Operators
and
Matrices
Volume 11, Number 3 (2017), 875–884
doi:10.7153/oam-11-61

C–E TYPE TOEPLITZ OPERATORS ON $L_a^2(\mathbb{D})$

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(*Communicated by I. M. Spitkovsky*)

Abstract. In this paper, we initiate the study of a new class of conditional type operators, which we call C-E type Toeplitz operators. Sufficient conditions for boundedness and compactness of C-E type Toeplitz operators on the Bergman space $L_a^2(\mathbb{D})$ will be presented. Also, some differences between C-E type Toeplitz operators and Toeplitz operators will be illustrated by examples.

1. Introduction and preliminaries

Let (X,Σ,μ) be a probability measure space and let $\mathscr{A}$ be a subalgebra of Σ. All sets and functions statements are to be interpreted as being valid almost everywhere with respect to μ. The collection of $\mathscr{A}$-measurable complex-valued functions on X will be denoted by $L^0(\mathscr{A})$. We take $L^2(\mathscr{A}) = L^2(X,\mathscr{A},\mu_{|_{\mathscr{A}}})$. For each non-negative function $f \in L^0(\Sigma)$ or $f \in L^2(\Sigma)$, by the Radon-Nikodym theorem, there exists a unique $\mathscr{A}$-measurable function $E^{\mathscr{A}}(f)$ such that

$$\int_A f d\mu = \int_A E^{\mathscr{A}}(f) d\mu,$$

where A is any $\mathscr{A}$-measurable set for which $\int_A f d\mu$ exists. Now associated with every subalgebra $\mathscr{A} \subseteq \Sigma$, the mapping $E^{\mathscr{A}} : L^2(\Sigma) \to L^2(\mathscr{A})$, uniquely defined by the assignment $f \mapsto E^{\mathscr{A}}(f)$, is called the conditional expectation operator with respect to $\mathscr{A}$. As an operator on $L^2(\Sigma)$, $E^{\mathscr{A}}$ is a contractive orthogonal projection onto $L^2(\mathscr{A})$. For fix $\mathscr{A} \subseteq \Sigma$, set $E^{\mathscr{A}} = E$. The domain of E contains $L^1(\Sigma) \cup L^0_+(\Sigma)$, where $L^0_+(\Sigma) = \{f \in L^0(\Sigma) : f \geqslant 0\}$. For more details on conditional expectation see [12]. Recall that an $\mathscr{A}$-atom of the measure μ is an element $C \in \mathscr{A}$ with $\mu(C) > 0$ such that for each $F \in \mathscr{A}$, if $F \subseteq C$ then either $\mu(F) = 0$ or $\mu(F) = \mu(C)$. A measure with no atoms is called non-atomic. It is well-known fact that every σ-finite measure space $(X,\mathscr{A},\mu_{|_{\mathscr{A}}})$ can be partitioned uniquely as $X = (\cup_{n\in\mathbb{N}} C_n) \cup B$, where $\{C_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint $\mathscr{A}$-atoms and B, being disjoint from each C_n, is non-atomic (see [13]). Note that every $L^2(\mathscr{A})$-function is constant on any $\mathscr{A}$-atom.

We now restrict our attention to the case $(\mathbb{D},\mathscr{M},A)$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathscr{M}$ is the sigma-algebra of Lebesgue-measurable sets in $\mathbb{D}$ and A =normalized

Mathematics subject classification (2010): Primary 47B20, Secondary 47B38.
Keywords and phrases: Bergman space, Toeplitz operator, conditional expectation.

Paper OaM-11-61

area measure in $\mathbb{D}$. For $1 \leqslant p < \infty$, the Bergman space $L_a^p(\mathbb{D}) = L_a^p(\mathscr{M})$ is a closed subspace of $L^p(\mathscr{M})$ consisting of analytic functions. Let P be the Bergman projection. For $u \in L^\infty(\mathscr{M})$, the operator T_u defined on $L_a^2(\mathbb{D})$ by $T_u f = P(uf)$ is called Toeplitz operator. When $u \in H^\infty(\mathbb{D})$, the space of bounded analytic functions on $\mathbb{D}$, then T_u is reduced to the multiplication operator M_u. For general information in this context one can refer to excellent monograph [14].

The study of conditional expectation operator and its applications on the space of analytic functions has a long history. In [3] Ball investigated conditional expectation operator induced by an inner function on the Hardy space and obtained some result about it. In [1] Aleksandrov proved that conditional expectation operator commutes with Riesz projection if and only if the measurable partition of circumference has been induced by an inner function. Attele [2] used properties of conditional expectation operator and obtained some results about the multiplier of range of composition operators. This type of operator in the Bergman space was studied for the first time in the paper by Carswell and Stessin in [4]. In [8] the first author and Hassanloo extend some result in [4] to larger classes of sigma-algebras.

The operator $T = EM_u$ have been defined as combination of multiplication operator and conditional expectation operator. Lambert, the first author of this note and others have obtained many property of T such as boundedness, compactness, spectrum and so on. For more details about this type of operators one can refer to [5] and [6]. In this paper, we introduce the concept of C-E type Toeplitz operators, PT, on the Bergman space $L_a^2(\mathbb{D})$ and present some algebraic and analytic properties of these types of operators. In Examples 2.7(i), 2.13, 2.16 and 2.19 we see that C-E type Toeplitz operators with same properties for u, have different behavior relative to Toepliz operators. In particular, a sufficient condition for boundedness and compactness of mentioned operators will be presented.

2. C-E type Toeplitz operators on $L_a^2(\mathbb{D})$

Suppose $\mathscr{M}$ is the σ-algebra of Lebesgue-measurable sets in $\mathbb{D}$ and $\mathscr{A}$ is a subalgebra of $\mathscr{M}$ and $E = E^{\mathscr{A}}$ is the related conditional expectation operator. For a nonconstant analytic self-map φ on $\mathbb{D}$, it may be happen that $\mathscr{A}(\varphi) := \varphi^{-1}(\mathscr{M})$. For $z \in \mathbb{D}$, put $c_z = \varphi^{-1}(\varphi(z)) \cap \mathbb{D}_0 = \{\xi \in \mathbb{D}_0 : \varphi(\xi) = \varphi(z)\}$, where $\mathbb{D}_0 = \{\xi \in \mathbb{D} : \varphi'(\xi) \neq 0\}$. We say that φ has finite multiplicity if there exists $N \in \mathbb{N}$ such that for each $z \in \mathbb{D}$, the level set c_z contains at most N points.

LEMMA 2.1. [8, Theorem 2.1] *Suppose that $\mathscr{A} = \mathscr{A}(\varphi)$ for some self-map $\varphi : \mathbb{D} \to \mathbb{D}$ with finite multiplicity. Then for each $f \in L_a^p(\mathbb{D})$ and $z \in \mathbb{D}_0 = \cup_{z\in\mathbb{D}} c_z$ we have*

$$E(f)(z) = \left(\sum_{\xi \in c_z} \frac{f(\xi)}{|\varphi'(\xi)|^2} \right) \left(\sum_{\xi \in c_z} \frac{1}{|\varphi'(\xi)|^2} \right)^{-1}.$$

Notice that if $EP = PE$, then $E(L_a^2(\mathbb{D})) \subseteq L_a^2(\mathbb{D})$.

DEFINITION 2.2. For $u \in L^{\infty}(\mathscr{M})$, the C-E type Toeplitz operator induced by the pair (u, E) is denoted by T_u^E and defined as follows:

$$T_u^E = PEM_u : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$$

$$f \to PE(uf),$$

where M_u is the multiplication operator. Note that, since $uf \in L^2(\mathscr{M}) \subseteq \mathscr{D}(E)$ and $E(uf) \in L^2(\mathscr{A}) \subseteq L^2(\mathscr{M})$, so the linear operator T_u^E is well defined.

Let $\mathfrak{T}_2 = \{u \in L^2(\mathscr{M}) : uL_a^2(\mathbb{D}) \subseteq L^2(\mathscr{M})\}$. Note that $L^{\infty}(\mathscr{M}) \subseteq \mathfrak{T}_2 \subseteq L^1(\mathscr{M})$ and that $\mathfrak{T}_2$ is a vector space. For $u \in \mathfrak{T}_2$, let $\mathscr{T}_u^E$ be the corresponding C-E type Toeplitz operator. For $u \in L^{\infty}(\mathscr{M})$, $\mathscr{T}_u^E = T_u^E$. So $\mathscr{T}_u^E$ is a generalization of T_u^E.

LEMMA 2.3. *Let $u \in L^2(\mathscr{M})$. Then the operator $EM_u : L^2(\mathscr{M}) \to L^2(\mathscr{A})$ is bounded if and only if $E(|u|^2) \in L^{\infty}(\mathscr{A})$, and in this case $\|EM_u\| = \|\sqrt{E(|u|^2)}\|_{\infty}$.*

Proof. Let $T : L^2(\mathscr{A}) \to L^2(\mathscr{M})$ defined by $Tf = \bar{u}f$. If $E(|u|^2) \in L^{\infty}(\mathscr{A})$, then for each $f \in L^2(\mathscr{A})$,

$$\|Tf\|^2 = \int_{\mathbb{D}} |\bar{u}f|^2 dA = \int_{\mathbb{D}} E(|u|^2)|f|^2 dA \leqslant \|E(|u|^2)\|_{\infty}\|f\|^2.$$

Conversely, let T is bounded. Then for each $B \in \mathscr{A}$,

$$\int_B E(|u|^2) dA = \int_{\mathbb{D}} |\bar{u}\chi_B|^2 dA = \|T\chi_B\|^2 \leqslant \|T\|^2 A(B).$$

Hence,

$$\|E(|u|^2)\|_{\infty} = \sup_{\{B \in \mathscr{A}, A(B)>0\}} \frac{1}{A(B)} \int_B E(|u|^2) dA \leqslant \|T\|^2.$$

Now, it is easy to show that the adjoint operator $T^* : L^2(\mathscr{M}) \to L^2(\mathscr{A})$ is given by $T^*f = E(uf)$. This completes the proof. □

PROPOSITION 2.4. *If $E(|u|^2) \in L^{\infty}(\mathscr{A})$, then $\mathscr{T}_u^E$ is a bounded linear operator on $L_a^2(\mathbb{D})$.*

Proof. Since the Bergman projection P has norm 1 and $E(|u|^2) \in L^{\infty}(\mathscr{A})$, then by Lemma 2.3 we have $\|\mathscr{T}_u^E\| \leqslant \|EM_u\| = \|\sqrt{E(|u|^2)}\|_{\infty}$. □

If $\mathscr{M} = \mathscr{A}$, then $E = I$, the identity operator. In this case $\mathscr{T}_u^E = \mathscr{T}_u$

EXAMPLE 2.5. (i) Let $\mathscr{A} = \{\emptyset, \mathbb{D}\}$. Then $E(f)(z) = \int_{\mathbb{D}} f(w) dA(w)$, and so

$$\mathscr{T}_u^E(f)(z) = P(E(uf))(z) = \int_{\mathbb{D}} u(w) f(w) dA(w).$$

It follows that if $u \in \mathfrak{T}_2 \setminus L^{\infty}(\mathscr{M})$, then $E(|u|^2) = \|u\|^2$, and hence $\mathscr{T}_u^E$ is bounded. Now, let $\mathscr{A} = \langle C_i \rangle$ be the algebra generated by the countable collection of the non-null disjoint Lebesgue measurable subsets of $\mathbb{D}$ such that their union is $\mathbb{D}$. In this case (see [12])

$$E(f)=\sum_{i=1}^{\infty}\frac{1}{A(C_i)}\left(\int_{C_i} f dA\right)\chi_{C_i}.$$

Then

$$\begin{aligned}\mathscr{T}_u^E(f)(z)&=\int_{\mathbb{D}}\frac{1}{(1-z\bar{w})^2}E(uf)(w)dA(w)\\&=\int_{\mathbb{D}}\frac{1}{(1-z\bar{w})^2}\left(\sum_{i=1}^{\infty}\frac{1}{A(C_i)}\left(\int_{C_i}(uf)(z)dA(z)\right)\chi_{C_i}(w)\right)dA(w).\end{aligned}$$

Since

$$\begin{aligned}\int_{\mathbb{D}}\frac{1}{|(1-z\bar{w})|^2}\left(\sum_{i=1}^{\infty}\frac{1}{A(C_i)}\left(\int_{C_i}|u(z)f(z)|dA(z)\right)\chi_{C_i}(w)\right)dA(w)\\ \leqslant \|K_z\|_2\|E(|uf|)\|_2,\end{aligned}$$

it holds that

$$\begin{aligned}\mathscr{T}_u^E(f)(z)&=\sum_{i=1}^{\infty}\frac{\chi_{C_i}(w)}{A(C_i)}\int_{C_i}\left(\int_{\mathbb{D}}\frac{u(z)f(z)}{(1-z\bar{w})^2}dA(z)\right)dA(w)\\&=E\overline{P(\overline{uf})}(z).\end{aligned}$$

(ii) For $1<n\in\mathbb{N}$, let $\varphi(z)=z^n$. For $z\in\mathbb{D}$ let $\varphi^{-1}(\varphi(z))=\{(z^n)^{1/n}\}=\{z_1,\cdots,z_n\}$, where $z_k=|z|e^{i\theta_k}$ with $\theta_k=(\arg z^n+2k\pi)/n$. So for $1\leqslant k\leqslant n$, $|z_k|=|z|$ and thus $|\varphi'(z_k)|^2=|nz_k^{n-1}|^2=n^2|z|^{2(n-1)}\neq 0$, for $z\in\mathbb{D}_0=\mathbb{D}\setminus\{0\}$. Let $\mathscr{A}=\mathscr{A}(\varphi)$ be the subalgebra of $\mathscr{M}$ generated by $\{(z^n)^{-1}(U):U\subset\mathbb{D}$ is open$\}$. Then by Lemma 2.1 we have

$$\begin{aligned}E(f)(z)&=\left(\sum_{k=1}^{n}\frac{f(z_k)}{n^2|z|^{2(n-1)}}\right)\left(\frac{1}{n|z|^{2(n-1)}}\right)^{-1}\\&=\frac{1}{n}\sum_{k=1}^{n}f(z_k)=\frac{1}{n}\sum_{\{\zeta:\zeta^n=z^n\}}f(\zeta),\qquad f\in L^2(\mathscr{M}),\ z\in\mathbb{D}_0.\end{aligned}$$

Note that the point $z=0$ is an isolated singularity for Ef. Since Ef is bounded in a deleted neighborhood of point 0 in $\mathbb{D}$, so we can obtain holomorphic extension of Ef and define it on $\mathbb{D}$. Furthermore E is an averaging operator. Hence $nE(|u|^2)(z)\geqslant |u(z)|^2$ for every $z\in\mathbb{D}_0$ and

$$\begin{aligned}(\mathscr{T}_u^E f)(z)=P(E(uf))(z)&=\int_{\mathbb{D}}K(z,w)E(uf)(w)dA(w)\\&=\frac{1}{n}\sum_{k=1}^{n}\int_{\mathbb{D}}\frac{u(w_k)f(w_k)}{(1-z\bar{w})^2}dA(w),\end{aligned}$$

where $w_k = |w|e^{i(\arg w^n + 2k\pi)/n}$ and $f \in L^2_a(\mathbb{D})$. Consequently, since E is a contraction and $nE(|u|^2) \geqslant |u|^2$, $u \in L^\infty(\mathscr{M})$ if and only if $E(|u|^2) \in L^\infty(\mathscr{M})$. In this case, by Proposition 2.4, $\mathscr{T}_u^E = T_u^E$ is bounded. In case $n=2$, $Ef(z) = \frac{f(z)+f(-z)}{2}$. So $\mathscr{T}_u^E f = \frac{1}{2}\{T_u(f) + T_w(f \circ g)\}$, where $w = u \circ g$ and $g(z) = -z$. Moreover, since $E(L^2_a(\mathbb{D})) \subseteq L^2_a(\mathbb{D})$ then $\mathscr{T}_u^E(f) = \frac{1}{2}\{M_u(f) + M_w(f \circ g)\}$ for every $u \in H^\infty(\mathbb{D})$. For $f \in L^2_a(\mathbb{D})$, put $A_\varphi(f)(z) = \frac{1}{n}\sum_{\{\zeta:\zeta^n = z\}} f(\zeta)$. Then $E(f) = A_\varphi(f) \circ \varphi$. The function $A_\varphi(f)$ is uniquely determined in $\mathbb{D}_0$. Therefore, even though φ is not invertible, the expression $A_\varphi(f) = (E(f)) \circ \varphi^{-1}$ is well defined (see [10]).

(iii) Let $\varphi(z) = az^2 + bz + c$ where $a \neq 0$, $|a| + |b| + |c| < 1$, and let $\mathscr{A} = \mathscr{A}(\varphi)$. For each $z \in \mathbb{D}$, $\varphi^{-1}(\varphi(z)) = \{z, -(az+b)/a\}$. But $-(az+b)/a$ may be in $\mathbb{D}$ or not. Put $\mathbb{D}_1 = \{z \in \mathbb{D}_0 : |az+b| < |a|\}$, where $\mathbb{D}_0 = \mathbb{D} \setminus \{-b/2a\}$. Since for each $z \in \mathbb{D}_1$, $|\varphi'(z)| = |2az+b| = |\varphi'(-(az+b)/a)|$ then according to Lemma 2.1 we have

$$Ef(z) = \begin{cases} \frac{1}{2}\{f(z) + f(-\frac{az+b}{a})\} & z \in \mathbb{D}_1 \\ f(z) & z \in \mathbb{D}_0 \setminus \mathbb{D}_1, \end{cases}$$

where $f \in L^2(\mathscr{M})$. It follows that for each $u \in \mathfrak{T}_2$ and $f \in L^2_a(\mathbb{D})$ we have

$$\mathscr{T}_u^E(f)(z) = \begin{cases} \int_{\mathbb{D}} \left\{ \frac{(uf)(w)}{2(1-z\overline{w})^2} + \frac{(uf)(-\frac{aw+b}{a})}{2(1-z\overline{w})^2} \right\} dA(w) & z \in \mathbb{D}_1 \\ \mathscr{T}_u(f)(z) & z \in \mathbb{D}_0 \setminus \mathbb{D}_1, \end{cases}$$

PROPOSITION 2.6. *Suppose a and b are complex numbers and u and v are in $\mathfrak{T}_2$ such that the C-E type Toeplitz operators induced by them are bounded. Then*
(a) $\mathscr{T}_{au+bv}^E = a\mathscr{T}_u^E + bT_v^E$ and $(\mathscr{T}_u^E)^* = PM_{\bar{u}}E$;
(b) If u be a $\mathscr{A}$-measurable and $u \geqslant 0$, then $\mathscr{T}_u^E \geqslant 0$.

Proof. (a) The first equality follows from $M_{au+bv} = aM_u + bM_v$. Let $f, g \in L^2_a(\mathbb{D})$. Then

$$\begin{aligned} \langle (\mathscr{T}_u^E)^* f, g\rangle &= \langle f, \mathscr{T}_u^E g\rangle = \langle f, PE(ug)\rangle \\ &= \langle f, E(ug)\rangle = \langle Ef, M_u g\rangle \\ &= \langle M_{\bar{u}}Ef, Pg\rangle = \langle PM_{\bar{u}}Ef, g\rangle. \end{aligned}$$

So $(\mathscr{T}_u^E)^* = PM_{\bar{u}}E$.

(b) Since E is an orthogonal projection and $E(uf) = uEf$ for all $f \in L^2_a(\mathbb{D})$, then

$$\begin{aligned} \langle \mathscr{T}_u^E f, f\rangle &= \langle PEM_u f, f\rangle = \langle EM_u f, Pf\rangle \\ &= \langle EM_u f, f\rangle = \langle EM_u f, Ef\rangle \\ &= \langle M_u Ef, Ef\rangle = \int_{\mathbb{D}} u|Ef|^2 dA \geqslant 0. \quad \square \end{aligned}$$

For classical Toeplitz operator $\mathscr{T}_u = PM_u$ on $L^2_a(\mathbb{D})$, $\mathscr{T}_u \equiv 0$ if and only if $u \equiv 0$. But the analogous fact does not hold for C-E type Toeplitz operators, in general.

EXAMPLE 2.7. (i) Suppose again that $\mathscr{A}=\{\emptyset,\mathbb{D}\}$ and u is a nonzero analytic function on $\mathbb{D}$ such that $u(0)=0$. According to Example 2.5(i) and mean value property of harmonic functions, $\mathscr{T}_u^E\equiv 0$ on $L_a^2(\mathbb{D})$, but $u\not\equiv 0$.

(ii) It seems that $\mathscr{T}_u^E\equiv 0$ whenever $Eu=0$. But it is not hold in general. For this, let $\varphi(z)=z^2$, $u(z)=z$ and $\mathscr{A}=\mathscr{A}(\varphi)$. According to Example 2.5(ii), it follows that $Eu=0$ and

$$\mathscr{T}_u^E f=\frac{1}{2}(P(zf(z))+P(-zf(-z))).$$

Now, if we put $f(z)=z$, then $\mathscr{T}_u^E f=P(z^2)\neq 0$.

PROPOSITION 2.8. *Suppose u is an $\mathscr{A}$-measurable function on $\mathbb{D}$. Then $\mathscr{T}_u^E\equiv 0$ implies that $u\equiv 0$ if and only if the linear combinations of $\{\overline{E(z^i)}E(z^j)\}_{i,j=0}^{\infty}$ is dense in $L^2(\mathscr{A})$.*

Proof. Let M denotes the linear combination of $\{\overline{E(z^i)}E(z^j)\}_{i,j=0}^{\infty}$. Suppose that $\mathscr{T}_u^E\equiv 0$ and M is dense in $L^2(\mathscr{A})$. Then $\langle\mathscr{T}_u^E f,g\rangle=0$ for every $f,g\in L_a^2(\mathbb{D})$. Since P and E are projection, u is an $\mathscr{A}$-measurable and $P(z^j)=z^j$, we obtain

$$\begin{aligned}\langle u,\overline{E(z^i)}E(z^j)\rangle &= \langle uE(z^i),E(z^j)\rangle=\langle E(uz^i),z^j\rangle\\ &=\langle PE(uz^i),z^j\rangle=\langle\mathscr{T}_u^E z^i,z^j\rangle=0,\end{aligned}$$

for all $i,j\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Since M is dense in $L^2(\mathscr{A})$, it concluded that $u\equiv 0$. Conversely suppose that M is not dense in $L^2(\mathscr{A})$, therefore $M^{\perp}\neq\{0\}$. Let $0\not\equiv u\in M^{\perp}$. Simple computations show that $\mathscr{T}_u^E\equiv 0$. $\square$

COROLLARY 2.9. *Suppose u is an $\mathscr{A}$-measurable function on $\mathbb{D}$ and M denotes the linear combinations of $\{\overline{E(z^i)}E(z^j)\}_{i,j=0}^{\infty}$. Then the following assertions hold.*

(i) $\mathscr{T}_u^E$ is self-adjoint if and only if $u-\bar{u}$ is perpendicular to $\overline{M}$.

(ii) If $\overline{M}=L^2(\mathscr{A})$, then $\mathscr{T}_u^E$ is self-adjoint if and only if $u=\bar{u}$.

We recall that for a bounded linear operator T on $L_a^2(\mathbb{D})$, the Berezin transform of T is denoted by $\widetilde{T}$ and defined as $\widetilde{T}(z)=\langle Tk_z,k_z\rangle$, for each $z\in\mathbb{D}$. It is proved in [11] that $T\equiv 0$ if and only if $\widetilde{T}\equiv 0$. If $u\in L^2(\mathscr{M})$ then the Berezin transform of u that denoted by $\widetilde{u}$ defined as the Berezin transform of $\mathscr{T}_u$. Denoted by B the transform that $B(u)=\widetilde{u}$. Then B is one-to-one on $L^1(\mathscr{M})$ (see [7]).

DEFINITION 2.10. Let $u\in\mathfrak{T}_2$. The C-E Berezin transform of u is denoted by $\widetilde{u^E}$ and defined on $\mathbb{D}$ as $\widetilde{u^E}(z)=\langle\mathscr{T}_u^E k_z,k_z\rangle$ and will be denoted by B^E the transform that $B^E(u)=\widetilde{u^E}$.

COROLLARY 2.11. *Let M denotes the linear combinations of $\{E(z^i)\overline{E(z^j)}\}$. Then B^E is one-to-one on $\overline{M}$.*

REMARK 2.12. It is clear that if $E=I$, previous corollary coincides with classical statement about B.

It is well-known fact that when $u \in L^1(\mathscr{M})$ is harmonic then $\widetilde{u} = u$ (see [14]). As will be shown in the next example, the analogous fact does not hold for C-E Berezin transform of u in general.

EXAMPLE 2.13. Let $\mathscr{A} = \{\emptyset, \mathbb{D}\}$. Then

$$\mathscr{T}_u^E(f)(z) = \int_{\mathbb{D}} u(w)f(w)dA(w).$$

Therefore

$$\begin{aligned}\widetilde{u^E}(z) &= \int_{\mathbb{D}} \mathscr{T}_u^E(k_z)(w)\overline{k_z}(w)dA(w) \\ &= \int_{\mathbb{D}} u(t)k_z(t)dA(t)\int_{\mathbb{D}} \overline{k_z}(w)dA(w) = (1-|z|^2)^2\overline{P(\bar{u})}(z).\end{aligned}$$

Putting $u(z) = z^2 + z$, we have $P(\bar{u}) = 0$. Hence $\widetilde{u^E}(z) = 0$, and so $\widetilde{u^E} \neq u$ in this case.

In the following we present a sufficient condition for compactness of some C-E type Toeplitz operators on $L_a^2(\mathbb{D})$ and by examples illustrate the difference between compactness of Toeplitz operators and C-E type Toepliz operators on $L_a^2(\mathbb{D})$.

THEOREM 2.14. *Suppose that* $\mathscr{A}$ *is a subalgebra of* $\mathscr{M}$, $(\mathbb{D}, \mathscr{A}, A_{|_{\mathscr{A}}})$ *can be partitioned as* $\mathbb{D} = (\cup_{n\in\mathbb{N}} C_n)\bigcup B$ *and* $T = \mathscr{T}_u^E$ *is bounded on* $L_a^2(\mathbb{D})$. *If* $u(B) = 0$ *(*$u(z) = 0$ *for all* $z \in B$*) and for any* $\varepsilon > 0$, $A(C_n \cap G^{\varepsilon}(u)) > 0$ *for finitely many* n, *where* $G^{\varepsilon}(u) = \{z \in \mathbb{D} : E(|u|)(z) \geqslant \varepsilon\}$, *then* $\mathscr{T}_u^E$ *is compact.*

Proof. Suppose that $u(B) = 0$ and for an arbitrary $\varepsilon > 0$, the number of $\mathscr{A}$-atoms $\{C_n^{\varepsilon}\}$ such that $A(C_n^{\varepsilon} \cap G^{\varepsilon}(u)) > 0$ is $k < \infty$. Put $B_{\varepsilon} = \cup_{n=1}^k C_n^{\varepsilon}$. It is clear that $E(|u|)(z) < \varepsilon$ on $\mathbb{D} \setminus B_{\varepsilon}$ and therefore $|u| < \varepsilon$ on $\mathbb{D} \setminus (B_{\varepsilon} \cup B)$. Let $T_1 = \mathscr{T}_v^E$ where $v = \chi_{B_{\varepsilon}} u$. Since $u = v = 0$ on B, $u = v$ on B_{ε} and T is bounded, hence T_1 is bounded. Using $|E(f)|^2 \leqslant E(|f|^2)$ and $B_{\varepsilon} \cup B \in \mathscr{A}$, for each $f \in L_a^2(\mathbb{D})$ we have

$$\begin{aligned}\|Tf - T_1 f\|^2 \leqslant \|Euf - Evf\|^2 &= \int_{\mathbb{D}} |E(u-v)f|^2 dA \\ &= \int_{\mathbb{D}\setminus(B_{\varepsilon}\cup B)} |Euf|^2 dA \leqslant \int_{\mathbb{D}\setminus(B_{\varepsilon}\cup B)} E(|uf|^2)dA \\ &= \int_{\mathbb{D}\setminus(B_{\varepsilon}\cup B)} |uf|^2 dA \leqslant \varepsilon^2 \int_{\mathbb{D}} |f|^2 dA = \varepsilon^2 \|f\|^2.\end{aligned}$$

But we have

$$T_1 f = PE(\chi_{B_{\varepsilon}} uf) = PE(\sum_{n=1}^k \chi_{C_n^{\varepsilon}} uf) = \sum_{n=1}^k PE(\chi_{C_n^{\varepsilon}} uf).$$

Since $\sum_{n=1}^k E(\chi_{C_n^{\varepsilon}} uf) = \sum_{n=1}^k E(uf)(C_n^{\varepsilon})\chi_{C_n^{\varepsilon}}$, so $S = EM_v$ is a finite rank operator and the set of all finite rank operators is a self-adjoint two-sided ideal of $B(L^2(\mathscr{M}))$, the set of all bounded operators on $L^2(\mathscr{M})$, thus $T_1 = PS$ has finite rank and hence T is compact. $\square$

EXAMPLE 2.15. As in Example 2.5(i), let $\mathscr{A} = \langle C_i \rangle$ be the algebra generated by the countable collection of the non-null disjoint Lebesgue measurable subsets of $\mathbb{D}$ such that their union is $\mathbb{D}$. It is clear that each C_i is an $\mathscr{A}$-atom and $(\mathbb{D}, \mathscr{A}, A_{|_{\mathscr{A}}})$ can be partitioned as $\mathbb{D} = (\cup_{n\in\mathbb{N}} C_n) \cup B$, where $B = \emptyset$. If $\mathscr{T}_u^E$ be bounded on $L_a^2(\mathbb{D})$ and u satisfies the conditions of Theorem 2.14, then $\mathscr{T}_u^E$ is compact.

EXAMPLE 2.16. Suppose $u \in L^1(\mathscr{M})$ is harmonic. Then $\mathscr{T}_u$ is compact if and only if $u \equiv 0$ (see [14]). Let $\mathscr{A} = \{\emptyset, \mathbb{D}\}$ and $u \in H^\infty(\mathbb{D})$, in Example 2.5(i) we saw that $\mathscr{T}_{\bar{u}}^E(f)(z) = \int_{\mathbb{D}} \overline{u(w)} f(w) dA(w)$. It is clear that $\mathscr{T}_{\bar{u}}^E$ is bounded, $\bar{u}$ and $\mathscr{A}$ satisfy conditions of Theorem 2.13, hence $\mathscr{T}_{\bar{u}}^E$ is compact, while $\bar{u}$ is non-zero harmonic $L^1(\mathscr{M})$-function. We can directly show that $\mathscr{T}_{\bar{u}}^E$ is compact. Suppose that $f_n \to 0$ weakly in $L_a^2(\mathbb{D})$, then $\int_{\mathbb{D}} \bar{g} f_n dA \to 0$ for each $g \in L_a^2(\mathbb{D})$. Thus $\|\mathscr{T}_{\bar{u}}^E f_n\|^2 = |\int_{\mathbb{D}} \bar{u} f_n dA|^2 \to 0$. Hence $\mathscr{T}_{\bar{u}}^E$ is compact.

At this stage, we consider diagonal operators and present some statements about diagonal Toeplitz and C-E type Toeplitz operators. Recall that an operator $T : \mathscr{H} \to \mathscr{H}$ is called a diagonal operator if $Te_j = \alpha_j e_j$, where $\{e_j\}$ is a basis for $\mathscr{H}$. According to definition, it is clear that $T : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ is diagonal if and only if $\langle Tz^i, z^j \rangle = 0$ for all $i \neq j$.

For $u \in L^2(\mathscr{M})$, Louhichi et al. [9] proved that $\mathscr{T}_u$ is a diagonal operator on $L_a^2(\mathbb{D})$ if and only if u is radial. Although it is not known if the C-E type Toepliz operator induced by radial symbol is diagonal, as we will see in next example when u is radial, operators in Example 2.5, are diagonal.

EXAMPLE 2.17. Let $u(z) = u(|z|)$, $\varphi(z) = z^2$ and let $\mathscr{A} = \mathscr{A}(\varphi)$. Then by Example 2.5(ii), we have

$$\mathscr{T}_u^E(f)(z) = \frac{1}{2}\{P((uf)\circ\varphi_1) + P((uf)\circ\varphi_2)\}(z),$$

where $\varphi_i(z) = (-1)^{i+1} z$. Now, take $f(z) = z^k$. So

$$\mathscr{T}_u^E(z^k)(z) = \frac{1}{2}\int_{\mathbb{D}} \frac{u(w)w^k}{(1-z\overline{w})^2} dA(w) + \frac{(-1)^k}{2}\int_{\mathbb{D}} \frac{u(-w)w^k}{(1-z\overline{w})^2} dA(w).$$

Since u is radial so for $k = 2n+1$, $\mathscr{T}_u^E(z^k) = 0$ and for $k = 2n$,

$$\begin{aligned}
\mathscr{T}_u^E(z^k)(z) &= \int_{\mathbb{D}} \frac{u(|w|)w^k}{(1-z\overline{w})^2} dA(w) \\
&= \sum_{j=0}^{\infty} (j+1) z^j \int_{\mathbb{D}} w^k \overline{w}^j u(|w|) dA(w) \\
&= \sum_{j=0}^{\infty} (j+1) z^j \int_0^1 \int_0^{2\pi} r^{j+k+1} u(r) \frac{dr d\theta}{\pi} \\
&= [2(k+1)\int_0^1 r^{2k+1} u(r) dr] z^k := \alpha_k z^k.
\end{aligned}$$

Thus, $\mathscr{T}_u^E(z^k) = c_k z^k$, where

$$c_k = \begin{cases} 0 & k = 2n+1 \\ \alpha_k & k = 2n. \end{cases}$$

A calculation shows that for operators in Example 2.5(i), we have $\mathscr{T}_u^E(z^k) = c_k z^k$ when u is a radial function.

THEOREM 2.18. *Suppose $u \in \mathfrak{T}_2$. If for $n \geqslant 0$, Eu and $|z|^{2n}\bar{u}$ are perpendicular to $zL_a^2(\mathbb{D})$ and for $n \geqslant 0$, $|z|^{2n}u$ is perpendicular to $L_a^2(\mathbb{D})$ then $\mathscr{T}_u^E$ is a diagonal operator on $L_a^2(\mathbb{D})$.*

Proof. Suppose that $\mathscr{T}_u^E$ is not a diagonal operator on $L_a^2(\mathbb{D})$. Thus there is $j \neq n$ such that $\langle Euz^n, z^j \rangle \neq 0$. If $n = 0$ or $j = 0$, it concluded that Eu or $\bar{u}$ has not the mentioned property respectively. If $n \neq 0$ and $j \neq 0$, then $\langle u, \bar{z}^n E z^j \rangle \neq 0$. Since $\bar{z}^n E z^j \in L^2(\mathscr{M})$ and $p(z,\bar{z})$'s are dense in $L^2(\mathscr{M})$, there is $z^l \bar{z}^k$ such that $\langle u, z^l \bar{z}^k \rangle \neq 0$. Putting $l = k$, $l > k$ and $l < k$ we have $\langle u, z^{2l} \rangle \neq 0$, $\langle u|z|^{2k}, z^{l-k} \rangle \neq 0$ and $\langle u|z|^{2l}, z^{k-l} \rangle \neq 0$ respectively, thus desired result is concluded. $\square$

When $u \in L^2(\mathscr{M})$ is not radial then $\mathscr{T}_u$ is not diagonal on $L_a^2(\mathbb{D})$. In spit of classical Toeplitz operator cases, there are functions $u \in L^2(\mathscr{M})$ such that u is not radial but the induced C-E type Toeplitz operator $\mathscr{T}_u^E$ is diagonal.

EXAMPLE 2.19. Let $u(z) = p_n(z)$, such that $p_n(0) \neq 0$. Suppose E and $\mathscr{A}$ are as in Example 2.5(i). Since $\mathscr{T}_u^E(z^k)(z) = \int_{\mathbb{D}} u(w) w^k dA(w)$, using mean value property for harmonic functions, we have

$$\mathscr{T}_u^E(z^k) = \begin{cases} p_n(0) & k = 0 \\ 0 & k \geqslant 1, \end{cases}$$

thus $\mathscr{T}_u^E(z^k) = c_k z^k$, where $c_0 = p_n(0)$ and $c_k = 0$ for $k \geqslant 1$, so $\mathscr{T}_u^E$ is diagonal while $\mathscr{T}_u$ is not diagonal.

Acknowledgement.

The authors would like to express their cordial thanks to the referee(s) for their time consuming.

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(Received January 12, 2017)

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