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ON DECOMPOSITION OF OPERATORS HAVING Γ_3 AS A SPECTRAL SET

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Abstract. The symmetrized polydisc of dimension three is the set

$$\Gamma_3 = \{(z_1+z_2+z_3, z_1z_2+z_2z_3+z_3z_1, z_1z_2z_3) : |z_i| \leqslant 1, i=1,2,3\} \subseteq \mathbb{C}^3.$$

A triple of commuting operators for which Γ_3 is a spectral set is called a Γ_3-contraction. We show that every Γ_3-contraction admits a decomposition into a Γ_3-unitary and a completely non-unitary Γ_3-contraction. This decomposition parallels the canonical decomposition of a contraction into a unitary and a completely non-unitary contraction. We also find new characterizations for the set Γ_3 and Γ_3-contractions.

1. Introduction

One of the most wonderful discoveries in one variable operator theory is the canonical decomposition of a contraction which ascertains that every contraction operator (i.e, an operator with norm not greater than 1) admits a unique decomposition into two orthogonal parts of which one is a unitary and the other is a completely non-unitary contraction. More precisely, for an operator T with norm not greater than one acting on a Hilbert space $\mathscr{H}$, there exist unique reducing subspaces $\mathscr{H}_1, \mathscr{H}_2$ of T such that $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, $T|_{\mathscr{H}_1}$ is a unitary and $T|_{\mathscr{H}_2}$ is a completely non-unitary contraction (see Theorem 3.2 in Ch-I, [8] for details). A contraction on a Hilbert space is said to be *completely non-unitary* if there is no reducing subspace on which the operator acts like a unitary. Following von Neumann's famous notion of spectral set for an operator (which we define below), a contraction is better understood as an operator having the closed unit disk $\overline{\mathbb{D}}$ of the complex plane as a spectral set. Indeed, in 1951 von Neumann proved the following theorem whose impact has been extraordinary.

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THEOREM 1.1. (von Neumann, [14]) *An operator T acting on a Hilbert space is a contraction if and only if the closed unit disk $\overline{\mathbb{D}}$ is a spectral set for T.*

Since an operator having $\overline{\mathbb{D}}$ as a spectral set admits a canonical decomposition, it is naturally asked whether we can decompose operators having a particular domain in $\mathbb{C}^n$ as a spectral set. In [2], Agler and Young answered this question by showing an explicit decomposition of a pair of commuting operators having the closed symmetrized bidisc

$$\Gamma_2 = \{(z_1+z_2, z_1z_2) : |z_i| \leqslant 1, i = 1,2\}$$

as a spectral set (Theorem 2.8, [2]). In this article, we provide an analogous decomposition for operators having the closed symmetrized tridisc

$$\Gamma_3 = \{(z_1+z_2+z_3, z_1z_2+z_2z_3+z_3z_1, z_1z_2z_3) : |z_i| \leqslant 1, i = 1,2,3\}$$

as a spectral set. The reason behind considering the symmetrized polydisc of dimension 3 in particular is that there are substantial variations in operator theory if we move from two to three dimensional symmetrized polydisc, e.g., rational dilation succeeds on the symmetrized bidisc [1, 5, 11] but fails on the symmetrized tridisc, [12]. This article can be considered as a sequel of [12].

A compact subset X of $\mathbb{C}^n$ is said to be a *spectral set* for a commuting n-tuple of bounded operators $\underline{T} = (T_1, \dots, T_n)$ defined on a Hilbert space $\mathscr{H}$ if the Taylor joint spectrum $\sigma_T(\underline{T})$ of $\underline{T}$ is a subset of X and

$$\|f(\underline{T})\| \leqslant \|f\|_{\infty,X} = \sup\{|f(z_1,\dots,z_n)| : (z_1,\dots,z_n) \in X\},$$

for all rational functions f in $\mathscr{R}(X)$. Here $\mathscr{R}(X)$ denotes the algebra of all rational functions on X, that is, all quotients p/q of holomorphic polynomials p,q in n-variables for which q has no zeros in X.

For $n \geqslant 2$, the symmetrization map in n-complex variables $z = (z_1,\dots,z_n)$ is the following proper holomorphic map

$$\pi_n(z) = (s_1(z), \dots, s_{n-1}(z), p(z))$$

where

$$s_i(z) = \sum_{1 \leqslant k_1 \leqslant k_2 \dots \leqslant k_i \leqslant n-1} z_{k_1} \dots z_{k_i} \quad \text{and } p(z) = \prod_{i=1}^{n} z_i .$$

The closed *symmetrized n-disk* (or simply closed *symmetrized polydisc*) is the image of the closed unit n-disc $\overline{\mathbb{D}^n}$ under the symmetrization map π_n, that is, $\Gamma_n := \pi_n(\overline{\mathbb{D}^n})$. Similarly the open symmetrized polydisc $\mathbb{G}_n$ is defined as the image of the open unit polydisc $\mathbb{D}^n$ under π_n. The set Γ_n is polynomially convex but not convex (see [10, 7]). So in particular the closed and open symmetrized tridisc are the sets

$$\Gamma_3 = \{(z_1+z_2+z_3, z_1z_2+z_2z_3+z_3z_1, z_1z_2z_3) : |z_i| \leqslant 1, i = 1,2,3\} \subseteq \mathbb{C}^3$$
$$\mathbb{G}_3 = \{(z_1+z_2+z_3, z_1z_2+z_2z_3+z_3z_1, z_1z_2z_3) : |z_i| < 1, i = 1,2,3\} \subseteq \Gamma_3.$$

We obtain from the literature (see [10, 7]) the fact that the distinguished boundary of the symmetrized polydisc is the symmetrization of the distinguished boundary of the n-dimensional polydisc, which is n-torus $\mathbb{T}^n$. Hence the distinguished boundary for Γ_3 is the set

$$b\Gamma_3 = \{(z_1+z_2+z_3, z_1z_2+z_2z_3+z_3z_1, z_1z_2z_3) : |z_i| = 1, i = 1,2,3\}.$$

Operator theory on the symmetrized polydiscs of dimension 2 and n have been extensively studied in past two decades [1, 2, 3, 5, 6, 7, 11, 13].

DEFINITION 1.2. A triple of commuting operators (S_1,S_2,P) on a Hilbert space $\mathscr{H}$ for which Γ_3 is a spectral set is called a Γ_3-*contraction*. A Γ_3-contraction (S_1,S_2,P) is said to a *completely non-unitary* if P is a completely non-unitary contraction.

It is evident from the definition that if (S_1,S_2,P) is a Γ_3-contraction then S_1,S_2 have norms not greater than 3 and P is a contraction. Unitaries, isometries and co-isometries are important special classes of contractions. There are natural analogues of these classes for Γ_3-contractions.

DEFINITION 1.3. Let S_1,S_2,P be commuting operators on a Hilbert space $\mathscr{H}$. We say that (S_1,S_2,P) is

(i) a Γ_3-*unitary* if S_1,S_2,P are normal operators and the Taylor joint spectrum $\sigma_T(S_1,S_2,P)$ is contained in $b\Gamma_3$;

(ii) a Γ_3-*isometry* if there exists a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and a Γ_3-unitary $(\tilde{S_1},\tilde{S_2},\tilde{P})$ on $\mathscr{K}$ such that $\mathscr{H}$ is a common invariant subspace for $\tilde{S_1},\tilde{S_2},\tilde{P}$ and that $S_i = \tilde{S_i}|_{\mathscr{H}}$ for $i=1,2$ and $\tilde{P}|_{\mathscr{H}} = P$;

(iii) a Γ_3-*co-isometry* if (S_1^*,S_2^*,P^*) is a Γ_3-isometry.

Moreover, a Γ_3-isometry (S_1,S_2,P) is said to be *pure* if P is a pure contraction, that is, $P^* \to 0$ strongly as $n \to \infty$.

The main result of this article is the following explicit orthogonal decomposition of a Γ_3-contraction which parallels the one-variable canonical decomposition.

THEOREM 1.4. *Let (S_1,S_2,P) be a Γ_3-contraction on a Hilbert space $\mathscr{H}$. Let $\mathscr{H}_1$ be the maximal subspace of $\mathscr{H}$ which reduces P and on which P is unitary. Let $\mathscr{H}_2 = \mathscr{H} \ominus \mathscr{H}_1$. Then $\mathscr{H}_1,\mathscr{H}_2$ reduce S_1,S_2; $(S_1|_{\mathscr{H}_1},S_2|_{\mathscr{H}_1},P|_{\mathscr{H}_1})$ is a Γ_3-unitary and $(S_1|_{\mathscr{H}_2},S_2|_{\mathscr{H}_2},P|_{\mathscr{H}_2})$ is a completely non-unitary Γ_3-contraction. The subspaces $\mathscr{H}_1$ or $\mathscr{H}_2$ may equal to the trivial subspace $\{0\}$.*

En route we find few characterizations for the set Γ_3 and also for the Γ_3-contractions which we accumulate in section 2.

2. Background material

In this section we recall some results from literature about the geometry and operator theory on the set Γ_3. Also we obtain few new results in the same direction which we accumulate here. We begin with a few characterizations of the set Γ_3.

THEOREM 2.1. *Let $(s_1,s_2,p)\in\mathbb{C}^3$. Then the following are equivalent:*

1. $(s_1,s_2,p)\in\Gamma_3$ *;*
2. $(\omega s_1,\omega^2 s_2,\omega^3 p)\in\Gamma_3$ *for all* $\omega\in\mathbb{T}$ *;*
3. $|p|\leqslant 1$ *and there exists* $(c_1,c_2)\in\Gamma_2$ *such that*
$$s_1=c_1+\bar{c_2}p \text{ and } s_2=c_2+\bar{c_1}p,$$
where Γ_2 is the closed symmetrized bidisc defined as
$$\Gamma_2=\{(z_1+z_2,z_1z_2)\,:\,z_1,z_2\in\overline{\mathbb{D}}\}.$$

Proof. (1)$\Leftrightarrow$(3) has been established in [9] (see Theorem 3.7 in [9] for a proof). We prove here (1)$\Leftrightarrow$(2). Let $(s_1,s_2,p)\in\Gamma_3$. Then by (1)$\Leftrightarrow$(3), $|p|\leqslant 1$ and there exist $(c_1,c_2)\in\Gamma_2$ such that
$$s_1=c_1+\bar{c_2}p,\quad s_2=c_2+\bar{c_1}p.$$
Since $(c_1,c_2)\in\Gamma_2$, there are complex numbers u_1,u_2 of modulus not greater than 1 such that $c_1=u_1+u_2$ and $c_2=u_1u_2$. For $\omega\in\mathbb{T}$ if we choose $d_1=\omega c_1$ and $d_2=\omega^2c_2$ we see that
$$d_1=\omega u_1+\omega u_2 \text{ and } d_2=(\omega u_1)(\omega u_2),$$
which means that $(d_1,d_2)\in\Gamma_2$. Now
$$\begin{aligned}\omega s_1&=\omega(c_1+\bar{c_2}p)=\omega c_1+\overline{\omega^2c_2}(\omega^3p)=d_1+\bar{d_2}(\omega^3p),\\ \omega^2s_2&=\omega^2(c_2+\bar{c_1}p)=\omega^2c_2+\overline{\omega c_1}(\omega^3p)=d_2+\bar{d_1}(\omega^3p).\end{aligned}$$
Therefore, by part (1)$\Leftrightarrow$(3), $(\omega s_1,\omega^2s_2,\omega^3p)\in\Gamma_3$. The other side of the proof is trivial. □

In a similar fashion, we have the following characterizations for Γ_3-contractions.

THEOREM 2.2. *Let (S_1,S_2,P) be a triple of commuting operators acting on a Hilbert space $\mathscr{H}$. Then the following are equivalent:*

1. (S_1,S_2,P) *is a* Γ_3*-contraction;*
2. *for all holomorphic polynomials f in three variables*
$$\|f(S_1,S_2,P)\|\leqslant\|f\|_{\infty,\Gamma_3}=\sup\{|f(s_1,s_2,p)|\,:\,(s_1,s_2,p)\in\Gamma_3\};$$

3. $(\omega S_1, \omega^2 S_2, \omega^3 P)$ *is a* Γ_3*-contraction for any* $\omega \in \mathbb{T}$.

Proof. $(1) \Rightarrow (2)$ follows from definition of spectral set and $(2) \Rightarrow (1)$ just requires polynomial convexity of the set Γ_3. We prove here $(1) \Rightarrow (3)$ because $(3) \Rightarrow (1)$ is obvious. Let $f(s_1, s_2, p)$ be a holomorphic polynomial in the co-ordinates of Γ_3 and for $\omega \in \mathbb{T}$ let $f_1(s_1, s_2, p) = f(\omega s_1, \omega^2 s_2, \omega^3 p)$. It is evident from part $(1) \Rightarrow (2)$ that

$$\sup\{|f(s_1,s_2,p)| : (s_1,s_2,p) \in \Gamma_3\} = \sup\{|f_1(s_1,s_2,p)| : (s_1,s_2,p) \in \Gamma_3\}.$$

Therefore,

$$\begin{aligned} \|f(\omega S_1, \omega^2 S_2, \omega^3 P)\| &= \|f_1(S_1, S_2, P)\| \\ &\leqslant \|f_1\|_{\infty, \Gamma_3} \\ &= \|f\|_{\infty, \Gamma_3}. \end{aligned}$$

Therefore, by $(1) \Rightarrow (2)$, $(\omega S_1, \omega^2 S_2, \omega^3 P)$ is a Γ_3-contraction. □

In [12], two operator pencils Φ_1, Φ_2 were introduced which played pivotal role in determining the classes of Γ_3-contractions for which rational dilation failed or succeeded. Here we recall the definition of Φ_1, Φ_2 for any three commuting operators S_1, S_2, P with $\|S_i\| \leqslant 3$ and P being a contraction.

$$\begin{aligned} \Phi_1(S_1, S_2, P) &= 9(I - P^*P) + (S_1^*S_1 - S_2^*S_2) - 6 \text{ Re } (S_1 - S_2^*P), \\ \Phi_2(S_1, S_2, P) &= 9(I - P^*P) + (S_2^*S_2 - S_1^*S_1) - 6 \text{ Re } (S_2 - S_1^*P). \end{aligned}$$

The following result whose proof could be found in [12] (Proposition 4.4, [12]) is useful for this paper.

PROPOSITION 2.3. *Let* (S_1, S_2, P) *be a* Γ_3*-contraction. Then for* $i = 1, 2$, $\Phi_i(\alpha S_1, \alpha^2 S_2, \alpha^3 P) \geqslant 0$ *for all* $\alpha \in \overline{\mathbb{D}}$.

Here is a set of characterizations for the Γ_3-unitaries and for a proof of this result see Theorem 5.2 in [12] or, Theorem 4.2 in [7].

THEOREM 2.4. *Let* (S_1, S_2, P) *be a commuting triple of bounded operators. Then the following are equivalent.*

1. (S_1, S_2, P) *is a* Γ_3*-unitary,*

2. P *is a unitary and* (S_1, S_2, P) *is a* Γ_3*-contraction,*

3. $\left(\frac{2}{3}S_1, \frac{1}{3}S_2\right)$ *is a* Γ_2*-contraction,* P *is a unitary and* $S_1 = S_2^*P$.

3. Proof of Theorem 1.4

First we consider the case when P is a completely non-unitary contraction. Then obviously $\mathscr{H}_1 = \{0\}$ and if P is a unitary then $\mathscr{H} = \mathscr{H}_1$ and so $\mathscr{H}_2 = \{0\}$. In such cases the theorem is trivial. So let us suppose that P is neither a unitary nor a completely non unitary contraction. With respect to the decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, let

$$S_1 = \begin{bmatrix} S_{111} & S_{112} \\ S_{121} & S_{122} \end{bmatrix}, S_2 = \begin{bmatrix} S_{211} & S_{212} \\ S_{221} & S_{222} \end{bmatrix} \text{ and } P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

so that P_1 is a unitary and P_2 is completely non-unitary. Since P_2 is completely non-unitary it follows that if $h \in \mathscr{H}$ and

$$\|P_2^n h\| = \|h\| = \|P_2^{*n} h\|, \quad n = 1, 2, \ldots$$

then $h = 0$.

By the commutativity of S_1 and P we obtain

$$S_{111}P_1 = P_1 S_{111} \qquad S_{112}P_2 = P_1 S_{112}, \tag{3.1}$$

$$S_{121}P_1 = P_2 S_{121} \qquad S_{122}P_2 = P_2 S_{122}. \tag{3.2}$$

Also the commutativity of S_2 and P gives

$$S_{211}P_1 = P_1 S_{211} \qquad S_{212}P_2 = P_1 S_{212}, \tag{3.3}$$

$$S_{221}P_1 = P_2 S_{221} \qquad S_{222}P_2 = P_2 S_{222}. \tag{3.4}$$

By Proposition 2.3, we have for all $\omega, \beta \in \mathbb{T}$,

$$\Phi_1(\omega S_1, \omega^2 S_2, \omega^3 P) = 9(I - P^*P) + (S_1^*S_1 - S_2^*S_2) - 6 \operatorname{Re} \omega(S_1 - S_2^*P) \geqslant 0,$$
$$\Phi_2(\beta S_1, \beta^2 S_2, \beta^3 P) = 9(I - P^*P) + (S_2^*S_2 - S_1^*S_1) - 6 \operatorname{Re} \beta^2(S_2 - S_1^*P) \geqslant 0.$$

Adding Φ_1 and Φ_2 we get

$$3(I - P^*P) - \operatorname{Re} \omega(S_1 - S_2^*P) - \operatorname{Re} \beta^2(S_2 - S_1^*P) \geqslant 0$$

that is

$$\begin{bmatrix} 0 & 0 \\ 0 & 3(I - P_2^*P_2) \end{bmatrix} - \operatorname{Re} \omega \begin{bmatrix} S_{111} - S_{211}^*P_1 & S_{112} - S_{221}^*P_2 \\ S_{121} - S_{212}^*P_1 & S_{122} - S_{222}^*P_2 \end{bmatrix} \tag{3.5}$$
$$- \operatorname{Re} \beta^2 \begin{bmatrix} S_{211} - S_{111}^*P_1 & S_{212} - S_{121}^*P_2 \\ S_{221} - S_{112}^*P_1 & S_{222} - S_{122}^*P_2 \end{bmatrix} \geqslant 0$$

for all $\omega, \beta \in \mathbb{T}$. Since the matrix in the left hand side of (3.5) is self-adjoint, if we write (3.5) as

$$\begin{bmatrix} R & X \\ X^* & Q \end{bmatrix} \geqslant 0, \tag{3.6}$$

then

$$\begin{cases} \text{(i) } R,Q \geqslant 0 \text{ and } R = -\text{ Re } \omega(S_{111}-S_{211}^*P_1) - \text{ Re } \beta^2(S_{211}-S_{111}^*P_1) \\ \text{(ii)} X = -\frac{1}{2}\{\omega(S_{112}-S_{221}^*P_2)+\bar{\omega}(S_{121}^*-P_1^*S_{212}) \\ \qquad\qquad +\beta^2(S_{212}-S_{121}^*P_2)+\bar{\beta^2}(S_{221}^*-P_1^*S_{112})\} \\ \text{(iii) } Q = 3(I-P_2^*P_2) - \text{ Re } \omega(S_{122}-S_{222}^*P_2) - \text{ Re } \beta^2(S_{222}-S_{122}^*P_2)\,. \end{cases}$$

Since the left hand side of (3.6) is a positive semi-definite matrix for every ω and β, if we choose $\beta^2 = 1$ and $\beta^2 = -1$ respectively then consideration of the $(1,1)$ block reveals that

$$\omega(S_{111}-S_{211}^*P_1)+\bar{\omega}(S_{111}^*-P_1^*S_{211}) \leqslant 0$$

for all $\omega \in \mathbb{T}$. Choosing $\omega = \pm 1$ we get

$$(S_{111}-S_{211}^*P_1)+(S_{111}^*-P_1^*S_{211}) = 0 \tag{3.7}$$

and choosing $\omega = \pm i$ we get

$$(S_{111}-S_{211}^*P_1)-(S_{111}^*-P_1^*S_{211}) = 0. \tag{3.8}$$

Therefore, from (3.7) and (3.8) we get

$$S_{111} = S_{211}^*P_1\,,$$

where P_1 is unitary. Similarly, we can show that

$$S_{211} = S_{111}^*P_1\,.$$

Therefore, $R = 0$. Since (S_1,S_2,P) is a Γ_3-contraction, $\|S_2\| \leqslant 3$ and hence $\|S_{211}\| \leqslant 3$. Also since (S_1,S_2,P) is a Γ_3-contraction, by Lemma 2.5 of [7] $\left(\frac{2}{3}S_1,\frac{1}{3}S_2\right)$ is a Γ_2-contraction and hence $\left(\frac{2}{3}S_{111},\frac{1}{3}S_{211}\right)$ is a Γ_2-contraction. Therefore, by part-(3) of Theorem 2.4, (S_{111},S_{211},P_1) is a Γ_3-unitary.

Now we apply Proposition 1.3.2 of [4] to the positive semi-definite matrix in the left hand side of (3.6). This Proposition states that if $R,Q \geqslant 0$ then $\begin{bmatrix} R & X \\ X^* & Q \end{bmatrix} \geqslant 0$ if and only if $X = R^{1/2}KQ^{1/2}$ for some contraction K.

Since $R = 0$, we have $X = 0$. Therefore,

$$\omega(S_{112}-S_{221}^*P_2)+\bar{\omega}(S_{121}^*-P_1^*S_{212})+\beta^2(S_{212}-S_{121}^*P_2)+\bar{\beta^2}(S_{221}^*-P_1^*S_{112}) = 0\,,$$

for all $\omega,\beta \in \mathbb{T}$. Choosing $\beta^2 = \pm 1$ we get

$$\omega(S_{112}-S_{221}^*P_2)+\bar{\omega}(S_{121}^*-P_1^*S_{212}) = 0\,,$$

for all $\omega \in \mathbb{T}$. With the choices $\omega = 1, i$, this gives

$$S_{112} = S_{221}^* P_2 .$$

Therefore, we also have

$$S_{121}^* = P_1^* S_{212} .$$

Similarly, we can prove that

$$S_{212} = S_{121}^* P_2 , \quad S_{221}^* = P_1^* S_{112} .$$

Thus, we have the following equations

$$S_{112} = S_{221}^* P_2 \qquad S_{121}^* = P_1^* S_{212} \tag{3.9}$$
$$S_{212} = S_{121}^* P_2 \qquad S_{221}^* = P_1^* S_{112} . \tag{3.10}$$

Thus from (3.9), $S_{121} = S_{212}^* P_1$ and together with the first equation in (3.2), this implies that

$$S_{212}^* P_1^2 = S_{121} P_1 = P_2 S_{121} = P_2 S_{212}^* P_1$$

and hence

$$S_{212}^* P_1 = P_2 S_{212}^* . \tag{3.11}$$

From equations in (3.3) and (3.11) we have that

$$S_{212} P_2 = P_1 S_{212} , \quad S_{212} P_2^* = P_1^* S_{212}.$$

Thus

$$S_{212} P_2 P_2^* = P_1 S_{212} P_2^* = P_1 P_1^* S_{212} = S_{212} ,$$
$$S_{212} P_2^* P_2 = P_1^* S_{212} P_2 = P_1^* P_1 S_{212} = S_{212} ,$$

and so we have

$$P_2 P_2^* S_{212}^* = S_{212}^* = P_2^* P_2 S_{212}^* .$$

This shows that P_2 is unitary on the range of S_{212}^* which can never happen because P_2 is completely non-unitary. Therefore, we must have $S_{212}^* = 0$ and so $S_{212} = 0$. Similarly we can prove that $S_{112} = 0$. Also from (3.9), $S_{121} = 0$ and from (3.10), $S_{221} = 0$. Thus with respect to the decomposition $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$

$$S_1 = \begin{bmatrix} S_{111} & 0 \\ 0 & S_{122} \end{bmatrix} , \quad S_2 = \begin{bmatrix} S_{211} & 0 \\ 0 & S_{222} \end{bmatrix} .$$

So, $\mathscr{H}_1$ and $\mathscr{H}_2$ reduce S_1 and S_2. Also (S_{122}, S_{222}, P_2), being the restriction of the $\mathbb{E}$-contraction (S_1, S_2, P) to the reducing subspace $\mathscr{H}_2$, is an Γ_3-contraction. Since P_2 is completely non-unitary, (S_{122}, S_{222}, P_2) is a completely non-unitary Γ_3-contraction.

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