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AN OBSERVATION ABOUT NORMALOID OPERATORS

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Abstract. Let H be a complex Hilbert space and $B(H)$ the Banach space of all bounded linear operators on H. For any $A \in B(H)$, let $w(A)$ denote the numerical radius of A. Then A is normaloid if $w(A) = \|A\|$. In this note, we show that A is normaloid if there is a sequence of unit vectors (x_n) such that $\lim_{n\to\infty} \|Ax_n\| = \|A\|$ and $\lim_{n\to\infty} |\langle Ax_n, x_n\rangle| = w(A)$ simultaneously. The result is then used to study the Davis-Wielandt radius.

1. Introduction

Let H be a complex Hilbert space and $B(H)$ the Banach space of all bounded linear operators on H. When $\dim H = n < \infty$, $B(H)$ will be identified with M_n, the space of all $n \times n$ complex matrices. For any operator $A \in B(H)$, the *numerical range* and *numerical radius* of A are defined respectively by

$$W(A) = \{\langle Ax, x\rangle : x \text{ is a unit vector in } H\} \quad \text{and} \quad w(A) = \sup\{|\lambda| : \lambda \in W(A)\}.$$

It is well-known that $w(\cdot)$ is a norm on $B(H)$ which satisfies

$$\frac{1}{2}\|A\| \leqslant w(A) \leqslant \|A\| \quad \text{for all } A \in B(H).$$

See, for example, [5, Problem 218]. When $w(A) = \|A\|$, A is called *normaloid*. All normal operators are normaloid, but not all normaloid operators are normal. It is an interesting topic to characterize normaloid operators. When H is finite dimensional, a characterization was given by Goldberg and Zwas in [4], which states that a matrix $A \in M_n$ is normaloid if and only if there exists an integer $1 \leqslant k \leqslant n$ and a unitary matrix U such that

$$A = U \begin{pmatrix} \Lambda & 0 \\ 0 & B \end{pmatrix} U^*,$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_k)$ with $|\lambda_1| = \cdots = |\lambda_k| = \|A\|$ and $B \in M_{n-k}$ is such that $\|B\| < \|A\|$. When H is infinite dimensional, characterizations of normaloid operator A can be found in [1], [9] and [10].

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2. Normaloid operators

Our main result is the following observation about normaloid operators.

THEOREM 1. *An operator $A \in B(H)$ is normaloid if and only if there is a sequence of unit vectors (x_n) in H such that*

$$\lim_{n\to\infty} \|Ax_n\| = \|A\| \quad \text{and} \quad \lim_{n\to\infty} |\langle Ax_n, x_n\rangle| = w(A). \tag{$*$}$$

Proof. The necessity is clear. If A is normaloid, then every sequence of unit vectors (x_n) such that $\lim_{n\to\infty} |\langle Ax_n, x_n\rangle| = w(A) = \|A\|$ satisfies $\lim_{n\to\infty} \|Ax_n\| = \|A\|$. This follows from

$$\|A\| \geqslant \|Ax_n\| \geqslant |\langle Ax_n, x_n\rangle| \geqslant w(A) - \varepsilon = \|A\| - \varepsilon,$$

for all $\varepsilon > 0$ and n large enough.

To prove the sufficiency, assume without loss of generality that $1 = \|A\| \geqslant w(A) > 0$ and that (x_n) is a sequence of unit vectors satisfying $(*)$. For each n, let y_n be a unit vector in H such that $\langle x_n, y_n\rangle = 0$ and $Ax_n = a_n x_n + c_n y_n$. Then from the hypothesis, we have

$$\lim_{n\to\infty} |a_n| = w(A) \quad \text{and} \quad \lim_{n\to\infty} (|a_n|^2 + |c_n|^2) = \|A\|^2 = 1.$$

Our aim is to show that $\lim_{n\to\infty} c_n = 0$ so that $w(A) = 1 = \|A\|$.

Consider the compression of A onto $\operatorname{span}\{x_n, y_n\}$, realized as the 2×2 matrix

$$A_n = \begin{pmatrix} \langle Ax_n, x_n\rangle & \langle Ay_n, x_n\rangle \\ \langle Ax_n, y_n\rangle & \langle Ay_n, y_n\rangle \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

Passing to a subsequence if necessary, we may further assume that

$$A_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A_0.$$

From

$$\lim_{n\to\infty} |a_n| \leqslant \lim_{n\to\infty} w(A_n) \leqslant w(A) = \lim_{n\to\infty} |a_n|,$$

and $w(A_0) = \lim_{n\to\infty} w(A_n)$, we get $w(A_0) = \lim_{n\to\infty} |a_n| = |a|$. Similarly, $\|A_0\|^2 = |a|^2 + |c|^2 = 1$.

We claim that $|b| = |c|$. For this purpose, consider the hermitian matrix

$$H = \frac{1}{2}(\overline{a}A_0 + aA_0^*) = \frac{1}{2}\begin{pmatrix} 2|a|^2 & \overline{a}b + a\overline{c} \\ \overline{a}c + a\overline{b} & \overline{a}d + a\overline{d} \end{pmatrix}.$$

As the $(1,1)$-entry of H is $|a|^2$,

$$|a|^2 \leqslant w(H) = w\left(\frac{1}{2}(\overline{a}A_0 + aA_0^*)\right) \leqslant |a|^2,$$

implying that $w(H) = |a|^2$. Since H is hermitian, $\|H\| = w(H) = |a|^2$. The (1, 2)-entry of H must be zero. In other words, $\overline{a}c + a\overline{b} = 0$, from which the claim $|b| = |c|$ follows.

We now show that $c = 0$. Assume on the contrary that $c \neq 0$. Consider

$$A_0^* A_0 = \begin{pmatrix} |a|^2 + |c|^2 & \overline{a}b + \overline{c}d \\ \overline{b}a + \overline{d}c & |b|^2 + |d|^2 \end{pmatrix}.$$

Since $\|A_0^* A_0\| = \|A_0\|^2 = |a|^2 + |c|^2$, the (2, 1)-entry of A_0 is zero. In other words, $\overline{b}a + \overline{d}c = 0$. Together with $|b| = |c| > 0$, we get $|a| = |d|$. Hence $|b|^2 + |d|^2 = |a|^2 + |c|^2 = 1$ and consequently $A_0^* A_0 = I_2$, the 2×2 identity matrix I_2. This means A_0 is a unitary matrix. As the numerical radius of a unitary matrix is one, $|a| = w(A_0) = 1$. It follows that $c = 0$, a contradiction.

From our construction,

$$\|A\| = \lim_{n\to\infty} \sqrt{|a_n|^2 + |c_n|^2} = \lim_{n\to\infty} |a_n| = w(A),$$

i.e., A is normaloid. $\square$

We remark that even if A is normaloid, not every (x_n) such that $\lim_{n\to\infty} \|Ax_n\| = \|A\|$ satisfies $\lim_{n\to\infty} \langle Ax_n, x_n \rangle = w(A)$. Let

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is not hard to see that A is normaloid with $\|A\| = w(A) = 1$. However for $x = (1\ \ 0\ \ 0)^t$, $\|Ax\| = 1 = \|A\|$ while $|\langle Ax, x \rangle| \neq w(A)$.

Theorem 1 can be stated in terms of the *maximal numerical range* of the operator A. The notion was introduced by Stampfli in [11] and defined by

$$W_0(A) = \{\lambda : \langle Ax_n, x_n \rangle \to \lambda \text{ for unit vectors } (x_n) \text{ such that } \|Ax_n\| \to \|A\|\}.$$

Call $w_0(A) = \sup\{|\lambda| : \lambda \in W_0(A)\}$ the *maximal numerical radius* of A. It is not hard to see that $W_0(A) \subseteq W(A)^-$, the closure of A, and therefore $w_0(A) \leqslant w(A)$. Some other properties of $w_0(\cdot)$ were given in [12]. We have

COROLLARY 1. *An operator $A \in B(H)$ is normaloid if and only if $w_0(A) = w(A)$.*

Proof. Suppose A is normaloid. Take a sequence (x_n) of unit vectors such that $\lim_{n\to\infty} |\langle Ax_n, x_n \rangle| = w(A) = \|A\|$. Then $\lim_{n\to\infty} \|Ax_n\| = \|A\|$. Any accumulation point of $(\langle Ax_n, x_n \rangle)$ belongs to $W_0(A)$ and has modulus $w(A)$. Since $w_0(A) \leqslant w(A)$, we must have $w_0(A) = w(A)$.

Now suppose that $w_0(A) = w(A)$. By definition of $w_0(A)$,

$$w(A) = w_0(A) = \lim_{n\to\infty} |\langle Ax_n, x_n \rangle|$$

for a sequence of unit vectors (x_n) such that $\|Ax_n\| \to \|A\|$. Therefore A is normaloid, by Theorem 1. $\square$

3. The Davis-Wielandt radius

For any $A \in B(H)$, its Davis-Wielandt shell is the set

$$DW(A) = \{(\langle Ax,x\rangle, \langle Ax,Ax\rangle) : x \in H \text{ and } \|x\| = 1\}.$$

It was introduced by Davis in [3] and has been studied extensively as a generalization of the numerical range. See, for example, [6], [7] and [8]. As in the case of the numerical range, we define the *Davis-Wielandt radius* of A by

$$\begin{aligned} r_{DW}(A) &= \sup\{\sqrt{|\langle Ax,x\rangle|^2 + |\langle A^*Ax,x\rangle|^2} : x \in H \text{ and } \|x\| = 1\} \\ &= \sup\{\sqrt{|\langle Ax,x\rangle|^2 + \|Ax\|^4} : x \in H \text{ and } \|x\| = 1\}. \end{aligned}$$

It is easy to see that $r_{DW}(\cdot)$ is not positive homogeneous and therefore cannot be a norm on $B(H)$. In spite of this, it has many interesting properties. A description of $r_{DW}(\cdot)$-distance preservers was given in [2]. Here, we are interested in the following inequalities, which will be stated without proof.

PROPOSITION 1. *For every* $A \in B(H)$, $\|A\|^2 \leqslant r_{DW}(A) \leqslant \sqrt{(w(A))^2 + \|A\|^4}$.

Both inequalities in Proposition 1 can be attained by a nonzero A. Clearly,

$$r_{DW}(I) = \sqrt{(w(I))^2 + \|I\|^4}.$$

For the other inequality, consider $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It is easy to see that $\|A\| = 1$. To compute $r_{DW}(A)$, write any unit vector in $\mathbb{C}^2$ as $(\lambda\cos\theta, \mu\sin\theta)^t$, where λ and μ are complex units. Then

$$\begin{aligned} r_{DW}(A)^2 &= \max\{|\langle Ay,y\rangle|^2 + \|Ay\|^4 : y \in H \text{ and } \|y\| = 1\} \\ &= \max\{\cos^2\theta\sin^2\theta + \sin^4\theta : \theta \in \mathbb{R}\} \\ &= \max\{\sin^2\theta : \theta \in \mathbb{R}\}. \end{aligned}$$

Therefore $r_{DW}(A) = 1 = \|A\|^2$.

One may wonder if (x_n) is a sequence of unit vectors such that

$$\lim_{n\to\infty} \sqrt{|\langle Ax_n,x_n\rangle|^2 + \|Ax_n\|^4} = r_{DW}(A),$$

would it be also true that $\lim_{n\to\infty}\|Ax_n\| = \|A\|$? The following example shows that this is not the case. Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus (r),$$

where $r \in (0,1)$. Then $\|A\| = 1$, which is attained only at unit multiples of $x_1 = (0\ \ 1\ \ 0)^t$. To compute $r_{DW}(A)$, we have by [8, Theorem 2.1 (e)],

$$r_{DW}(A) = \max\left\{ r_{DW}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right), r_{DW}((r)) \right\} = \max\left\{ 1, \sqrt{r^2 + r^4} \right\}.$$

Clearly, we can choose r large enough so that $r_{DW}(\cdot)$ is attained at $x_2 = (0\ \ 0\ \ 1)^t$, which is not a multiple of x_1.

PROPOSITION 2. *Suppose $A \in B(H)$. Then $r_{DW}(A) = \sqrt{(w(A))^2 + \|A\|^4}$ if and only if A is normaloid.*

Proof. Suppose that $r_{DW}(A) = \sqrt{(w(A))^2 + \|A\|^4}$. Take a sequence of unit vectors (x_n) such that

$$\lim_{n\to\infty} \sqrt{|\langle Ax_n, x_n\rangle|^2 + \|Ax_n\|^4} = r_{DW}(A) = \sqrt{(w(A))^2 + \|A\|^4}.$$

Then we have

$$\lim_{n\to\infty} \|Ax_n\| = \|A\| \quad \text{and} \quad \lim_{n\to\infty} |\langle Ax_n, x_n\rangle| = w(A).$$

By Theorem 1, A is normaloid.

Conversely, suppose A is normaloid. Take any sequence of unit vectors (x_n) such that $\lim_{n\to\infty} |\langle Ax_n, x_n\rangle| = w(A) = \|A\|$. Then $\lim_{n\to\infty} \|Ax_n\| = \|A\|$ and

$$\lim_{n\to\infty} \sqrt{|\langle Ax_n, x_n\rangle|^2 + \|Ax_n\|^4} = \sqrt{(w(A))^2 + \|A\|^4}.$$

As $r_{DW}(A) \leqslant \sqrt{(w(A))^2 + \|A\|^4}$, equality follows. $\square$

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