrightarrow K^\perp$, viewed as an operator on $K^\perp$, belongs to $\mathcal{N}$, which in turn happens if and only if $s_{m+1}(A)$ is an eigenvalue of $A|_{K^\perp}$, since $\|A|_{K^\perp}\| = s_{m+1}(A)$. But $s_{m+1}(A) \neq s_m(A)$ implies that $s_{m+1}(A)$ is an eigenvalue of $A|_{K^\perp}$ if and only if $s_{m+1}(A)$ is an eigenvalue of A. This proves the assertion. □

4.1. Necessary conditions for positive operators in $\mathcal{AN}_{[k]}$

The purpose of this subsection is to study the necessary conditions for a positive operator on complex Hilbert space of arbitrary dimension to be absolutely $[k]$-norming for any $k \in \mathbb{N}$.

PROPOSITION 4.4. *Let $A \in \mathcal{B}(\mathcal{H})$ be a positive operator and $k \in \mathbb{N}$. If $A \in \mathcal{N}_{[k]}$, then $s_1(A), \dots, s_k(A)$ are eigenvalues of A.*

Proof. The proof is by contrapositive. Assuming that at least one of the elements from the set $\{s_1(A), \dots, s_k(A)\}$ is not an eigenvalue of A, we show that $A \notin \mathcal{N}_{[k]}$. Suppose that $s_1(A)$ is not an eigenvalue of A. Then it must be an accumulation point of the spectrum of A in which case none of the singular values of A is an eigenvalue of A and that $s_j(A) = s_1(A)$ for every $j \geqslant 2$. Since $s_1(A) = \|A\|$, it follows from [11, Theorem 2.3] that $A \notin \mathcal{N}$ which means that for every $x \in \mathcal{H}$, $\|x\| = 1$, we have

$\|Ax\| < \|A\| = s_1(A)$. Consequently, for every orthonormal set $\{x_1, \ldots, x_k\} \subseteq \mathscr{H}$ we have $\sum_{j=1}^{k} \|Ax_j\| < k\|A\| = \sum_{j=1}^{k} s_j(A)$ so that $A \notin \mathscr{N}_{[k]}$.

Next suppose that $s_1(A)$ is an eigenvalue of A but $s_2(A)$ is not. Clearly then $s_1(A)$ is an eigenvalue with multiplicity 1, $s_2(A) \neq s_1(A)$ and $s_j(A) = s_2(A)$ for every $j \geqslant 3$ in which case Proposition 4.3 ascertains that $A \in \mathscr{N}$ but $A \notin \mathscr{N}_{[2]}$. This implies that there exists $y_1 \in \mathscr{H}$ with $\|y_1\| = 1$ such that $\|Ay_1\| = \|A\|$ and for every $y \in \operatorname{span}\{y_1\}^{\perp}$ with $\|y\| = 1$ we have $\|Ay\| < s_2(A)$ which in turn implies that for every orthonormal set $\{y_2, \ldots y_k\} \subseteq \operatorname{span}\{y_1\}^{\perp}$ we have $\sum_{j=2}^{k} \|Ay_j\| < (k-1)s_2(A) = \sum_{j=2}^{k} s_j(A)$. This yields $\sum_{j=1}^{k} \|Ay_j\| < \sum_{j=1}^{k} s_j(A)$ for every orthonormal set $\{y_1, \ldots, y_k\} \subseteq \mathscr{H}$ which implies that $A \notin \mathscr{N}_{[k]}$.

If we continue in this way, we can show at every step that $A \notin \mathscr{N}_{[k]}$. We conclude the proof by discussing the final case when $s_1(A), \ldots, s_{k-1}(A)$ are all eigenvalues of A but $s_k(A)$ is not in which case $s_k(A) \neq s_{k-1}(A)$ and thus by Proposition 4.3, we infer that $A \notin \mathscr{N}_{[k]}$. This exhausts all the possibilities and the assertion is thus proved contrapositively. □

The converse of the above proposition is not necessarily true as the following example shows.

EXAMPLE 4.5. Consider the diagonal operator $T = \operatorname{diag}\{1, 1, 1/2, 2/3, \ldots, 1 - 1/n, \ldots\}$ on ℓ^2 with respect to an orthonormal basis $B = \{v_i : i \in \mathbb{N}\}$. That T is positive diagonalizable operator with $\|T\| = 1$ is obvious. The spectrum $\sigma(T)$ of T is given by the set $\{1 - \frac{1}{n} : n \in \mathbb{N}, n > 1\} \cup \{1\}$ where $1 \in \sigma(T)$ is an accumulation point of the spectrum as well as an eigenvalue of T with multiplicity 2 and hence $s_j(T) = 1$ for each $j \in \mathbb{N}$. Notice that $\{v_1, v_2\} \subseteq B$ serves to be an orthonormal set such that $\|T\|_{[2]} = \|Tv_1\| + \|Tv_2\|$ which implies that $T \in \mathscr{N}_{[2]}$. Also, if there exists an orthonormal set $\{w_1, w_2\} \subseteq \ell^2$ of two vectors such that $\|T\|_{[2]} = \|Tw_1\| + \|Tw_2\|$, then this set has to be contained in $\operatorname{span}\{v_1, v_2\}$. T is, however, not $[3]$-norming. To show that there does not exist a unit vector $x \in \operatorname{span}\{v_1, v_2\}^{\perp}$ such that $\|Tx\| = 1$, we consider the diagonal operator $A := T - P_{\operatorname{span}\{v_1, v_2\}} = \operatorname{diag}\{0, 0, 1/2, 2/3, \ldots, 1 - 1/n, \ldots\}$ where $P_{\operatorname{span}\{v_1, v_2\}}$ is the orthogonal projection of ℓ^2 onto the space $\operatorname{span}\{v_1, v_2\}$. It is not very hard to see that there exists a unit vector $x \in \operatorname{span}\{v_1, v_2\}^{\perp}$ with $\|Tx\| = 1$ if and only if $A|_{\operatorname{span}\{v_1, v_2\}^{\perp}} : \operatorname{span}\{v_1, v_2\}^{\perp} \longrightarrow \operatorname{span}\{v_1, v_2\}^{\perp}$ achieves its norm on $\operatorname{span}\{v_1, v_2\}^{\perp}$. Since $A|_{\operatorname{span}\{v_1, v_2\}^{\perp}}$ is positive on $\operatorname{span}\{v_1, v_2\}^{\perp}$, it follows that $A|_{\operatorname{span}\{v_1, v_2\}^{\perp}} \in \mathscr{N}$ if and only if $\|A|_{\operatorname{span}\{v_1, v_2\}^{\perp}}\| = 1$ is an eigenvalue of $A|_{\operatorname{span}\{v_1, v_2\}^{\perp}}$ which is indeed not the case.

PROPOSITION 4.6. *Let $A \in \mathscr{B}(\mathscr{H})$ be a positive operator and $k \in \mathbb{N}$. If $s_1(A), \ldots, s_k(A)$ are mutually distinct eigenvalues of A, then there exists an orthonormal set $\{v_1, \ldots, v_k\} \subseteq \mathscr{H}$ such that $Av_j = s_j(A)v_j$ for every $j \in \{1, \ldots, k\}$. Thus $A \in \mathscr{N}_{[k]}$.*

Proof. This is a direct consequence of the fact that the eigenvectors of a normal operator corresponding to distinct eigenvalues are mutually orthogonal. □

An immediate question that arises here is the following: suppose that $s_1(A),\dots,$ $s_k(A)$ are eigenvalues of the positive operator A with $s_1(A)=s_2(A)=\dots=s_k(A)$. Is it possible for A to be in $\mathscr{N}_{[k]}$, and if yes, then under what circumstances? The answer is affirmative and it happens if and only if the geometric multiplicity of the eigenvalue $s_1(A)$ is at least k.

PROPOSITION 4.7. *Let $A\in\mathscr{B}(\mathscr{H})$ be a positive operator, $k\in\mathbb{N}$ and let $s_1(A),\dots,$ $s_k(A)$ be the first k singular values of A that are also the eigenvalues of A and are not necessarily distinct. Then either $s_1(A)=\dots=s_k(A)$, in which case, $A\in\mathscr{N}_{[k]}$ if and only if the multiplicity of $\alpha:=s_1(A)$ is at least k; or there exists $t\in\{2,\dots,k\}$ such that $s_{t-1}(A)\neq s_t(A)=s_{t+1}(A)=\dots=s_k(A)$, in which case, $A\in\mathscr{N}_{[k]}$ if and only if the multiplicity of $\beta:=s_t(A)$ is at least $k-t+1$.*

Proof. It suffices to establish the assertion of the first case; the second case follows similarly. We thus assume that $s_1(A)=\dots=s_k(A)$ and prove that $A\in\mathscr{N}_{[k]}$ if and only if the multiplicity of $\alpha:=s_1(A)$ is at least k. The backward implication is trivial. To see the forward implication, let us assume contrapositively that the geometric multiplicity of α is strictly less that k, that is, the dimension of the eigenspace $E_\alpha=\ker(A-\alpha I)$ associated with the eigenvalue α is $m<k$. Then α has to be an accumulation point of the spectrum $\sigma(A)$ of A as well; for the number of times an eigenvalue with finite multiplicity appears in the sequence $(s_j(A))_{j\in\mathbb{N}}$ exceeds its multiplicity only when it is also an accumulation point of the spectrum. It is easy to see that $A\in\mathscr{N}_{[m]}$ since there exists an orthonormal set $\{v_1,\dots,v_m\}\subseteq E_\alpha$ such that $\|T\|_{[m]}=\|Tv_1\|+\dots+\|Tv_m\|$. Even more, if there exists any orthonormal set $\{w_1,\dots,w_m\}\subseteq\mathscr{H}$ such that $\|T\|_{[m]}=\|Tw_1\|+\dots+\|Tw_m\|$, then this set has to be contained in E_α. We now show that $A\notin\mathscr{N}_{[k]}$. Let P_{E_α} denote the orthogonal projection of $\mathscr{H}$ onto the eigenspace E_α. Now consider the positive operator $A-\alpha P_{E_\alpha}$ on $\mathscr{B}(\mathscr{H})$ and note that $E_\alpha^\perp$ is a closed subspace of $\mathscr{H}$ which is invariant under $A-\alpha P_{E_\alpha}$ which implies that $(A-\alpha P_{E_\alpha})|_{E_\alpha^\perp}:E_\alpha^\perp\longrightarrow E_\alpha^\perp$, viewed as an operator on $E_\alpha^\perp$, is positive and that $\|(A-\alpha P_{E_\alpha})|_{E_\alpha^\perp}\|=s_{m+1}(A)=\alpha$. It is easy to see that α is not an eigenvalue of the positive operator $(A-\alpha P_{E_\alpha})|_{E_\alpha^\perp}$ on $E_\alpha^\perp$. Consequently, this operator does not achieve its norm on $E_\alpha^\perp$ which means that for every $x\in E_\alpha^\perp$ with $\|x\|=1$ we have $\|(A-\alpha P_{E_\alpha})|_{E_\alpha^\perp}x\|<s_{m+1}(A)=\alpha$. Thus for every orthonormal set $\{v_{m+1},v_{m+2},\dots,v_k\}\subseteq E_\alpha^\perp$ we have $\|(A-\alpha P_{E_\alpha})|_{E_\alpha^\perp}v_j\|<s_j(A)=\alpha$, $m+1\leqslant j\leqslant k$ so that $\sum_{j=m+1}^k\|(A-\alpha P_{E_\alpha})|_{E_\alpha^\perp}v_j\|<\sum_{j=m+1}^k s_j(A)$. It now follows that for every orthonormal set $\{x_1,\dots,x_k\}\subseteq\mathscr{H}$, $\sum j=1^k\|Ax_j\|<\sum_{j=1}^k s_j(A)=\|A\|_{[k]}$ which implies that $A\notin\mathscr{N}_{[k]}$. This proves the proposition. □

THEOREM 4.8. *Let $A\in\mathscr{B}(\mathscr{H})$ be a positive operator and $k\in\mathbb{N}$. Then the following statements are equivalent.*

1. $A\in\mathscr{N}_{[k]}$.
2. $s_1(A),\dots,s_k(A)$ *are eigenvalues of A and there exists an orthonormal set $\{v_1,\dots,v_k\}\subseteq\mathscr{H}$ such that $Av_j=s_j(A)v_j$ for every $j\in\{1,\dots,k\}$.*

Proof. (1) follows from (2) trivially. Assume that $A \in \mathcal{N}_{[k]}$. Since $A \geqslant 0$, by Proposition 4.4, $s_1(A), \ldots, s_k(A)$ are all eigenvalues of A. If $s_1(A), \ldots, s_k(A)$ are mutually distinct, then by Proposition 4.6 $A \in \mathcal{N}_{[k]}$. However, if $s_1(A), \ldots, s_k(A)$ are not necessarily distinct then the Proposition 4.7 yields $A \in \mathcal{N}_{[k]}$. This completes the proof. □

COROLLARY 4.9. *Let $k \in \mathbb{N}$. If $A \in \mathcal{N}_{[k+1]}(\mathcal{H}, \mathcal{K})$ is positive, then $A \in \mathcal{N}_{[k]}(\mathcal{H}, \mathcal{K})$.*

THEOREM 4.10. *Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $k \in \mathbb{N}$. Then the following statements are equivalent.*

(1) $T \in \mathcal{N}_{[k]}$.
(2) $|T| \in \mathcal{N}_{[k]}$.
(3) $T^*T \in \mathcal{N}_{[k]}$.

Proof. The equivalence of (1) and (2) follows from facts that for every j, $s_j(T) = \lambda_j(|T|) = s_j(|T|)$ and for every $x \in \mathcal{H}$, $\|Tx\| = \||T|x\|$. The equivalence of (2) and (3) can be established by using Theorem 4.8 and observing that if $|T| \in \mathcal{N}_{[k]}$, then $s_1(|T|), \ldots, s_k(|T|)$ are eigenvalues of $|T|$ whence $s_j(T^*T) = s_j^2(|T|)$ are eigenvalues of $T^*T = |T|^2$. □

THEOREM 4.11. *Let $k \in \mathbb{N}$. Then $\mathcal{AN}_{[k+1]}(\mathcal{H}, \mathcal{K}) \subseteq \mathcal{AN}_{[k]}(\mathcal{H}, \mathcal{K})$.*

Proof. If $A \in \mathcal{AN}(\mathcal{H}, \mathcal{K})$, then Theorem 4.10 along with Corollary 4.9 implies that for every nontrivial closed subspace $\mathcal{M}$ of $\mathcal{H}$, $AV_{\mathcal{M}} \in \mathcal{N}_{[k+1]} \iff |AV_{\mathcal{M}}| \in \mathcal{N}_{[k+1]} \implies |AV_{\mathcal{M}}| \in \mathcal{N}_{[k]} \iff AV_{\mathcal{M}} \in \mathcal{N}_{[k]}$. Since the above implications (and both way implications) hold for every $\mathcal{M}$, the assertion is proved. □

THEOREM 4.12. *Let A be a positive operator on $\mathcal{H}$, and $k \in \mathbb{N}$. If $A \in \mathcal{AN}_{[k]}$, then A is of the form $A = \alpha I + K + F$, where $\alpha \geqslant 0$, K is a positive compact operator and F is self-adjoint finite-rank operator.*

Proof. Since $A \in \mathcal{AN}_{[k]}$, $A \in \mathcal{AN}$. The forward implication of [11, Theorem 5.1], hence, implies the assertion. □

We finish this subsection by proving a result which will be useful later in section 6 for establishing the notion of absolutely norming operators in symmetrically-normed ideals.

THEOREM 4.13. *For a closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ let $V_{\mathcal{M}} : \mathcal{M} \longrightarrow \mathcal{H}$ be the inclusion map from $\mathcal{M}$ to $\mathcal{H}$ defined as $V_{\mathcal{M}}(x) = x$ for each $x \in \mathcal{M}$, let $P_{\mathcal{M}} \in \mathcal{B}(\mathcal{H})$ be the orthogonal projection onto $\mathcal{M}$, and let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. For any $k \in \mathbb{N}$, the following statements are equivalent.*

1. $T \in \mathcal{AN}_{[k]}(\mathcal{H}, \mathcal{K})$.

2. *$TV_{\mathscr{M}} \in \mathscr{N}_{[k]}(\mathscr{M}, \mathscr{K})$ for every nontrivial closed linear subspace $\mathscr{M}$ of $\mathscr{H}$.*

3. *$TP_{\mathscr{M}} \in \mathscr{N}_{[k]}(\mathscr{H}, \mathscr{K})$ for every nontrivial closed linear subspace $\mathscr{M}$ of $\mathscr{H}$.*

Proof. The equivalence of (1) and (2) follows from Lemma 3.5. We will prove $(1) \iff (3)$. Fix $\mathscr{M}$ to be a nontrivial closed subspace of $\mathscr{H}$. A trivial verification shows that $\sigma(|T|_{\mathscr{M}}|) \setminus \{0\} = \sigma(|TP_{\mathscr{M}}|) \setminus \{0\}$ which implies that the singular values of $T|_{\mathscr{M}}$ and $TP_{\mathscr{M}}$ are identical, which gives $\|T|_{\mathscr{M}}\|_{[k]} = \|TP_{\mathscr{M}}\|_{[k]}$. Of course, $\|T|_{\mathscr{M}}x\| = \|TP_{\mathscr{M}}x\|$ for each $x \in \mathscr{M}$. This establishes the implication $(1) \implies (3)$. All that remains is to prove $(3) \implies (1)$. Assume that $TP_{\mathscr{M}} \in \mathscr{N}_{[k]}(\mathscr{H}, \mathscr{K})$. Then by Theorem 4.10 $|TP_{\mathscr{M}}| \in \mathscr{N}_{[k]}(\mathscr{H})$. Theorem 4.8 guarantees the existence of an orthonormal set $\{x_1, \dots, x_k\} \subseteq \mathscr{H}$ with $|TP_{\mathscr{M}}|x_j = s_j(|TP_{\mathscr{M}}|)x_j$ for every $j \in \{1, \dots, k\}$ which implies that $|TP_{\mathscr{M}}|^2 x_j = s_j^2(|TP_{\mathscr{M}}|)x_j = s_j(|TP_{\mathscr{M}}|^2)x_j$ for every $j \in \{1, \dots, k\}$. Without loss of generality we assume that for each $j, s_j(|TP_{\mathscr{M}}|^2) \neq 0$. Then $x_j \in \mathscr{M}$ for each j and consequently $\|T|_{\mathscr{M}}\|_{[k]} = \|TP_{\mathscr{M}}\|_{[k]} = \|\,|TP_{\mathscr{M}}|\,\|_{[k]} = \sum_{j=1}^k \|\,|TP_{\mathscr{M}}|x_j\| = \sum_{j=1}^k \|TP_{\mathscr{M}}x_j\| = \sum_{j=1}^k \|Tx_j\| = \sum_{j=1}^k \|T|_{\mathscr{M}}x_j\|$, where $\{x_1, \dots, x_k\}$ is an orthonormal set contained in $\mathscr{M}$. Using the fact $s_j(|TP_{\mathscr{M}}|) = s_j(TP_{\mathscr{M}})$ we conclude $T|_{\mathscr{M}} \in \mathscr{N}_{[k]}$. But $\mathscr{M}$ is arbitrary, so $T \in \mathscr{A}\mathscr{N}_{[k]}(\mathscr{H}, \mathscr{K})$. Since $k \in \mathscr{N}$ is arbitrary, the assertion holds for each $k \in \mathscr{N}$. □

4.2. Sufficient conditions for operators in $\mathscr{A}\mathscr{N}_{[k]}$

In this subsection, we discuss the sufficient conditions for an operator (not necessarily positive) to be absolutely $[k]$-norming for every $k \in \mathbb{N}$. We begin with a relatively easy proposition, the proof of which is trivial and thus omitted, that gives a sufficient condition for a positive diagonalizable operator to be in $\mathscr{N}_{[k]}$.

PROPOSITION 4.14. *Let $A \in \mathscr{B}(\mathscr{H})$ be a positive diagonal operator with respect to an orthonormal basis $B = \{v_\beta : \beta \in \Lambda\}$ and $k \in \mathbb{N}$. If there exists a subset $\{\beta_1, \dots, \beta_k\} \subseteq \Lambda$ such that $A(v_{\beta_j}) = \lambda_j(A)v_{\beta_j}$ for each $j \in \{1, \dots, k\}$, then $A \in \mathscr{N}_{[k]}$. If $\dim(\mathscr{H}) = r < k$, then the existence of a subset $\{\beta_1, \dots, \beta_r\} \subseteq \Lambda$ is required with the condition that for each $j \in \{1, \dots, r\}$ $A(v_{\beta_j}) = \lambda_j(A)v_{\beta_j}$, for the operator A to be in $\mathscr{N}_{[k]}$. Here $\lambda_j(A)$ is as introduced in the Definition 2.3.*

THEOREM 4.15. (The Courant-Fischer Theorem for positive compact operators) *Let $A \in \mathscr{B}(\mathscr{H})$ be a positive compact operator on $\mathscr{H}$ and let $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \dots$ be its algebraically ordered eigenvalues (counting multiplicities) in nonincreasing sense. Let $k \in \mathbb{N}$ and let S denote a subspace of $\mathscr{H}$.Then*

$$\lambda_1(A) = \max_{\{x : x \in \mathscr{H} \text{ and } \|x\|=1\}} \langle Ax, x\rangle \tag{1}$$

$$\lambda_{k+1}(A) = \min_{\{S : \dim(S)=k\}} \left(\max_{\{x : x \in S^\perp \text{ and } \|x\|=1\}} \langle Ax, x\rangle \right) \tag{2}$$

where the maximum in (1) is attained only at those eigenvectors of A which correspond to $\lambda_1(A)$ and the minimum in (2) is attained when S coincides with the k-dimensional subspace spanned by the eigenvectors $\{u_j : 1 \leqslant j \leqslant k\}$ of A corresponding to the eigenvalues $\{\lambda_j : 1 \leqslant j \leqslant k\}$, so that

$$\lambda_{k+1}(A) = \max_{\{x: x\in S^{\perp} \text{ and } \|x\|=1\}} \langle Ax, x\rangle . \tag{3}$$

PROPOSITION 4.16. *If $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$ is a compact operator, then $T \in \mathscr{A}\mathscr{N}_{[k]}$ for every $k \in \mathbb{N}$.*

Proof. If T is a compact operator from $\mathscr{H}$ to $\mathscr{K}$ then the restriction of T to any closed subspace $\mathscr{M}$ is a compact operator from $\mathscr{M}$ to $\mathscr{K}$. So it will be sufficient to prove that if T is a compact operator then $T \in \mathscr{N}_{[k]}$ for each $k \in \mathbb{N}$.

The assertion is trivial if $\mathscr{H}$ is finite-dimensional; for then $|T| \in M_n(\mathbb{C})$ for some $n \in \mathbb{N}$ is a positive diagonal matrix. We thus assume $\mathscr{H}$ to be infinite-dimensional. Let us fix $k \in \mathbb{N}$. Since $|T|$ is positive compact operator the singular values of T are precisely the eigenvalues of $|T|$, the Courant Fisher Theorem (4.15) guarantees the existence of an orthonormal set $\{v_1,\ldots,v_k\} \subseteq \mathscr{H}$ such that for every $j \in \{1,\ldots,k\}$, $|T|v_j = \lambda_j(|T|)v_j$ which implies that $\|T\|_{[k]} = \sum_{j=1}^{k} s_j(T) = \sum_{j=1}^{k} \lambda_j(|T|) = \sum_{j=1}^{k} \||T|v_j\|$ $= \sum_{j=1}^{k} \|Tv_j\|$ so that $T \in \mathscr{N}_{[k]}$. Since $k \in \mathbb{N}$ is arbitrary, it follows that $T \in \mathscr{N}_{[k]}$ for every $k \in \mathbb{N}$. □

LEMMA 4.17. *If $F \in \mathscr{B}(\mathscr{H})$ is a self-adjoint finite-rank operator and $\alpha \geqslant 0$, then $\alpha I + F \in \mathscr{N}_{[k]}$ for every $k \in \mathbb{N}$.*

Proof. $\alpha I + F$ has no accumulation point in its spectrum, so by Proposition 4.14 $\alpha I + F \in \mathscr{N}_{[k]}$ for every $k \in \mathbb{N}$. □

PROPOSITION 4.18. *Let $K \in \mathscr{B}(\mathscr{H})$ be a positive compact operator, $F \in \mathscr{B}(\mathscr{H})$ be a self-adjoint finite-rank operator, and $\alpha \geqslant 0$. Then $\alpha I + K + F \in \mathscr{N}_{[k]}$ for every $k \in \mathbb{N}$.*

Proof. The assertion is trivial if $\alpha = 0$; for then $K + F$ is a compact operator which sits in $\mathscr{N}_{[k]}$ for every $k \in \mathbb{N}$. We fix k and assume that $\alpha > 0$. There is no loss of generality in assuming that $\mathscr{H}$ is infinite-dimensional and that $\dim(\operatorname{ran} K) > n$ for every $n \in \mathbb{N}$. Due to the equivalence of (1) and (2) of Theorem 4.10, it suffices to show that $|\alpha I + K + F| \in \mathscr{N}_{[k]}$.

Notice that $K + F$ is a self-adjoint compact operator on $\mathscr{H}$ and thus there exists an orthonormal basis B of $\mathscr{H}$ consisting entirely of eigenvectors of $K + F$ corresponding to which it is diagonalizable. From [11, Lemma 4.8], $K + F$ can have at most finitely many negative eigenvalues. Let $\{\nu_1, \nu_2, \ldots, \nu_m\}$ be the set of all negative eigenvalues of $K + F$ with $\{v_1, v_2, \ldots, v_m\}$ as the corresponding eigenvectors in basis B; and let $\{\mu_\beta : \beta \in \Lambda\}$ be the set of all remaining nonnegative eigenvalues of $K + F$ with $\{w_\beta : \beta \in \Lambda\}$ as the corresponding eigenvectors in B. We have $B := \{v_1, v_2, \ldots, v_n\} \cup \{w_\beta :$

$\beta \in \Lambda\}$ and the matrix $M_B(K+F)$ of $K+F$ with respect to B is given by $K+F = \operatorname{diag}\{\nu_1,\dots,\nu_m\} \oplus \operatorname{diag}\{\mu_\beta\}_{\beta\in\Lambda}$. Because $K+F$ is compact, the multiplicity of each nonzero eigenvalue is finite and there are at most countably many nonzero eigenvalues, counting multiplicities. In fact, we can safely assume that there are countably infinite nonzero eigenvalues (counting multiplicities) of $K+F$; for if there are only finitely many nonzero eigenvalues, then $K+F$ would be a self-adjoint finite-rank operator which, by Lemma 4.17, belongs to $\mathscr{N}_{[k]}$. With this observation, the set $\Gamma := \Lambda \setminus \{\beta \in \Lambda : \mu_\beta = 0\}$ is countably infinite and can be safely replaced by $\mathbb{N}$. This essentially redefines the spectrum $\sigma(K+F) = \{\nu_n\}_{n=1}^m \cup \{\mu_n\}_{n=1}^\infty \cup \{0\}$ of $K+F$ and allows us to enumerate the positive eigenvalues $\{\mu_n\}_{n=1}^\infty$ in nonincreasing order $\mu_1 \geqslant \mu_2 \geqslant \dots$ so that each eigenvalue appears as many times as is its multiplicity. This ensures that the set of all positive eigenvalues of $K+F$ has been exhausted in the process of constructing the sequence $\{\mu_n\}_{n\in\mathbb{N}}$. That the sequence $\{\mu_n\}$ converges to 0 is a trivial observation. So, 0 is an accumulation point of the spectrum. However, it can also be an eigenvalue with infinite multiplicity. At this point, we rename and denote by $\{v_n\}_{n=1}^m$, $\{w_n\}_{n=1}^\infty$, and $\{z_\beta\}_{\beta\in\Lambda\setminus\Gamma}$ the eigenvectors corresponding to the eigenvalues $\{\nu_n\}_{n=1}^m$, $\{\mu_n\}_{n=1}^\infty$, and $\{0\}$ respectively. With the reordering, we now have $B := \{v_n\}_{n=1}^m \cup \{w_n\}_{n=1}^\infty \cup \{z_\beta\}_{\beta\in\Lambda\setminus\Gamma}$ and the matrix $M_B(K+F)$ of $K+F$ with respect to B is given by $K+F = \operatorname{diag}\{\nu_1,\dots,\nu_m\} \oplus \operatorname{diag}\{\mu_1,\mu_2,\dots,\mu_k,\dots\} \oplus \operatorname{diag}\{0,0,\dots\}$. Consider the operator $|\alpha I+K+F|$. The matrix $M_B(|\alpha I+K+F|)$ is given by $|\alpha I+K+F| = \operatorname{diag}\{|\alpha+\nu_1|,\dots,|\alpha+\nu_m|\} \oplus \operatorname{diag}\{\alpha+\mu_1,\alpha+\mu_2,\dots,\alpha+\mu_k,\dots\} \oplus \operatorname{diag}\{\alpha,\alpha,\dots\}$.

Observe that $\sigma_e(|\alpha I+K+F|) = \{\alpha\}$ and that for any given $k\in\mathbb{N}$, $\lambda_j(|\alpha I+K+F|) > \alpha$ for every $j\in\{1,\dots,k\}$. In fact, $\lambda_j(|\alpha I+K+F|) \in \{|\alpha+\nu_n|\}_{n=1}^m \cup \{\alpha+\mu_n\}_{n=1}^\infty$ for every $j\in\{1,\dots,k\}$. It then immediately follows that there exist k orthogonal eigenvectors in $\{v_n\}_{n=1}^m \cup \{w_n\}_{n=1}^\infty$ with $\lambda_j(|\alpha I+K+F|)$, $j\in\{1,\dots,k\}$ being their correspoding eigenvalues. This proves the assertion. Since $k\in\mathbb{N}$ is arbitrary in the above proof, the propostion holds for every $k\in\mathbb{N}$. □

This result is the key to the following theorem.

THEOREM 4.19. *Let $K\in\mathscr{B}(\mathscr{H})$ be a positive compact operator, $F\in\mathscr{B}(\mathscr{H})$ be a self-adjoint finite-rank operator, and $\alpha \geqslant 0$. Then $\alpha I+K+F \in \mathscr{A}\mathscr{N}_{[k]}$ for every $k\in\mathbb{N}$.*

Proof. Let us define $T := \alpha I+K+F$ so that we have $|T| = |\alpha I+K+F|$ and $|T|^*|T| = \beta I+\tilde{K}+\tilde{F}$ where $\beta = \alpha^2 \geqslant 0$, $\tilde{K} = 2\alpha K+K^2$ and $\tilde{F} = 2\alpha F+FK+KF+F^2$ are respectively positive compact and self-adjoint finite-rank operators. Further, let $\mathscr{M}$ be an arbitrary nonempty closed linear subspace of $\mathscr{H}$ and $V_{\mathscr{M}} : \mathscr{M} \longrightarrow \mathscr{H}$ be the inclusion map from $\mathscr{M}$ to $\mathscr{H}$ defined as $V_{\mathscr{M}}(x) = x$ for each $x\in\mathscr{M}$. We fix $k\in\mathbb{N}$ and observe that $|T|V_{\mathscr{M}} \in \mathscr{N}_{[k]} \iff (|T|V_{\mathscr{M}})^*(|T|V_{\mathscr{M}}) \in \mathscr{N}_{[k]} \iff V_{\mathscr{M}}^*(\beta I+\tilde{K}+\tilde{F})V_{\mathscr{M}} \in \mathscr{N}_{[k]}$. It suffices to show that $V_{\mathscr{M}}^*(\beta I+\tilde{K}+\tilde{F})V_{\mathscr{M}} \in \mathscr{N}_{[k]}$; for then, since $\mathscr{M}$ is arbitrary, it immediately follows from lemma 3.5 that $|T| \in \mathscr{A}\mathscr{N}_{[k]}$ and so does T due to the equivalence of (1) and (2) of Theorem 4.10. To this end, notice that $V_{\mathscr{M}}^*(\beta I+\tilde{K}+\tilde{F})V_{\mathscr{M}}$ is an operator on $\mathscr{M}$ and $V_{\mathscr{M}}^*(\beta I+\tilde{K}+\tilde{F})V_{\mathscr{M}} = \beta I_{\mathscr{M}}+\tilde{K}_{\mathscr{M}}+\tilde{F}_{\mathscr{M}}$ is the sum of a nonnegative scalar multiple of Identity, a positive compact operator and a self-adjoint finite-rank operator on a Hilbert space $\mathscr{M}$ which, by the preceding proposition, belongs to $\mathscr{N}_{[k]}$. This proves the assertion. Moreover, since $k \in \mathbb{N}$ is arbitrary, the result holds for every $k \in \mathbb{N}$. □

We are now ready to establish the spectral theorem for positive operators that belong to $\mathscr{AN}_{[k]}$ for every $k \in \mathbb{N}$. Note that the Theorem 4.19 we just proved is the stronger version of the backward implication of our spectral theorem for positive $\mathscr{AN}_{[k]}$ operators. If the operator $\alpha I + K + F$ is also positive then the implication can be reversed and the two conditions are equivalent. This is what the next theorem states.

THEOREM 4.20. (Spectral Theorem for positive operators in $\mathscr{AN}_{[k]}$) *Let P be a positive operator on $\mathscr{H}$. Then the following statements are equivalent.*

1. *$P \in \mathscr{AN}_{[k]}$ for every $k \in \mathbb{N}$.*
2. *$P \in \mathscr{AN}_{[k]}$ for some $k \in \mathbb{N}$.*
3. *P is of the form $P = \alpha I + K + F$, where $\alpha \geqslant 0$, K is a positive compact operator and F is self-adjoint finite-rank operator.*

Proof. (1) implies (2) trivially. (2) implies (3) is due to Theorem 4.12. (1) follows from (3) due to Theorem 4.19. □

4.3. Characterization of operators in $\mathscr{AN}_{[k]}$

In this subsection we extend the preceding theorem to bounded operators. We first state the polar decomposition theorem. Let W be a subspace of $\mathscr{H}$. We use $\text{clos}[W]$ to denote the norm closure of W in $\mathscr{H}$.

THEOREM 4.21. (Polar decomposition Theorem) [7, Page 15] *If $T \in \mathscr{B}(\mathscr{H}, \mathscr{K})$, then there exists a unique partial isometry $U : \mathscr{H} \longrightarrow \mathscr{K}$ with final space $\text{clos}[\text{ran}\, T]$ and initial space $\text{clos}[\text{ran}\, |T|]$ such that $T = U|T|$ and $|T| = U^*T$. If T is invertible, then U is unitary.*

THEOREM 4.22. (Spectral Theorem for operators in $\mathscr{AN}_{[k]}$) *Let $T \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ and let $T = U|T|$ be its polar decomposition. Then the following statements are equivalent.*

1. *$T \in \mathscr{AN}_{[k]}$ for every $k \in \mathbb{N}$.*
2. *$T \in \mathscr{AN}_{[k]}$ for some $k \in \mathbb{N}$.*
3. *$|T|$ is of the form $|T| = \alpha I + F + K$, where $\alpha \geqslant 0$, K is a positive compact operator and F is self-adjoint finite-rank operator.*

Proof. The proof follows from the Proposition 3.6, the polar decomposition theorem and the spectral theorem for positive $\mathscr{AN}_{[k]}$ operators. □

5. Spectral characterization of operators in $\mathscr{AN}_{[\pi,k]}$ and $\mathscr{AN}_{(p,k)}$

5.1. The classes $\mathscr{N}_{[\pi,k]}$ and $\mathscr{AN}_{[\pi,k]}$

We first introduce the class of weighted Ky Fan π, k-norm.

DEFINITION 5.1. (Weighted Ky Fan π, k-norm) Let $(\pi_j)_{j\in\mathbb{N}}$ be a nonincreasing sequence of positive numbers with $\pi_1 = 1$ and let $k \in \mathbb{N}$. The weighted Ky Fan π, k-norm $\|\cdot\|_{[\pi,k]}$ of an operator $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$ is defined to be the weighted sum of the k largest singular values of T, the weights being the first k terms of the sequence $(\pi_j)_{j\in\mathbb{N}}$, that is,

$$\|T\|_{[\pi,k]} = \sum_{j=1}^{k} \pi_j s_j(T).$$

The weighted Ky Fan π, k-norm on $\mathscr{B}(\mathscr{H},\mathscr{K})$ is, indeed, norm; the proof is similar to that of the Ky Fan k-norm. It can be easily shown that on $\mathscr{B}(\mathscr{H})$ it is a symmetric norm.

If we choose $(\pi_j)_{j\in\mathbb{N}}$ to be a constant sequence with each term equals to 1, then weighted Ky Fan π, k-norm $\|\cdot\|_{[\pi,k]}$ is simply the Ky Fan k-norm $\|\cdot\|_{[k]}$. If in addition, we also choose $k = 1$, we get the operator norm.

DEFINITION 5.2. Let $(\pi_j)_{j\in\mathbb{N}}$ be a nonincreasing sequence of positive numbers with $\pi_1 = 1$ and let $k \in \mathbb{N}$. An operator $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$ is said to be $[\pi,k]$-*norming* if there are orthonormal elements $x_1,\dots,x_k \in \mathscr{H}$ such that $\|T\|_{[\pi,k]} = \|Tx_1\| + \pi_2\|Tx_2\| + \dots + \pi_k\|Tx_k\|$. If $\dim(\mathscr{H}) = r < k$, we define T to be $[\pi,k]$-*norming* if there exist orthonormal elements $x_1,\dots,x_r \in \mathscr{H}$ such that $\|T\|_{[\pi,k]} = \|Tx_1\| + \pi_2\|Tx_2\| + \dots + \pi_r\|Tx_r\|$. We let $\mathscr{N}_{[\pi,k]}(\mathscr{H},\mathscr{K})$ denote the set of $[\pi,k]$-norming operators in $\mathscr{B}(\mathscr{H},\mathscr{K})$.

DEFINITION 5.3. Let $(\pi_j)_{j\in\mathbb{N}}$ be a nonincreasing sequence of positive numbers with $\pi_1 = 1$ and let $k \in \mathbb{N}$. An operator $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$ is said to be *absolutely* $[\pi,k]$-*norming* if for every nontrivial closed subspace $\mathscr{M}$ of $\mathscr{H}$, $T|_{\mathscr{M}}$ is $[\pi,k]$-norming. We let $\mathscr{AN}_{[\pi,k]}(\mathscr{H},\mathscr{K})$ denote the set of absolutely $[\pi,k]$-norming operators in $\mathscr{B}(\mathscr{H},\mathscr{K})$. Note that $\mathscr{AN}_{[\pi,k]}(\mathscr{H},\mathscr{K}) \subseteq \mathscr{N}_{[\pi,k]}(\mathscr{H},\mathscr{K})$.

REMARK 5.4. We let Π denote the set of all nonincreasing sequences of positive numbers with their first term equal to 1. Every operator on a finite-dimensional Hilbert space is $[\pi,k]$-norming for any $\pi \in \Pi$ and for any $k \in \mathbb{N}$. However, this is not true when the Hilbert space in question is not finite-dimensional. The operator in Example 4.5 is one such operator. There exists $\tilde{\pi} = (1,1,1,1,\dots)$ such that $A \in \mathscr{N}_{[\tilde{\pi},2]}(\mathscr{H},\mathscr{K})$ but $A \notin \mathscr{N}_{[\tilde{\pi},3]}(\mathscr{H},\mathscr{K})$.

5.2. The classes $\mathcal{N}_{(p,k)}$ and $\mathcal{AN}_{(p,k)}$

Govind S. Mudholkar and Marshall Freimer focussed on a particular class of norms in [10] — the vector p norm of the first k singular values — and found specific results about these norms. Nathaniel Johnston, in one of his blogs *Ky Fan Norms, Schatten Norms, and Everything in Between*, discusses these norms as the natural generalization of two well known families of norms, the Ky Fan norms and the Schatten norms. He coined in the term "(p,k)-singular norm" for this class of norms.

DEFINITION 5.5. ((p,k)-singular norm) [10] Let $p \in [1,\infty)$ and let $k \in \mathbb{N}$. The (p,k)-singular norm $\|\cdot\|_{(p,k)}$ of an operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is defined by

$$\|T\|_{(p,k)} = \left(\sum_{j=1}^{k} s_j^p(T)\right)^{1/p}.$$

The (p,k)-singular norm on $\mathcal{B}(\mathcal{H},\mathcal{K})$ is, indeed, a norm. When $\mathcal{K}=\mathcal{H}$, it can be shown that this norm is symmetric.

If we choose $p=1$, then the $(1,k)$-singular norm $\|\cdot\|_{(1,k)}$ is simply the Ky Fan k-norm $\|\cdot\|_{[k]}$. If in addition, we also choose $k=1$, we get the operator norm.

DEFINITION 5.6. Let $p \in [1,\infty)$ and let $k \in \mathbb{N}$. An operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is said to be (p,k)*-norming* if there are orthonormal elements $x_1,\ldots,x_k \in \mathcal{H}$ such that

$$\|T\|_{(p,k)}^p = \sum_{j=1}^{k} \|Tx_j\|^p.$$

If $\dim(\mathcal{H}) = r < k$, we define T to be (p,k)-norming if there exist orthonormal elements $x_1,\ldots,x_r \in \mathcal{H}$ such that $\|T\|_{(p,k)}^p = \sum_{j=1}^{r} \|Tx_j\|^p$. We let $\mathcal{N}_{(p,k)}(\mathcal{H},\mathcal{K})$ denote the set of (p,k)-norming operators in $\mathcal{B}(\mathcal{H},\mathcal{K})$.

DEFINITION 5.7. Let $p \in [1,\infty)$ and let $k \in \mathbb{N}$. An operator $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is said to be *absolutely* (p,k)*-norming* if for every nontrivial closed subspace $\mathcal{M}$ of $\mathcal{H}$, $T|_{\mathcal{M}}$ is (p,k)-norming. We let $\mathcal{AN}_{(p,k)}(\mathcal{H},\mathcal{K})$ denote the set of absolutely (p,k)-norming operators in $\mathcal{B}(\mathcal{H},\mathcal{K})$. Note that $\mathcal{AN}_{(p,k)}(\mathcal{H},\mathcal{K}) \subseteq \mathcal{N}_{(p,k)}(\mathcal{H},\mathcal{K})$.

REMARK 5.8. Every operator on a finite-dimensional Hilbert space is (p,k)-norming for each $p \in [1,\infty)$ and for each $k \in \mathbb{N}$. However, this is not true when the Hilbert space in question is not finite-dimensional. The operator in Example 4.5 is one such operator. There exists $p_0 = 1$ such that $A \notin \mathcal{N}_{(p_0,3)}(\mathcal{H},\mathcal{K})$.

Henceforth, we use $\mathcal{N}_{[\pi,k]}$, $\mathcal{AN}_{[\pi,k]}$, $\mathcal{N}_{(p,k)}$ and $\mathcal{AN}_{(p,k)}$ for $\mathcal{N}_{[\pi,k]}(\mathcal{H},\mathcal{K})$, $\mathcal{AN}_{[\pi,k]}(\mathcal{H},\mathcal{K})$, $\mathcal{N}_{(p,k)}(\mathcal{H},\mathcal{K})$ and $\mathcal{AN}_{(p,k)}(\mathcal{H},\mathcal{K})$ respectively, as long as the domain and codomain spaces are obvious from the context.

5.3. Characterization of operators in these families

Our purpose of introducing the classes $\mathscr{AN}_{[\pi,k]}$ and $\mathscr{AN}_{(p,k)}$ in the last two sections is to characterize the operators in each of these classes and this subsection is devoted to that purpose. We will omit proofs in order to maintain the reader's orientation in the midst of mazes of detail. However, we will do our best to explain the key ideas used to obtain results. For a detailed exposition of results that lead to characterization theorems of this section we refer the reader to the extended version of this paper (see [12]), where sections 6, 7, and 8 discuss the class $\mathscr{AN}_{[\pi,k]}$ and sections 9, 10, and 11 deals with the class $\mathscr{AN}_{(p,k)}$.

Without going into details, we mention that Proposition 3.6 carries over word for word to operators in $\mathscr{AN}_{[\pi,k]}$ and $\mathscr{AN}_{(p,k)}$.

PROPOSITION 5.9. *Let $\mathscr{H}$ and $\mathscr{K}$ be complex Hilbert spaces and let $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$. Then the following statements are true.*

1. *For every $\pi \in \Pi$ and for every $k \in \mathbb{N}$, $T \in \mathscr{AN}_{[\pi,k]}(\mathscr{H},\mathscr{K}) \iff |T| \in \mathscr{AN}_{[\pi,k]}(\mathscr{H},\mathscr{K})$.*

2. *For every $p \in [1,\infty)$ and for every $k \in \mathbb{N}$, $T \in \mathscr{AN}_{(p,k)}(\mathscr{H},\mathscr{K}) \iff |T| \in \mathscr{AN}_{(p,k)}(\mathscr{H},\mathscr{K})$.*

The following result may be considered as an analogue of Theorem 4.12 and can be proved in much the same way. It exhibits the necessary conditions for a positive operator to be absolutely $[\pi,k]$-norming and absolutely (p,k)-norming respectively.

THEOREM 5.10. *Let $\mathscr{H}$ be a complex Hilbert space and A be a positive operator on $\mathscr{H}$. Then the following statements are true.*

1. *If $\pi \in \Pi$, $k \in \mathbb{N}$, and $A \in \mathscr{AN}_{[\pi,k]}$, then A is of the form $A = \alpha I + K + F$, where $\alpha \geqslant 0$, K is a positive compact operator and F is self-adjoint finite-rank operator.*

2. *If $p \in [1,\infty)$, $k \in \mathbb{N}$, and $A \in \mathscr{AN}_{(p,k)}$, then A is of the form $A = \alpha I + K + F$, where $\alpha \geqslant 0$, K is a positive compact operator and F is self-adjoint finite-rank operator.*

The following result mentions the sufficient conditions for an operator to be absolutely $[\pi,k]$-norming and absolutely (p,k)-norming respectively and hence can be considered as an analogue of Theorem 4.19.

THEOREM 5.11. *Let $K \in \mathscr{B}(\mathscr{H})$ be a positive compact operator, $F \in \mathscr{B}(\mathscr{H})$ be a self-adjoint finite-rank operator, and $\alpha \geqslant 0$. Then the following statements are true.*

1. *$\alpha I + K + F \in \mathscr{AN}_{[\pi,k]}$ for every $\pi \in \Pi$ and for every $k \in \mathbb{N}$.*

2. *$\alpha I + K + F \in \mathscr{AN}_{(p,k)}$ for every $p \in [1,\infty)$ and for every $k \in \mathbb{N}$.*

By Proposition 5.9, Theorem 5.10, Theorem 5.11, and the polar decomposition theorem (see Theorem 4.21), we can safely consider the following theorems to be fully proved.

THEOREM 5.12. (Spectral Theorem for operators in $\mathscr{AN}_{[\pi,k]}$) *Let $\mathscr{H}$ and $\mathscr{K}$ be complex Hilbert spaces of arbitrary dimensions, let $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$ and let $T = U|T|$ be its polar decomposition. Then the following statements are equivalent.*

1. *$T \in \mathscr{AN}_{[\pi,k]}$ for every $\pi \in \Pi$ and for every $k \in \mathbb{N}$.*
2. *$T \in \mathscr{AN}_{[\pi,k]}$ for some $\pi \in \Pi$ and for some $k \in \mathbb{N}$.*
3. *$|T|$ is of the form $|T| = \alpha I + K + F$, where $\alpha \geqslant 0$, K is a positive compact operator and F is self-adjoint finite-rank operator.*

THEOREM 5.13. (Spectral Theorem for operators in $\mathscr{AN}_{(p,k)}$) *Let $\mathscr{H}$ and $\mathscr{K}$ be complex Hilbert spaces of arbitrary dimensions, let $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$ and let $T = U|T|$ be its polar decomposition. Then the following statements are equivalent.*

1. *$T \in \mathscr{AN}_{(p,k)}$ for every $p \in [1,\infty)$ and for every $k \in \mathbb{N}$.*
2. *$T \in \mathscr{AN}_{(p,k)}$ for some $p \in [1,\infty)$ and for some $k \in \mathbb{N}$.*
3. *$|T|$ is of the form $|T| = \alpha I + K + F$, where $\alpha \geqslant 0$, K is a positive compact operator and F is self-adjoint finite-rank operator.*

6. Absolutely norming operators on symmetrically-normed ideals

All the spectral characterization theorems we established in previous sections exhibit a common phenomenon: if an operator in $\mathscr{B}(\mathscr{H},\mathscr{K})$ belongs to one of the families, it belongs to all of them and its absolute value is of the form $\alpha I + K + F$, where $\alpha \geqslant 0$, K is a positive compact operator and F is self-adjoint finite-rank operator. Conversely, any operator in $\mathscr{B}(\mathscr{H},\mathscr{K})$ with its absolute value of the form $\alpha I + K + F$ is absolutely norming with respect to each of the norms we discussed. As a corollary of these results, we have that every positive operator of the form $\alpha I + K + F$ belongs to each of the families $\mathscr{AN}_{[k]}\mathscr{B}(\mathscr{H}), \mathscr{AN}_{[\pi,k]}\mathscr{B}(\mathscr{H})$ and $\mathscr{AN}_{(p,k)}\mathscr{B}(\mathscr{H})$. So, it might appear at this stage that with respect to *every* symmetric norm $\|\cdot\|_s$ on $\mathscr{B}(\mathscr{H})$, the positive operators on $\mathscr{B}(\mathscr{H})$, that are of the above form, are "absolutely s-norming".

If this were true, if the end result of the analysis of absolutely norming operators with respect to various symmetric norms were that they are all of the same form, then this theory would be relatively straightforward. But this is not the case, for we prove the existence of a symmetric norm $\|\cdot\|_{\Phi_\pi^*}$ on $\mathscr{B}(\ell^2(\mathbb{N}))$ with respect to which the identity operator does not attain its norm.

In order to discover this not-so-usual symmetric norm we need to put down some definitions and collect some facts that we will be using in the remaining portion of this paper. Henceforth, we assume $\mathscr{H}$ to be a separable Hilbert space.

DEFINITION 6.1. (Symmetrically-normed ideals) An ideal $\mathfrak{S}$ of the algebra $\mathscr{B}(\mathscr{H})$ is said to be a *symmetrically-normed ideal* (or an *s.n.ideal*) of $\mathscr{B}(\mathscr{H})$ if there is defined on it a symmetric norm $\|.\|_{\mathfrak{S}}$ which makes $\mathfrak{S}$ a Banach space.

We say that two ideals $\mathfrak{S}_I$ and $\mathfrak{S}_{II}$ *coincide elementwise* if $\mathfrak{S}_I$ and $\mathfrak{S}_{II}$ consist of the same elements. If the s.n. ideals $(\mathfrak{S}_I, \|\cdot\|_I)$ and $(\mathfrak{S}_{II}, \|\cdot\|_{II})$ coincide elementwise, then their norms are topologically equivalent.

DEFINITION 6.2. (Symmetric norming function) [9, Chapter 3, Page 71] A function $\Phi : c_{00} \longrightarrow [0,\infty)$ is said to be *symmetric norming function* (or, *s.n. function*) if it satisfies the following properties.

1. $\Phi(\xi) \geqslant 0$ for every $\xi := (\xi_j)_j \in c_{00}$.
2. $\Phi(\xi) = 0 \iff \xi = 0$
3. $\Phi(\alpha\xi) = |\alpha|\Phi(\xi)$ for every scalar $\alpha \in \mathbb{R}$ and $\xi \in c_{00}$.
4. $\Phi(\xi+\psi) \leqslant \Phi(\xi) + \Phi(\psi)$ for every $\xi, \psi \in c_{00}$.
5. $\Phi(1,0,0,\ldots) = 1$.
6. $\Phi(\xi_1, \xi_2, \ldots, \xi_n, 0, 0, \ldots) = \Phi(|\xi_{j_1}|, |\xi_{j_2}|, \ldots, |\xi_{j_n}|, 0, 0, \ldots)$ for every $\xi \in c_{00}$ and $n \in \mathbb{N}$, where $j_1, j_2, \ldots, j_n$ is any permutation of the integers $1, 2, \ldots, n$.

DEFINITION 6.3. [9, Chapter 3, page 76] Two s.n. functions Φ and Ψ are said to be *equivalent* if

$$\sup_{\xi \in c_{00}} \frac{\Phi(\xi)}{\Psi(\xi)} < \infty \quad \text{and} \quad \sup_{\xi \in c_{00}} \frac{\Psi(\xi)}{\Phi(\xi)} < \infty.$$

We say that $\Phi \leqslant \Psi$ if for every every $\xi \in c_{00}$, we have $\Phi(\xi) \leqslant \Psi(\xi)$.

In Gohberg and Krein's text [9], the reader can find a nice exposition on the theory of s.n. ideals, and since we will be dealing with this theory extensively, we have adopted their text as a reader's companion to the remaining portion of this article. Consequently, we have attempted to duplicate their notation wherever possible. We also will frequently use objects that are not defined here but whose definitions may be found in [9].

NOTATION 6.4. (Notations and terminologies) Consider the algebra $\mathscr{B}(\mathscr{H})$ of operators on a separable Hilbert space $\mathscr{H}$. We use $\mathscr{B}_{00}(\mathscr{H})$ to denote the set of all finite-rank operators on $\mathscr{H}$. $\mathscr{B}_0(\mathscr{H})$ and $\mathscr{B}_1(\mathscr{H})$ are, respectively, used to denote the set of all compacts and trace class operators on $\mathscr{H}$. The trace norm is denoted by $\|\cdot\|_1$. These are indeed s.n. ideals. We denote by c_0 the space of all convergent sequences of real numbers with limit 0. We let $c_{00} \subseteq c_0$ denote the linear subspace of c_0 consisting of all sequences with a finite number of nonzero terms. By c_{00}^+ we denote the positive cone of c_{00}. Finally, we let $c_{00}^* \subseteq c_{00}^+$ denote the cone of all nonincreasing nonnegative sequences from c_{00}. To every vector $\xi = (\xi_j)_j \in c_{00}$, we associate the unique vector $\xi^* = (\xi_j^*)_j \in c_{00}^*$, where $\xi_j^* = |\xi_{n_j}|$ for every $j \in \mathbb{N}$ and $n_1, n_2, \ldots, n_j, \ldots$ is a permutation of the positive integers such that the sequence $(|\xi_{n_j}|)_j$ is nonincreasing. Since for any s.n. function Φ, we have $\Phi(\xi) = \Phi(\xi^*)$ for every $\xi \in c_{00}$, it follows that an s.n. function can be uniquely defined by its values on the cone c_{00}^*. Consider the function $\Phi_\infty : c_{00}^* \longrightarrow [0,\infty)$ defined by $\Phi_\infty(\xi) = \xi_1$ for every $\xi = (\xi_j)_j \in c_{00}^*$. This is an s.n. function and is called the *minimal* s.n. function. Next we consider the function $\Phi_1 : c_{00}^* \longrightarrow [0,\infty)$ defined by $\Phi_1(\xi) = \sum_j \xi_j$ for every $\xi = (\xi_j)_j \in c_{00}^*$. This is also an s.n. function and is called the *maximal* s.n. function. If Φ is any s.n. function, then it has been shown that $\Phi_\infty \leqslant \Phi \leqslant \Phi_1$ (see [9, Chapter 3, Section 3, Relation 3.12, Page 76]), which justifies the name "minimal" and "maximal" given to the s.n. functions Φ_∞ and Φ_1 respectively.

We would like to mention few classical results on s.n. ideals generated by an s.n. function. Let Φ be an arbitrary s.n. function and c_Φ be its natural domain (see [9, Chapter 3, Page 80] for the definition of natural domain of an s.n. function). To this function we associate the set $\mathfrak{S}_\Phi$ of all operators $X \in \mathscr{B}_0(\mathscr{H})$ for which $s(X) = (s_j(X))_j \in c_\Phi$. Next we define a (symmetric) norm $\|\cdot\|_\Phi$ on $\mathfrak{S}_\Phi$ by $\|X\|_\Phi := \Phi(s(X))$ for every $X \in \mathfrak{S}_\Phi$. $\mathfrak{S}_\Phi$ thus denotes the s.n. ideal associated to Φ. If Φ, Ψ are s.n. functions and $\mathfrak{S}_\Phi, \mathfrak{S}_\Psi$ are the s.n. ideals generated by these s.n. functions respectively, then $\mathfrak{S}_\Phi$ and $\mathfrak{S}_\Psi$ *coincide elementwise* (that is, consist of the same elements) if and only if Φ and Ψ are equivalent. In particular, if Φ is an s.n. function equivalent to Φ_1, then $\mathfrak{S}_\Phi$ and $\mathscr{B}_1(\mathscr{H})$ coincide elementwise and when Φ is equivalent to Φ_∞, $\mathfrak{S}_\Phi$ and $\mathscr{B}_0(\mathscr{H})$ coincide elementwise. For a given s.n. function there is a notion of its adjoint. The *adjoint* Φ^* of the s.n. function Φ is given by

$$\Phi^*(\eta) = \max\left\{\sum_j \eta_j \xi_j : \xi \in c_{00}^*, \Phi(\xi) = 1\right\}, \text{ for every } \eta \in c_{00}^*,$$

and is itself an s.n. function. The adjoint of Φ^* is Φ. In particular, the minimal and maximal s.n. functions are the adjoint of each other, that is, $\Phi_1^* = \Phi_\infty$ and $\Phi_\infty^* = \Phi_1$. Therefore, when an s.n. function is equivalent to the maximal(minimal) one, its adjoint is equivalent to the minimal(maximal) one. In [9, Chapter 3, section 14] there are examples of s.n. ideals in which the set $\mathscr{B}_{00}(\mathscr{H})$ of finite-rank operators is not dense. This circumstance suggests the necessity of introducing the subspace $\mathfrak{S}_\Phi^{(0)}$, the norm closure of the set $\mathscr{B}_{00}(\mathscr{H})$ in the norm of $\mathfrak{S}_\Phi$, that is,

$$\mathfrak{S}_\Phi^{(0)} := \operatorname{clos}_{\|\cdot\|_\Phi}[\mathscr{B}_{00}(\mathscr{H})].$$

In our exposition we will need the following elementary piece of folklore from [9], whose proof we leave to the reader.

PROPOSITION 6.5. [9, Chapter 3, Theorems 12.2 and 12.4] *Let Φ be an arbitrary s.n. function.*

1. *If Φ is not equivalent to the maximal s.n. function, then the general form of a continuous linear functional f on the separable space $\mathfrak{S}_\Phi^{(0)}$ is given by $f(X) =$*

$Tr(AX)$ for some $A \in \mathfrak{S}_{\Phi^*}$ and

$$\|f\| := \sup\{|Tr(AX)| : X \in \mathfrak{S}_{\Phi}^{(0)}, \|X\|_{\Phi} \leqslant 1\} = \|A\|_{\Phi^*}.$$

Thus, the space adjoint to the space $\mathfrak{S}_{\Phi}^{(0)}$ is isometrically isomorphic to $\mathfrak{S}_{\Phi^*}$, that is,

$$\mathfrak{S}_{\Phi}^{(0)^*} \cong \mathfrak{S}_{\Phi^*}.$$

In particular, if both functions Φ and Φ^* are mononormalizing, the space $\mathfrak{S}_{\Phi}$ is reflexive.

2. *If Φ is equivalent to the maximal s.n. function, then the general form of a continuous linear functional f on the separable space $\mathfrak{S}_{\Phi}$ is given by $f(X) = Tr(AX)$ for some $A \in \mathscr{B}(\mathscr{H})$ and*

$$\|f\| := \sup\{|Tr(AX)| : X \in \mathfrak{S}_{\Phi}, \|X\|_{\Phi} \leqslant 1\} = \|A\|_{\Phi^*}.$$

Thus, the dual space $\mathfrak{S}_{\Phi}^*$ is isometrically isomorphic to $(\mathscr{B}(\mathscr{H}), \|\cdot\|_{\Phi^*})$, that is, $\mathfrak{S}_{\Phi}^* \cong (\mathscr{B}(\mathscr{H}), \|\cdot\|_{\Phi^*})$.

Let Φ be arbitrary s.n. function equivalent to the maximal one. The remaining part of this section is intended to establish the notion of "Φ^*-norming" operators on the s.n. ideal $(\mathscr{B}(\mathscr{H}), \|\cdot\|_{\Phi^*})$ which agrees with the Definitions 3.2, 5.2, and 5.6, and in essentially the same spirit, generalizes the concept. To meet this purpose we need to establish a sequence of propositions which provide us with the machinery required to convert the concept of norming operators in the language of s.n. ideals.

6.1. Norming operators on $\mathscr{B}(\mathscr{H})$

THEOREM 6.6. *Let $T \in \mathscr{B}(\mathscr{H})$. Then $T \in \mathscr{N}$ in the sense of the Definition 1.1 if and only if there exists an operator $K \in \mathscr{B}_1(\mathscr{H})$ with $\|K\|_1 = 1$ such that $|Tr(TK)| = \|T\|$.*

Proof. We first assume that $T \in \mathscr{N}$. Then there exists $x \in \mathscr{H}$, $\|x\| = 1$ such that $\|Tx\| = \|T\|$. Let $y = \frac{Tx}{\|Tx\|}$ and define a rank-one operator $K := x \otimes y \in \mathscr{B}_1(\mathscr{H})$. Notice that $\|K\|_1 = 1$ and $|\mathrm{Tr}(TK)| = |\mathrm{Tr}(T(x \otimes y))| = |\mathrm{Tr}(Tx \otimes y)| = |\langle Tx, y\rangle| = \|Tx\| = \|T\|$ which proves the forward implication.

To see the backward implication, we assume that there exists an operator $K \in \mathscr{B}_1(\mathscr{H})$ such that $\|K\|_1 = 1$ and $|\mathrm{Tr}(TK)| = \|T\|$. Since $\mathscr{B}_1(\mathscr{H}) \subseteq \mathscr{B}_0(\mathscr{H})$, Schmidt expansion allows us to write $K = \sum_{j=1}^{\mathrm{rank}\, K} s_j(K)(x_j \otimes y_j)$, where $\{x_j\}$ is an orthonormal basis of $\operatorname{clos}[\operatorname{ran} K]$ and $\{y_j\}$ is an orthonormal basis of $\operatorname{clos}[\operatorname{ran}|K|]$. We now have

$$\begin{aligned}
\|T\| = |\operatorname{Tr}(TK)| &= \left|\operatorname{Tr}\left(T\left(\sum_{j=1}^{\operatorname{rank} K} s_j(K)(x_j \otimes y_j)\right)\right)\right| \\
&= \left|\sum_{j=1}^{\operatorname{rank} K} s_j(K)\operatorname{Tr}(Tx_j \otimes y_j)\right| = \left|\sum_{j=1}^{\operatorname{rank} K} s_j(K)\left\langle Tx_j, y_j\right\rangle\right| \\
&\leqslant \sum_{j=1}^{\operatorname{rank} K} s_j(K)|\left\langle Tx_j, y_j\right\rangle| \leqslant \sum_{j=1}^{\operatorname{rank} K} s_j(K)\|T\| = \|T\|\|K\|_1 = \|T\|,
\end{aligned}$$

which forces all inequalities to be equalities, so

$$\sum_{j=1}^{\operatorname{rank} K} s_j(K)\|Tx_j\| = \sum_{j=1}^{\operatorname{rank} K} s_j(K)\|T\|,$$

which implies that $\sum_{j=1}^{\operatorname{rank} K} s_j(K)(\|T\| - \|Tx_j\|) = 0$. Notice that, for every j, $s_j(K) > 0$ and $\|T\| - \|Tx_j\| \geqslant 0$. Thus for every j we have $\|T\| = \|Tx_j\|$ which implies that $T \in \mathscr{N}$. This completes the proof. $\square$

6.2. $[2]$-norming operators on $(\mathscr{B}(\mathscr{H}), \|\cdot\|_{[2]})$

LEMMA 6.7. *If Φ and Ψ are s.n. functions defined as*

$$\Phi(\eta) = \max\left\{\eta_1, \frac{\sum_j \eta_j}{2}\right\}, \qquad \Psi(\xi) = \xi_1 + \xi_2,$$

with $\eta = (\eta_i)_{i\in\mathbb{N}}$, and $\xi = (\xi_j)_{j\in\mathbb{N}} \in c_{00}^*$, then Φ and Ψ are mutually adjoint, that is, $\Phi^* = \Psi$ and $\Psi^* = \Phi$.

The above lemma is a special case of Lemma 6.9 and its proof is thus omitted. It is easy to see that Ψ is equivalent to the minimal s.n. function and that it corresponds to the Ky Fan 2-norm on $\mathscr{B}(\mathscr{H})$. Notice that

$$\sup_n \left\{\frac{n}{\Psi^*(\underbrace{1,1,\ldots,1}_{n},0,0,\ldots)}\right\} = \sup_n \left\{\frac{n}{\max\{1,\frac{n}{2}\}}\right\} = 2 < \infty,$$

which implies that $\Phi = \Psi^*$ is equivalent to the maximal s.n. function Φ_1. Consequently, $\mathfrak{S}_\Phi^* \cong (\mathscr{B}(\mathscr{H}), \|\cdot\|_{\Phi^*})$. Moreover, the s.n. ideal $\mathfrak{S}_\Phi$ generated by Φ and the ideal $\mathscr{B}_1(\mathscr{H})$ of trace class operators coincide elementwise. Clearly, Φ and Φ^* are s.n. functions considered on their natural domain instead of merely c_{00}^*.

THEOREM 6.8. *Let $T \in \mathscr{B}(\mathscr{H})$, Φ be an s.n. function equivalent to the maximal s.n. function, defined by*

$$\Phi(\eta) = \max\left\{\eta_1, \frac{\sum_j \eta_j}{2}\right\},$$

where $\eta = (\eta_i)_{i\in\mathbb{N}} \in c_\Phi$*, and let* Φ^* *be its dual norm so that*

$$\mathfrak{S}_\Phi^* \cong (\mathscr{B}(\mathscr{H}), \|\cdot\|_{\Phi^*}),$$

with $\|T\|_{\Phi^*} = \|T\|_{[2]}$ *for every* $T \in \mathscr{B}(\mathscr{H})$*. Then* $T \in \mathscr{N}_{[2]}$ *in the sense of the Definition 3.2 if and only if there exists an operator* $K \in \mathfrak{S}_\Phi = \mathscr{B}_1(\mathscr{H})$ *with* $\|K\|_\Phi = 1$ *such that* $|Tr(TK)| = \|T\|_{\Phi^*}$*.*

Proof. First we assume that $T \in \mathscr{N}_{[2]}$. There exist $x_1, x_2 \in \mathscr{H}$ with $\|x_1\| = \|x_2\| = 1$ and $x_1 \perp x_2$ such that $\|T\|_{\Phi^*} = \|T\|_{[2]} = \|Tx_1\| + \|Tx_2\|$. Let

$$y_1 = \frac{Tx_1}{\|Tx_1\|}, \quad y_2 = \frac{Tx_2}{\|Tx_2\|}$$

and define

$$K := \sum_{j=1}^{2} x_j \otimes y_j.$$

That $K \in \mathscr{B}_1(\mathscr{H})$ and $s_1(K) = s_2(K) = 1$ is obvious, so $\|K\|_\Phi = 1$. Then $|\mathrm{Tr}(TK)| = |\mathrm{Tr}(\sum_{j=1}^2 Tx_j \otimes y_j)| = |\sum_{j=1}^2 \langle Tx_j, y_j\rangle| = \sum_{j=1}^2 \|Tx_j\| = \|T\|_{\Phi^*}$, finishes the proof of the forward implication.

To see the backward implication, we assume that there exists an operator $K \in \mathscr{B}_1(\mathscr{H})$ with $\|K\|_\Phi = 1$ such that $|\mathrm{Tr}(TK)| = \|T\|_{\Phi^*}$. Let $\alpha := \|T\|_{\Phi^*} = \|T\|_{[2]} = s_1(T) + s_2(T)$. Consequently,

$$\begin{aligned}
\alpha = \|T\|_{\Phi^*} = |\mathrm{Tr}(TK)| &= \left|\mathrm{Tr}\left(T\left(\sum_{j=1}^{\mathrm{rank}\,K} s_j(K)(x_j \otimes y_j)\right)\right)\right| \\
&= \left|\sum_{j=1}^{\mathrm{rank}\,K} s_j(K)\langle Tx_j, y_j\rangle\right| \leqslant \sum_{j=1}^{\mathrm{rank}\,K} s_j(K)|\langle Tx_j, y_j\rangle| \\
&= \left\langle \begin{bmatrix} s_1(K) \\ \vdots \\ s_j(K) \\ \vdots \end{bmatrix}, \begin{bmatrix} |\langle Tx_1, y_1\rangle| \\ \vdots \\ |\langle Tx_j, y_j\rangle| \\ \vdots \end{bmatrix} \right\rangle \\
&\leqslant \Phi((s_j(K))_j)\Phi^*((|\langle Tx_j, y_j\rangle|)_j) \\
&= \Phi^*((|\langle Tx_j, y_j\rangle|)_j) \leqslant \|T\|_{\Phi^*} = \alpha.
\end{aligned}$$

This forces $\Phi^*((|\langle Tx_j, y_j\rangle|)_j) = \alpha$. That is, $\|(|\langle Tx_j, y_j\rangle|)_j\|_{\Phi^*} = \alpha$. This observation along with the fact that the sequence $(s_j(K))_j$ is nonincreasing implies that the sequence $(|\langle Tx_j, y_j\rangle|)_j$ is also nonincreasing, that is, $|\langle Tx_1, y_1\rangle| \geqslant |\langle Tx_2, y_2\rangle| \geqslant \ldots \geqslant |\langle Tx_j, y_j\rangle| \geqslant \ldots$; for if it is not, then there exists $\ell \in \mathbb{N}$ such that $|\langle Tx_\ell, y_\ell\rangle| <$

$|\langle Tx_{\ell+1}, y_{\ell+1}\rangle|$ which yields

$$\begin{aligned}\alpha &= \left\langle \begin{bmatrix} \vdots \\ s_\ell(K) \\ s_{\ell+1}(K) \\ \vdots \end{bmatrix}, \begin{bmatrix} \vdots \\ |\langle Tx_\ell, y_\ell\rangle| \\ |\langle Tx_{\ell+1}, y_{\ell+1}\rangle| \\ \vdots \end{bmatrix} \right\rangle < \left\langle \begin{bmatrix} \vdots \\ s_\ell(K) \\ s_{\ell+1}(K) \\ \vdots \end{bmatrix}, \begin{bmatrix} \vdots \\ |\langle Tx_{\ell+1}, y_{\ell+1}\rangle| \\ |\langle Tx_\ell, y_\ell\rangle| \\ \vdots \end{bmatrix} \right\rangle \\ &\leqslant \Phi\left(\begin{bmatrix} \vdots \\ s_\ell(K) \\ s_{\ell+1}(K) \\ \vdots \end{bmatrix}\right) \Phi^*\left(\begin{bmatrix} \vdots \\ |\langle Tx_{\ell+1}, y_{\ell+1}\rangle| \\ |\langle Tx_\ell, y_\ell\rangle| \\ \vdots \end{bmatrix}\right) \\ &= \Phi\left(\begin{bmatrix} \vdots \\ s_\ell(K) \\ s_{\ell+1}(K) \\ \vdots \end{bmatrix}\right) \Phi^*\left(\begin{bmatrix} \vdots \\ |\langle Tx_\ell, y_\ell\rangle| \\ |\langle Tx_{\ell+1}, y_{\ell+1}\rangle| \\ \vdots \end{bmatrix}\right) = \alpha,\end{aligned}$$

which is indeed a contradiction. Consequently, $\alpha = \|(|\langle Tx_j, y_j\rangle|)_j\|_{\Phi^*} = |\langle Tx_1, y_1\rangle| + |\langle Tx_2, y_2\rangle| \leqslant \|Tx_1\| + \|Tx_2\| \leqslant s_1(T) + s_2(T) = \alpha$, which forces $\|Tx_1\| + \|Tx_2\| = s_1(T) + s_2(T)$ thereby establishing that $T \in \mathcal{N}_{[2]}$. This completes the proof. □

6.3. $[\pi, 2]$-norming operators on $(\mathscr{B}(\mathscr{H}), \|\cdot\|_{[\pi,2]})$

LEMMA 6.9. *Given $\pi = (\pi_j)_{j\in\mathbb{N}} \in \Pi$, if Φ and Ψ are s.n. functions defined as*

$$\Phi(\eta) = \max\left\{\eta_1, \frac{\sum_j \eta_j}{1+\pi_2}\right\}, \qquad \Psi(\xi) = \xi_1 + \pi_2\xi_2,$$

where $\eta = (\eta_i)_{i\in\mathbb{N}}$, and $\xi = (\xi_j)_{j\in\mathbb{N}} \in c_{00}^*$, then Φ and Ψ are mutually adjoint, that is, $\Phi^* = \Psi$ and $\Psi^* = \Phi$. Moreover, ϕ is equivalent to the maximal s.n. function and Ψ to the minimal.

Proof. Without much hassle it can be shown that

$$\Psi^*(\eta) = \max\left\{\eta_1 t + \frac{(\sum_{j\neq 1}\eta_j)(1-t)}{\pi_2} : t \in \left[\frac{1}{1+\pi_2}, 1\right]\right\}.$$

But

$$\begin{aligned}\eta_1 t + \frac{(\sum_{j\neq 1}\eta_j)(1-t)}{\pi_2} &= \left(\eta_1 - \left(\frac{\sum_{j\neq 1}\eta_j}{\pi_2}\right)\right)t + \left(\frac{\sum_{j\neq 1}\eta_j}{\pi_2}\right) \\ &= \begin{cases} \eta_1 & \text{when } t = 1 \\ \frac{\sum_j \eta_j}{1+\pi_2} & \text{when } t = \frac{1}{1+\pi_2}, \end{cases}\end{aligned}$$

which implies that

$$\Phi(\eta) = \Psi^*(\eta) = \max\left\{\eta_1, \frac{\sum_j \eta_j}{1+\pi_2}\right\}.$$

The final part of the assertion is trivial. □

Using this result we establish the following theorem.

THEOREM 6.10. *Given $\pi = (\pi_j)_{j\in\mathbb{N}} \in \Pi$, let $T \in \mathscr{B}(\mathscr{H})$ and Φ be an s.n. function equivalent to the maximal s.n. function, defined by*

$$\Phi(\eta) = \max\left\{\eta_1, \frac{\sum_j \eta_j}{1+\pi_2}\right\},$$

where $\eta = (\eta_i)_{i\in\mathbb{N}} \in c_\Phi$, and let Φ^* be its dual norm so that

$$\mathfrak{S}_\Phi^* \cong (\mathscr{B}(\mathscr{H}), \|\cdot\|_{\Phi^*}),$$

with $\|T\|_{\Phi^*} = \|T\|_{[\pi,2]}$ for every $T \in \mathscr{B}(\mathscr{H})$. Then $T \in \mathscr{N}_{[\pi,2]}$ in the sense of the Definition 5.2 if and only if there exists an operator $K \in \mathfrak{S}_\Phi = \mathscr{B}_1(\mathscr{H})$ with $\|K\|_\Phi = 1$ such that $|Tr(TK)| = \|T\|_{\Phi^*}$.

Proof. Let $\{x_1, x_2\} \in \mathscr{H}$ be an orthonormal set such that $\|T\|_{\Phi^*} = \|T\|_{[\pi,2]} = \|Tx_1\| + \pi_2\|Tx_2\|$ and let $y_1 = \frac{Tx_1}{\|Tx_1\|}$, $y_2 = \frac{Tx_2}{\|Tx_2\|}$. Define $K := (x_1 \otimes y_1) + \pi_2(x_2 \otimes y_2)$. Clearly, $K \in \mathscr{B}_1(\mathscr{H})$ and $s_1(K) = 1$, $s_2(K) = \pi_2$ with $\|K\|_\Phi = 1$. Then $|\mathrm{Tr}(TK)| = \left|\left\langle Tx_1, \frac{Tx_1}{\|Tx_1\|}\right\rangle + \pi_2\left\langle Tx_2, \frac{Tx_2}{\|Tx_2\|}\right\rangle\right| = \|T\|_{\Phi^*}$, proves the forward implication. Next we assume that there exists an operator $K \in \mathscr{B}_1(\mathscr{H})$ with $\|K\|_\Phi = 1$ such that $|\mathrm{Tr}(TK)| = \|T\|_{\Phi^*}$. Let $\alpha := \|T\|_{\Phi^*} = \|T\|_{[\pi,2]} = s_1(T) + \pi_2 s_2(T)$. Slightly tweaking the proof of Theorem 6.8 allows us to infer $\|(|\langle Tx_j, y_j\rangle|)_j\|_{\Phi^*} = \alpha$ and that the sequence $(|\langle Tx_j, y_j\rangle|)_j$ is nonincreasing. This yields

$$\begin{aligned}\alpha = \|(|\langle Tx_j, y_j\rangle|)_j\|_{\Phi^*} &= |\langle Tx_1, y_1\rangle| + \pi_2|\langle Tx_2, y_2\rangle| \\ &\leqslant \|Tx_1\| + \pi_2\|Tx_2\| \leqslant s_1(T) + \pi_2 s_2(T) = \alpha,\end{aligned}$$

which forces $\|Tx_1\| + \pi_2\|Tx_2\| = s_1(T) + \pi_2 s_2(T)$ thereby establishing that $T \in \mathscr{N}_{[\pi,2]}$. This completes the proof. □

Given an arbitrary s.n. function Φ that is equivalent to the maximal s.n. function, we are now ready to establish the definition of operators in $\mathscr{B}(\mathscr{H})$ that attain their Φ^*-norm.

DEFINITION 6.11. Let Φ be an s.n. function equivalent to the maximal s.n. function. An operator $T \in (\mathscr{B}(\mathscr{H}), \|\cdot\|_{\Phi^*})$ is said to be Φ^**-norming* if there exists an operator $K \in \mathfrak{S}_\Phi = \mathscr{B}_1(\mathscr{H})$ with $\|K\|_\Phi = 1$ such that $|\mathrm{Tr}(TK)| = \|T\|_{\Phi^*}$. We let $\mathscr{N}_{\Phi^*}(\mathscr{H})$ denote the set of Φ^*-norming operators in $\mathscr{B}(\mathscr{H})$.

The following proposition is a trivial observation and its prinicipal significance lies in the fact that it can be taken as a new equivalent definition of Φ^*-norming operators in $\mathscr{B}(\mathscr{H})$.

PROPOSITION 6.12. *Let Φ be an s.n. function equivalent to the maximal s.n. function and let Φ^* be its dual norm so that $\mathfrak{S}^*_\Phi \cong (\mathscr{B}(\mathscr{H}), \|\cdot\|_{\Phi^*})$. If $T \in \mathscr{B}(\mathscr{H})$ is identified with $f_T \in \mathfrak{S}^*_\Phi$, then the following statements are equivalent.*

1. $T \in \mathscr{N}_{\Phi^*}(\mathscr{H})$.

2. f_T *attains its norm.*

NOTATION 6.13. We let $\mathscr{N}(\mathfrak{S}_\Phi, \mathbb{C})$ denote the set of functionals on $\mathfrak{S}_\Phi$ that attain their norm.

Notice that Theorem 4.13 is a reformulation of the definition of an absolutely $[k]$-norming operators in $\mathscr{B}(\mathscr{H},\mathscr{K})$ by identifying $T|_{\mathscr{M}} \in \mathscr{B}(\mathscr{M},\mathscr{K})$ with $TP_{\mathscr{M}} \in \mathscr{B}(\mathscr{H},\mathscr{K})$; and so are Theorems 7.12 and 10.12 from the extended version of this paper (see [12]) for absolutely $[\pi,k]$-norming and absolutely (p,k)-norming operators, respectively. These reformulations motivate the following definition.

DEFINITION 6.14. Let Φ be an s.n. function equivalent to the maximal s.n. function. An operator $T \in (\mathscr{B}(\mathscr{H}), \|\cdot\|_{\Phi^*})$ is said to be *absolutely Φ^*-norming* if for every nontrivial closed subspace $\mathscr{M}$ of $\mathscr{H}$, $TP_{\mathscr{M}} \in \mathscr{B}(\mathscr{H})$ is Φ^*-norming. We let $\mathscr{A}\mathscr{N}_{\Phi^*}(\mathscr{H})$ denote the set of absolutely Φ^*-norming operators in $\mathscr{B}(\mathscr{H})$.

EXAMPLE 6.15. For any $\pi \in \Pi$ and for any $k \in \mathbb{N}$, choose Φ to be the s.n. function such that $\Phi^* = \|\cdot\|_{[\pi,k]}$. Then $T \in (\mathscr{B}(\mathscr{H}), \|\cdot\|_{\Phi^*})$ belongs to $\mathscr{A}\mathscr{N}_{\Phi^*}(\mathscr{H})$ if and only if $|T|$ is of the form $|T| = \alpha I + F + K$, where $\alpha \geqslant 0$, K is a positive compact operator and F is self-adjoint finite-rank operator.

Given $p \in [1,\infty)$ and $k \in \mathbb{N}$, choose Φ to be the s.n. function such that $\Phi^* = \|\cdot\|_{(p,k)}$. Then $T \in \mathscr{A}\mathscr{N}_{\Phi^*}(\mathscr{H})$ if and only if $|T|$ is of the form $|T| = \alpha I + F + K$, where $\alpha \geqslant 0$, K is a positive compact operator and F is self-adjoint finite-rank operator.

Proof of Proposition 1.3. Let $\{e_i\}_{i\in\mathbb{N}}$ be the canonical orthonormal basis of the Hilbert space $\ell^2(\mathbb{N})$, and let $\pi = (\pi_n)_{n\in\mathbb{N}}$ be a strictly decreasing convergent sequence of positive numbers with $\pi_1 = 1$ such that $\lim_n \pi_n > 0$. Let us define a symmetrically norming function Φ_π by $\Phi_\pi(\xi_1,\xi_2,\ldots) = \sum_j \pi_j \xi_j$. Notice that for every $n \in \mathbb{N}$, we have

$$\frac{n}{\Phi_\pi(\underbrace{1,\ldots,1}_{n \text{ times}},0,0,\ldots)} = \frac{n}{1+\pi_2+\ldots+\pi_n} < \frac{1}{\lim_n \pi_n},$$

which implies $\sup_n \left\{ \frac{n}{\Phi_\pi(\underbrace{1,\ldots,1}_{n \text{ times}},0,0,\ldots)} \right\} < \sup_n \left\{ \frac{1}{\lim_n \pi_n} \right\} < \infty$. Φ_π is thus equivalent to the maximal symmetric norming function Φ_1. The dual $\mathfrak{S}^*_{\Phi_\pi}$ of the symmetrically normed ideal $\mathfrak{S}_{\Phi_\pi}$ is thus isometrically isomorphic to $(\mathscr{B}(\ell^2), \|\cdot\|_{\Phi_\pi^*})$, that is, $\mathfrak{S}_{\Phi_\pi}^* \cong (\mathscr{B}(\ell^2), \|\cdot\|_{\Phi_\pi^*})$, and the $\|\cdot\|_{\Phi_\pi^*}$ norm for any operator $T \in \mathscr{B}(\ell^2)$ is given by $\|T\|_{\Phi_\pi^*} = \sup\{|\mathrm{Tr}(TK)| : K \in \mathscr{B}_1(\ell^2), \|K\|_{\Phi_\pi} = 1\}$, since the ideal $\mathscr{B}_1(\mathscr{H})$ and $\mathfrak{S}_{\Phi_\pi}$ coincide elementwise. We will show that I does not attain its Φ_π^*-norm in $\mathscr{B}(\ell^2)$. To show this, we assume that $I \in \mathscr{N}_{\Phi_\pi^*}(\ell^2)$, and we deduce a contradiction from this assumption.

We first claim that $\alpha := \sup\{|\mathrm{Tr}(K)| : K \in \mathscr{B}_1(\ell^2), \|K\|_{\Phi_\pi} = 1\} = \sup\{|\mathrm{Tr}(K)| : K \in \mathscr{B}_1(\ell^2), K = \mathrm{diag}\{s_1(K), s_2(K), \dots\}, \|K\|_{\Phi_\pi} = 1\} =: \beta$. That $\beta \leqslant \alpha$, is a trivial observation. Let us choose an operator $T \in \mathscr{B}_1(\ell^2)$ with $\|T\|_{\Phi_\pi} = 1$. We define $\tilde{T} := \mathrm{diag}\{s_1(T), s_2(T), \dots, s_j(T), \dots\}$. Notice that for every j, we have $s_j(\tilde{T}) = s_j(T)$ and thus $\|\tilde{T}\|_{\Phi_\pi} = \|T\|_{\Phi_\pi}$ which implies that $\tilde{T} \in \mathscr{B}_1(\ell^2)$. Consequently, $|\mathrm{Tr}(T)| \leqslant |\mathrm{Tr}(\tilde{T})| \leqslant \sup\{|\mathrm{Tr}(K)| : K \in \mathscr{B}_1(\ell^2), K = \mathrm{diag}\{s_1(K), s_2(K), \dots\}, \|K\|_{\Phi_\pi} = 1\} = \beta$. It follows then that $\alpha \leqslant \beta$, thereby establishing our claim. Next we observe that since the trace of a positive trace class diagonal operator is precisely the sum of its singular values, we have $\sup\{|\mathrm{Tr}(K)| : K \in \mathscr{B}_1(\ell^2), K = \mathrm{diag}\{s_1(K), s_2(K), \dots\}, \|K\|_{\Phi_\pi} = 1\} = \sup\{\sum_j s_j(K) : K \in \mathscr{B}_1(\ell^2), K = \mathrm{diag}\{s_1(K), s_2(K), \dots\}, \|K\|_{\Phi_\pi} = 1\}$.

The above two observations lead us to realize that if $I \in \mathscr{N}_{\Phi_\pi^*}$, then the supremum, $\sup\{\sum_j s_j(K) : K \in \mathscr{B}_1(\ell^2), K = \mathrm{diag}\{s_1(K), s_2(K), \dots\}, \|K\|_{\Phi_\pi} = 1\}$, is attained, that is, there exists $K_0 = \mathrm{diag}\{s_1(K_0), s_2(K_0), \dots\} \in \mathscr{B}_1(\mathscr{H})$ with $\sum_j a_j s_j(K_0) = 1$ such that $\|I\|_{\Phi_\pi^*} = |\mathrm{Tr}(K_0)| = \sum_j s_j(K_0)$. Since $K_0 \in \mathscr{B}_1(\ell^2) \subseteq \mathscr{B}_0(\ell^2)$, we have $\lim_{j\to\infty} s_j(K_0) = 0$. This forces the existence of a natural number M such that $s_M(K_0) > s_{M+1}(K_0)$. All that remains is to show the existence of an operator $\tilde{K} \in \mathscr{B}_1(\ell^2)$, $\|\tilde{K}\|_{\Phi_\pi} = 1$ of the form $\tilde{K} = \mathrm{diag}\{s_1(\tilde{K}), s_2(\tilde{K}), \dots\}$ such that $\sum_i s_i(\tilde{K}) > \sum_j s_j(K_0)$. If we define a sequence $(t_i)_{i\in\mathbb{N}}$ by

$$t_i = \begin{cases} \frac{\sum_{j=M}^{M+1} \pi_j s_j(K_0)}{\sum_{j=M}^{M+1} \pi_j} & \text{if } M \leqslant i \leqslant M+1, \\ s_i & \text{if } i < M \text{ or } i > M+1, \end{cases}$$

then it follows that $s_{M+1}(K_0) < t_M = t_{M+1} < s_M(K_0)$, and that $\sum_i \pi_i t_i = \sum_j \pi_j s_j(K_0) = 1$ so that

$$\sum_{i=M}^{M+1} t_i = 2\left(\frac{\sum_{j=M}^{M+1} \pi_j s_j(K_0)}{\sum_{j=M}^{M+1} \pi_j}\right) > \sum_{j=M}^{M+1} s_j(K_0),$$

which implies that

$$\sum_i t_i = \sum_{i=1}^{M+1} t_i + \sum_{i>M+1} t_i > \sum_{j=1}^{M+1} s_j(K_0) + \sum_{j>M+1} s_j(K_0) = \sum_j s_j(K_0).$$

Setting $\tilde{K} := \mathrm{diag}\{t_1, t_2, \dots, t_i, \dots\}$ we observe that $\|\tilde{K}\|_{\Phi_\pi} = 1 < \infty$ so that $\tilde{K} \in \mathscr{B}_1(\ell^2)$ and that it is of the form $\tilde{K} = \mathrm{diag}\{s_1(\tilde{K}), s_2(\tilde{K}), \dots\}$ where $s_i(\tilde{K}) = t_i$ for every i. But then $|\mathrm{Tr}(\tilde{K})| = \sum_i s_i(\tilde{K}) = \sum_i t_i > \sum_j s_j(K_0) = |\mathrm{Tr}(K_0)| = \|I\|_{\Phi_\pi^*}$, which contradicts the assumption that $|\mathrm{Tr}(K_0)|$ is the supremum of the set

$$\left\{\sum_j s_j(K) : K \in \mathfrak{S}_{\Phi_\pi}, K = \mathrm{diag}\{s_1(K), s_2(K), \dots\} \|K\|_{\Phi_\pi} = 1\right\}.$$

Since the operator K_0 with which we began our discussion is arbitrary, it follows that for any given operator in $\mathfrak{S}_{\Phi_\pi}$ with unit norm, one can find another operator in $\mathfrak{S}_{\Phi_\pi}$ with unit norm with trace of larger magnitude and hence the supremum of the above set can never be attained. This shows that the identity operator I does not attain its norm. $\square$

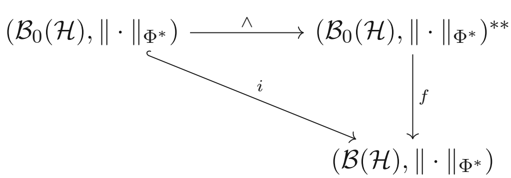
We wish to prove Theorem 6.17, which can be thought of as an analogue of Proposition 4.16 except that we are in the setting of $(\mathscr{B}(\mathscr{H}), \|\cdot\|_{\Phi^*})$ instead of $(\mathscr{B}(\mathscr{H}, \mathscr{K}), \|\cdot\|)$ with Φ being an s.n. function equivalent to the maximal s.n.function. Before proving this theorem we will need the following lemma.

LEMMA 6.16. *Let Φ be an arbitrary s.n. function equivalent to the maximal s.n. function. Then* $(\mathscr{B}_0(\mathscr{H}), \|\cdot\|_{\Phi^*})^{**} \cong (\mathscr{B}(\mathscr{H}), \|\cdot\|_{\Phi^*})$.

Proof. Since Φ is equivalent to the maximal s.n. function Φ_1, the first part of the Proposition 6.5 guarantees $(\mathfrak{S}_\Phi, \|\cdot\|_\Phi)^* \cong (\mathscr{B}(\mathscr{H}), \|\cdot\|_{\Phi^*})$ and $(\mathfrak{S}_{\Phi^*}, \|\cdot\|_{\Phi^*})^* \cong (\mathfrak{S}_\Phi, \|\cdot\|_\Phi)$. Moreover, $\mathfrak{S}_\Phi$ and $\mathscr{B}_1(\mathscr{H})$ coincide elementwise and so does $\mathfrak{S}_{\Phi^*}$ and $\mathscr{B}_0(\mathscr{H})$. Consequently, $(\mathscr{B}_1(\mathscr{H}), \|\cdot\|_\Phi)^* \cong (\mathscr{B}(\mathscr{H}), \|\cdot\|_{\Phi^*})$ and $(\mathscr{B}_0(\mathscr{H}), \|\cdot\|_{\Phi^*})^* \cong (\mathscr{B}_1(\mathscr{H}), \|\cdot\|_\Phi)$ which implies $(\mathscr{B}_0(\mathscr{H}), \|\cdot\|_{\Phi^*})^{**} \cong (\mathscr{B}(\mathscr{H}), \|\cdot\|_{\Phi^*})$. This completes the proof. $\square$

THEOREM 6.17. *Let Φ be an arbitrary s.n. function equivalent to the maximal s.n. function. If $T \in (\mathscr{B}(\mathscr{H}), \|\cdot\|_{\Phi^*})$ is a compact operator, then $T \in \mathscr{A}\mathscr{N}_{\Phi^*}$.*

Proof. If $T \in \mathscr{B}_0(\mathscr{H})$, then $TP_{\mathscr{M}} \in \mathscr{B}_0(\mathscr{H})$ for any closed subspace $\mathscr{M}$ of $\mathscr{H}$. So it suffices to show that $T \in \mathscr{N}_{\Phi^*}(\mathscr{H})$. Since $(\mathscr{B}_0(\mathscr{H}), \|\cdot\|_{\Phi^*})$ and $(\mathscr{B}_0(\mathscr{H}), \|\cdot\|_{\Phi^*})^{**}$ are Banach spaces, the Banach space theory guarantees the existence of the canonical map $\wedge : (\mathscr{B}_0(\mathscr{H}), \|\cdot\|_{\Phi^*}) \longrightarrow (\mathscr{B}_0(\mathscr{H}), \|\cdot\|_{\Phi^*})^{**}$ given by $T \mapsto \hat{T}$ and $\|\hat{T}\| = \max\{|\hat{T}(\varphi)| : \varphi \in (\mathscr{B}_0(\mathscr{H}), \|\cdot\|_{\Phi^*})^*, \|\varphi\| = 1\}$ so that there exists $\varphi_0 \in (\mathscr{B}_0(\mathscr{H}), \|\cdot\|_{\Phi^*})^*$ with $\|\varphi_0\| = 1$ such that $\|\hat{T}\| = |\hat{T}(\varphi_0)| = |\varphi_0(T)|$. Corresponding to this φ_0 there exists a unique $A_0 \in (\mathscr{B}_1(\mathscr{H}), \|\cdot\|_\Phi)$ so that $\varphi_0(X) = \mathrm{Tr}(A_0X)$ for every $X \in (\mathscr{B}_0(\mathscr{H}), \|\cdot\|_{\Phi^*})$ with $\|\varphi_0\| = \|A_0\|_\Phi$. Since the diagram below commutes



where $\wedge$ is the canonical map between the space $(\mathscr{B}_0(\mathscr{H}), \|\cdot\|_{\Phi^*})$ and its double dual, f is the isometric isomorphism resulting from Lemma 6.16, and i is the inclusion map. The operator in $(\mathscr{B}(\mathscr{H}), \|\cdot\|_{\Phi^*})$ associated with $\hat{T} \in (\mathscr{B}_0(\mathscr{H}), \|\cdot\|_{\Phi^*})^{**}$ is the operator T itself. So, $\|T\|_{\Phi^*} = \|\hat{T}\|$ which implies that there exists $A_0 \in (\mathscr{B}_1(\mathscr{H}), \|\cdot\|_\Phi)$ such that $\|\hat{T}\| = |\mathrm{Tr}(A_0T)|$ with $\|A_0\|_\Phi = 1$. This proves that $T \in \mathscr{N}_{\Phi^*}(\mathscr{H})$. $\square$

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REFERENCES

[1] M. D. ACOSTA, *Denseness of norm-attaining operators into strictly convex spaces*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics **129** (1999), no. 06, 1107–1114.

[2] M. D. ACOSTA, F. J. AGUIRRE, AND R. PAY, *A new sufficient condition for the denseness of norm attaining operators*, Journal of Mathematics **26** (1996), no. 2.

[3] M. D. ACOSTA AND M. R. GALAN, *Norm attaining operators and reflexivity*, Rend. Circ. Mat. Palermo **56** (1998), no. 2 Suppl, 171–177.

[4] M. D. ACOSTA AND C. RUIZ, *Norm attaining operators on some classical Banach spaces*, Mathematische Nachrichten **235** (2002), 17–28.

[5] F. J. AGUIRRE, *Norm-attaining operators into strictly convex Banach spaces*, Journal of mathematical analysis and applications **222** (1998), no. 2, 431–437.

[6] X. CARVAJAL AND W. NEVES, *Operators that achieve the norm*, Integral Equations and Operator Theory **72** (2012), no. 2, 179–195.

[7] J. B. CONWAY, *A course in operator theory*, American Mathematical Society, Providence, R. I, 2000.

[8] K. FAN, *Maximum properties and inequalities for the eigenvalues of completely continuous operators*, Proceedings of the National Academy of Sciences **37** (1951), no. 11, 760–766.

[9] I. C. GOHBERG AND M. G. KREIN, *Introduction to the theory of linear nonselfadjoint operators*, American Mathematical Society, Providence, R. I, 1969.

[10] G. S. MUDHOLKAR AND M. FREIMER, *A structure theorem for the polars of unitarily invariant norms*, Proceedings of the American Mathematical Society **95** (1985), no. 3, 331–337.

[11] S. K. PANDEY AND V. I. PAULSEN, *A spectral characterization of $\mathscr{A}\mathscr{N}$ operators*, Journal of the Australian Mathematical Society (2016), 1–23.

[12] S. K. PANDEY, *A spectral characterization of absolutely norming operators on s. n. ideals*, arXiv preprint arXiv:1610.02095 (2016).

[13] J. R. PARTINGTON, *Norm attaining operators*, Israel Journal of Mathematics **43** (1982), no. 3, 273–276.

[14] G. RAMESH, *Structure theorem for $\mathscr{A}\mathscr{N}$-operators*, Journal of the Australian Mathematical Society **96** (2014), no. 3, 386–395.

[15] W. SCHACHERMAYER, *Norm attaining operators and renormings of Banach spaces*, Israel Journal of Mathematics **44** (1983), no. 3, 201–212.

[16] W. SCHACHERMAYER, *Norm attaining operators on some classical Banach spaces*, Pacific Journal of Mathematics **105** (1983), no. 2, 427–438.

[17] S. SHKARIN, *Norm attaining operators and pseudospectrum*, Integral Equations and Operator Theory **64** (2009), no. 1, 115–136.

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