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TRUNCATED MOMENT PROBLEMS IN $\mathbb{R}^2$ AND RECURSIVENESS

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Abstract. This paper concerns the study of the truncated moment problems in $\mathbb{R}^2$. More precisely, some Curto-Fialkow's results, on the truncated complex moment problems, combined with properties of generalized Fibonacci sequences, allow us to establish some results on the truncated moment problems in $\mathbb{R}^2$ and its extension to the full moment problems. Furthermore, with the aid of the combinatorial formula of the generalized Fibonacci sequences, we infer a combinatorial expression for each term of the associated moment matrix. Which in turn permits us to construct a process for expressing the terms of the extension of the truncated moment problem in $\mathbb{R}^2$ to the full moment problems. As a consequence, we are able to provide a detailed study of the truncated quartic moment problem in $\mathbb{R}^2$, linked firmly to the associated moment matrix. Illustrative examples and applications are given.

1. Introduction

Let $\beta \equiv \beta^{(2d)} \equiv \{\beta_{ij}\}_{\{(i,j)\in\mathbb{Z}_+^2,\ i+j\leqslant 2d\}}$, where d is a nonzero integer, be a 2-dimensional real multisequence of degree $2d$. Let $K \subset \mathbb{R}^2$ be a closed subset, the *truncated K-moment problem in $\mathbb{R}^2$* or *truncated 2-variable K-moment problem* (T2MP for short) for the sequence β consists of finding a positive Borel measure μ on $\mathbb{R}^2$ such that,

$$\beta_{ij} = \int_{\mathbb{R}^2} x^i y^j d\mu(x,y), \ \ (0 \leqslant i+j \leqslant 2d) \text{ with } supp(\mu) \subset K. \tag{1}$$

A measure satisfying (1) is said a representing (or K-representing) measure for the sequence $\beta \equiv \beta^{(2d)}$. The K-moment problem (1) is called *full* if $d = +\infty$. Currently, in [9], Bayer-Teichmann provide a general setting for the truncated multivariable moment problem when $\beta \equiv \beta^{(m)} = \{\beta_i\}_{i\in\mathbb{Z}_+^N, |i|\leqslant m}$, where $|i| = i_1 + i_2 + \ldots + i_N$ for $i = (i_1, \ldots, i_N)$. More precisely, whether a positive measure μ solution of the truncated multivariable moment problem is found, then μ is a finitely-atomic representing measure. And such measure may be presented as the sum $\mu = \sum_{k=1}^{D} \rho_k \delta_{x_k}$, where $1 \leqslant D < +\infty$, $\rho_k > 0$ for $k = 1, \ldots, D$, and δ_{x_k} is the point mass at $x_k \in \mathbb{R}^N$. In the sequel, we apply this result in the case of $N = 2$.

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Let $K = K_{\mathbb{C}}$ be a nonempty subset of the complex plane $\mathbb{C}$, in general K is compact. The truncated K-complex moment problem for a sequence $\gamma \equiv \gamma^{(2d)} = \{\gamma_{0,0}, \gamma_{0,1}, \gamma_{1,0}, \dots, \gamma_{0,2d}, \dots, \gamma_{2d,0}\}$ $(\gamma_{0,0} > 0, \gamma_{i,j} = \overline{\gamma_{ji}})$, consists of finding a positive Borelean measure ν such that

$$\gamma_{ij} = \int_K \bar{z}^i z^j d\nu(z, \bar{z}), \ (0 \leqslant i + j \leqslant 2d) \text{ with } supp(\nu) \subset K. \tag{2}$$

Similarly, to the moment problem (1), if $d = +\infty$ the $K_{\mathbb{C}}$-moment problem (2) is called *full* complex moment problem. The solution of the complex moment problem has been explored, using various methods developped by Atzmon [2], Putinar [16], Curto-Fialkow [10, 11, 12], and others. In [12], Curto-Fialkow characterize the existence of a finitely atomic $K_{\mathbb{C}}$-representing measure. Their method is based on the employment of the moment matrix $M(d)(\gamma)$, associated to the sequence γ, which is defined by $M(d)(\gamma) = (M[i,j])_{0 \leqslant i,j \leqslant d}$, where $M[i,j]$ is the $(i+1) \times (j+1)$ block of Toeplitz form (i.e. with constant diagonals) such that the entries of the first row are given by $\gamma_{i,j}$, $\gamma_{i+1,j-1}$, $\cdots$, $\gamma_{i+j,0}$ and the entries of the first column are $\gamma_{i,j}$, $\gamma_{i-1,j+1}$, $\cdots$, $\gamma_{0,i+j}$ (see [10, 12] for more details). Moreover, the notion of flat data that hid the notion of recursiveness, plays an intrinsic role in the Curto-Fialkow's approach. On the other hands, in [4] Ben Taher-Rachidi consider the recursiveness of Fibonacci type as primordial tool to establish sufficient conditions for insuring when γ is a truncated moment sequence.

It is well known that the full and the truncated complex moment problems in $\mathbb{C}$ are equivalent, respectively, to corresponding analogous moment problem in $\mathbb{R}^2$, via the degree-one transformation $z \equiv x + iy$ (see [12]). Discussion on this closed connection has been emphasized in [8], where the representing measures for the truncated moment problems (1) and those of the Curto-Fialkow schemes for the truncated complex moment problems (2) are related. Even if the moment problem on $\mathbb{R}^2$ is more exploited in the literature, our main idea here is to give an approach by formulating the crucial bridge between the truncated moment problem in $\mathbb{R}^2$ and the linear generalized Fibonacci sequence. To this aim we define the notion of the *bi-indexed Fibonacci sequence* for $\{\beta_{i,j}\}_{(i,j) \in \mathbb{Z}_+^2}$ of order (r,s), by considering the two following recursive relations,

$$\begin{cases} \beta_{(k_1+1,k_2)} = \sum_{i=0}^{r-1} a_i \beta_{(k_1-i,k_2)} \\ \beta_{(k_1,k_2+1)} = \sum_{j=0}^{s-1} b_j \beta_{(k_1,k_2-j)}, \end{cases} \tag{3}$$

where $k_1 \geqslant r-1$ $(r \geqslant 2)$ and $k_2 \geqslant s-1$ $(s \geqslant 2)$. For a fixed $j \in \mathbb{Z}_+$, the sequences $\{\beta_{i,j}\}_{i \in \mathbb{Z}_+}$ are known in the literature as the generalized Fibonacci sequences of order r of characteristic polynomial $P(x) = x^r - a_0 x^{r-1} - \cdots - a_{r-1}$, as well as for a fixed $i \in \mathbb{Z}_+$, sequences $\{\beta_{i,j}\}_{j \in \mathbb{Z}_+}$ are also generalized Fibonacci sequences of order s of characteristic polynomial $Q(y) = y^s - b_0 y^{s-1} - \cdots - b_{s-1}$. For simplicity, the polynomials P and Q will be called the (characteristic) polynomials associated to the 2-dimensional real multisequence $\{\beta_{i,j}\}_{(i,j) \in \mathbb{Z}_+^2}$.

The rest of this study is organized as follows. First of all, we convert the T2MP to the truncated K-complex moment problem. Since the former moment problem is extensively studied via the generalized Fibonacci sequence method, we can derive some properties for the T2MP. The bi-indexed generalized Fibonacci sequence are involved, in order to introduce straightly the notion of recursiveness and discussing the T2MP. The results established by Curto-Fialkow, using the associated moment matrix, and the combinatorial expression of generalized Fibonacci sequences, are of great importance for providing the combinatorial expression of the columns of the associated moment matrix. We also give rise to the combinatorial expression for each term of the T2MP, which admits a finitely-atomic representing measure. In particular, we look into a detailed study for the truncated quartic moment problem.

2. The Truncated 2-variable moment problem for bi-indexed generalized Fibonacci sequence

2.1. The 2-variable truncated moment problem and the bi-indexed generalized Fibonacci sequence

In this section, we first give tractable formulas to emphasize on the closed connection between the truncated moment problem in $\mathbb{R}^2$ and the truncated complex moment problems in $\mathbb{C}$. The Binet formula of generalized Fibonacci sequences and the classical Newton's binomial formula are the key of this exploration. Appealing the relationship between the recursively representing measures for complex moment problems in $\mathbb{C}$ (see [10, 11, 12]) and the generalized Fibonacci sequence's properties established in [4], we arrive to supply the analogous link between the moment problem in $\mathbb{R}^2$, which admits a finitely atomic representing measure, and the bi-indexed generalized Fibonacci sequence (3).

Let $\beta \equiv \beta^{(2d)} \equiv \{\beta_{i,j}\}_{\{(i,j)\in\mathbb{Z}_+^2,\ i+j\leqslant 2d\}}$ be a 2-dimensional real multisequence of degree $2d$, and let $\mathscr{P}_d \equiv \mathbb{R}[x,y]_d$ denote the bivariate real polynomials of total degree at most d. For $p(x,y) \equiv \sum_{i,j\geqslant 0, i+j\leqslant d} a_{ij}x^iy^j \in \mathscr{P}_d$, and let $\hat{p} := (a_{ij})$ denote the vector coefficients with respect to the basis for $\mathscr{P}_d$, consisting of monomials in degree-lexicographic order, i.e., $\mathbf{1}$, x, y, x^2, xy, $\cdots$, x^d, $\cdots$, y^d. The moment matrix $\mathscr{M}_d \equiv \mathscr{M}_d(\beta)$, whose rows and columns are indexed by the monomials in $\mathscr{P}_d$, is defined by $L_\beta(pq) :=< \mathscr{M}_d\hat{p}, \hat{q} >$ $(p,q \in \mathscr{P}_d)$, where $L_\beta : \mathscr{P}_{2d} \to \mathbb{R}$ is the well known Riesz functional defined by

$$L_\beta\Big(\sum_{i,j\geqslant 0, i+j\leqslant d} a_{ij}x^iy^j\Big) = \sum a_{ij}\beta_{i,j}.$$

The successive rows and columns of the matrix $\mathscr{M}_d$ are labelled $\mathbf{1}$, X, Y, $\cdots$, X^d, $\cdots$, Y^d. Thus, the entry in row X^iY^j, column X^kY^l, which we denote by $\langle X^iY^j, X^kY^l\rangle$, is equal to $\beta_{(i+k,j+l)}$. A linear combination of rows or columns is written as follows $p(X,Y) := \sum a_{ij}X^iY^j$. For a polynomial $p \equiv \sum a_{ij}x^iy^j$ we set $p(X,Y) = \mathscr{M}_d\hat{p}$. We say that $\mathscr{M}_d$ is recursively generated if it satisfies the following property,

$$p, q, pq \in \mathscr{P}_d, p(X,Y) = 0 \Rightarrow (pq)(X,Y) = 0, \tag{4}$$

If β has a representing measure, then $\mathscr{M}_d$ is positive semi-definite and recursively generated (see [13]). In addition, it follows from [9] that β admits a finitely-atomic representing measure μ, which therefore has finite moments of all orders. As a consequence, $\mathscr{M}_d$ admits a positive recursively generated moment matrix extensions of all orders, namely $\mathscr{M}_{d+1}[\mu], \cdots, \mathscr{M}_{d+k}[\mu], \cdots$.

Assume that β admits a finitely-atomic representing measure μ; and set simply $\mu = \sum_{\ell=1}^{n} \rho_\ell \delta_{(x_\ell, y_\ell)}$. Put $S_1 = \{k;\ \ 1 \leqslant k \leqslant n \text{ with } x_k \neq x_i \text{ for } k \neq i\}$, $S_2 = \{k; 1 \leqslant k \leqslant n, \text{ with } y_k \neq y_j \text{ for } k \neq j\}$ and suppose that S_1 and S_2 are of sizes $r \leqslant n$ and $s \leqslant n$ (respectively). Therefore, the measure μ takes the form $\mu := \sum_{1 \leqslant k \leqslant r, 1 \leqslant h \leqslant s} \rho_{k,h} \delta_{(x_k, y_h)}$ (with $\rho_{k,h} \geqslant 0$), which means that $\beta_{i,j} = \sum_{1 \leqslant k \leqslant r, 1 \leqslant h \leqslant s} \rho_{k,h} x_k^i y_h^j$, for every $i, j \geqslant 0$ satisfying $i + j \leqslant 2d$. Accordingly, the truncated moment sequence $\beta \equiv \beta^{(2d)} := \{\beta_{i,j}\}_{i,j \geqslant 0, i+j \leqslant 2d}$ can be extended to the full moment sequence $\hat{\beta} := \{\hat{\beta}_{ij}\}_{i,j \geqslant 0}$ such that $\hat{\beta}_{ij} = \sum_{k,h; 1 \leqslant k \leqslant r, 1 \leqslant h \leqslant s} \rho_{k,h} x_k^i y_h^j$, for every $i, j \geqslant 0$. Hence, by the aid of the Binet formula, it is easily inferred that $\hat{\beta} := \{\hat{\beta}_{ij}\}_{i,j \geqslant 0}$ is a bi-indexed generalized Fibonacci sequence, whose the characteristic polynomial of the family of sequences $\{\hat{\beta}_{ij}\}_{j \geqslant 0}$ ($i \geqslant 0$ fixed) is $Q(y) = (y - y_1) \cdots (y - y_s)$ and that of the family of sequences $\{\hat{\beta}_{ij}\}_{i \geqslant 0}$ ($j \geqslant 0$ fixed) corresponds to $P(x) = (x - x_1) \cdots (x - x_r)$.

Suppose that $\beta_{0,0} > 0$, we can associate to β the complex sequence $\gamma \equiv \gamma^{(2d)} \equiv \{\gamma_{i,j}\}_{\{(i,j) \in \mathbb{Z}_+^2,\ i+j \leqslant 2d\}}$. Indeed, for every $k, \ell \in \mathbb{Z}_+^2$ with $k + \ell \leqslant 2d$, set

$$\gamma_{k,\ell} := L_\beta((x - iy)^k (x + iy)^\ell). \tag{5}$$

Clearly, we have $\gamma_{0,0} = L_\beta(1) = \beta_{0,0} > 0$, and $\gamma_{k,\ell} = \overline{\gamma_{\ell,k}}$. Let introduce the maps $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C} \times \mathbb{C}$ and $\tau : \text{Ran } \psi \to \mathbb{R} \times \mathbb{R}$ defined, respectively, by $\psi(x, y) = (z, \overline{z})$ and $\tau(z, \overline{z}) = \left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right)$, where $z \equiv x + iy$ and $\overline{z} = x - iy$. Some useful properties of L_β, are presented in the following lemma.

LEMMA 1. (see Proposition 2.8 of [13]) *Given a real sequence $\beta \equiv \beta^{(2d)}$ and consider $\gamma \equiv \gamma^{(2d)}$ its associated sequence obtained via (5). Then, we have the following assertions,*

(i) $\mathscr{M}_d(\beta) \geqslant 0 \Leftrightarrow M(d)(\gamma) \geqslant 0$.

(ii) $\text{rank } \mathscr{M}_d(\beta) = \text{rank } [M(d)(\gamma)]$.

(iii) If ν is a representing measure for γ, then $\mu = \nu \circ \psi$ is a representing measure for β, of the same measure class and cardinality of support. Moreover, we have $supp(\mu) = \tau(supp\ \nu)$.

Furthermore, a straightforward computation, using the Newton's binomial formula, leads to provide the combinatorial expression of β_{k_1,k_2} $(k_1 + k_2 \leqslant 2d)$ in terms of the $\gamma_{i,j}$ (with $0 \leqslant i \leqslant k_1 + k_2$, $0 \leqslant j \leqslant k_1 + k_2$) and conversely. Thus, we get the following property.

PROPOSITION 1. Let $\beta \equiv \beta^{(2d)}$, with $\beta_{0,0} > 0$, be a 2-dimensional real multisequence of degree $2d$ and $\gamma \equiv \gamma^{(2d)}$ its associated complex moment sequence. Then, we have

$$\begin{cases} \beta_{k_1,k_2} = \frac{1}{2^{k_1+k_2}}(-i)^{k_2} \sum_{k=0}^{k_1+k_2} \left[\sum_{S_k} \binom{k_1}{h}\binom{k_2}{j}(-1)^{k_2-j} \right] \gamma_{k,k_1+k_2-k} \\ \textit{where } S_k = \{(h,j)\ h+j=k,\ 0 \leqslant h \leqslant k_1,\ 0 \leqslant j \leqslant k_2\} \textit{ and} \\ \gamma_{k_1,k_2} = \sum_{k=0}^{k_1+k_2} \left[\sum_{S_k} \binom{k_1}{h}\binom{k_2}{j}(-1)^{k_1-h} \right] (i)^{k_1+k_2-k} \beta_{k,k_1+k_2-k}, \end{cases}$$

with $k_1, k_2 \in \mathbb{Z}_+$, $k_1 + k_2 \leqslant 2d$ and $i^2 = -1$.

Proposition 1 shows the closed link between the matrices $\mathscr{M}_2(\gamma)$ and $\mathscr{M}_2(\beta)$. To illustrate the preceding result, we examine the following example.

EXAMPLE 1. Let take again the following complex moment matrix associated to the sequence $\gamma^{(4)} \equiv \{\gamma_{i,j}\}_{0 \leqslant i+j \leqslant 4}$ (considered in [4]) defined as follows,

$$\mathscr{M}_2(\gamma) = \begin{array}{c} \begin{matrix} \mathbf{1} & Z & \overline{Z} & Z^2 & Z\overline{Z} & \overline{Z}^2 \end{matrix} \\ \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{-1}{3} & 1 & \frac{-1}{3} \\ \frac{1}{3} & 1 & \frac{-1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-1}{3} & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & 1 & \frac{1}{3} & 1 & \frac{-1}{3} & 1 \\ 1 & 1 & \frac{1}{3} & \frac{-1}{3} & 1 & \frac{-1}{3} \\ \frac{-1}{3} & 1 & \frac{1}{3} & 1 & \frac{-1}{3} & 1 \end{pmatrix} \end{array}.$$

In view of Proposition 1, we infer the terms of $\beta^{(4)}$, its associated 2- variable moment sequence as follows

$$\mathscr{M}_2(\beta) = \begin{array}{c} \begin{matrix} \mathbf{1} & X & Y & X^2 & XY & Y^2 \end{matrix} \\ \begin{pmatrix} 1 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \end{array}.$$

Let $\{\gamma_{i,j}\}_{i,\, j \geqslant 0}$ be a sequence of complex numbers with $\gamma_{i,j} = \overline{\gamma_{j,i}}$. The main result established in [4] is restated through the equivalence of the two following assertions,

1. *Assertion* 1: $\{\gamma_{i,j}\}_{i,j \geqslant 0}$ is a moment sequence which owns an r-atomic representing measure μ.

2. *Assertion* 2: The family of sequences $\{\gamma_{i,j}\}_{j \geqslant 0}$, where $i \geqslant 0$ (fixed), are linear recursive sequences of order r, with the same characteristic polynomial which owns distinct complex roots $z_0, \cdots, z_{r-1}$. More precisely, $\gamma_{i,n} = \sum_{j=0}^{r-1} \rho_{i,j} z_j^n$, with $\rho_{0,j} \geqslant 0$ and $\rho_{i,j} = \rho_{0,j} \overline{z_j}^i$.

Consider a real sequence $\beta \equiv \{\beta_{i,j}\}_{\{(i,j) \in \mathbb{Z}_+^2\}}$ and let $\gamma \equiv \{\gamma_{i,j}\}_{\{(i,j) \in \mathbb{Z}_+^2\}}$ be the associated complex sequence obtained by the transformation (5). Assume further that β admits an r-atomic representing measure μ, such that $\{(x_i, y_i)\}_{i=1}^r$ are its distinct atoms.

Hence, the representing measure associated to $\gamma = \gamma^{(2d)}$ has a finite atomic representing measure with exactly r atoms $\{x_k + iy_k)\}_{k=1}^{r}$. In this case, according to *Assertion* 1 or *Assertion* 2 above, the family of sequences $\{\gamma_{i,j}\}_{j\geqslant 0}$ ($i \geqslant 0$ fixed) are generalized Fibonacci sequences, with the same characteristic polynomial which owns the distinct complex roots $x_1 + iy_1, \cdots, x_r + iy_r$. More precisely, we have $\gamma_{i,\ell} = \sum_{j=0}^{r-1} \rho_{i,j} z_j^{\ell}$, with $z_j = x_j + iy_j$, $\rho_{0,j} \geqslant 0$ and $\rho_{i,j} = \rho_{0,j}\overline{z_j}^{i}$.

Employing the equivalence of *Assertion* 1 and *Assertion* 2 (see [4]), and utilizing the equivalence between the complex moment problems and the 2-variable moment problems via the degree-one transformation $z \equiv x + iy$, we manage to have an analogous result.

PROPOSITION 2. Let $\{\beta_{i,j}\}_{i,\, j\geqslant 0}$ be a sequence of real numbers. Then, the two following affirmations are equivalent,

1. $\{\beta_{i,j}\}_{i,j\geqslant 0}$ is a moment sequence which owns an r-atomic representing measure μ, whose atoms are $\{(x_i, y_i)\}_{i=1}^{r}$.
2. The sequence $\{\beta_{i,j}\}_{j\geqslant 0}$ is a bi-indexed generalized Fibonacci sequence, such that the characteristic polynomial of the family of sequences $\{\beta_{i,j}\}_{j\geqslant 0}$ ($i \geqslant 0$) owns the all distinct values of $\{y_1, \cdots, y_r\}$ in $\mathbb{R}$ as roots, and the characteristic polynomial of the family of sequences $\{\beta_{i,j}\}_{i\geqslant 0}$ ($j \geqslant 0$) owns the all distinct values of $\{x_1, \cdots, x_r\}$ in $\mathbb{R}$ as roots. More precisely, we have $\beta_{nm} = \sum_{i=1}^{r} \rho_i x_i^n y_i^m$ with $\rho_i \geqslant 0$.

Proof. First, the implication $1) \Rightarrow 2)$ is obvious. To prove $2) \Rightarrow 1)$, we use the expression

$$\gamma_{k_1,k_2} = \sum_{k=0}^{k_1+k_2} [\sum_{h+j=k, 0\leqslant h\leqslant k_1, 0\leqslant j\leqslant k_2} \binom{k_1}{h}\binom{k_2}{j}(-1)^{k_1-h}](i)^{k_1+k_2-k}\beta_{k,k_1+k_2-k}$$

and the fact that the family of sequences $\{\gamma_{i,j}\}_{j\geqslant 0}$ (for $i \geqslant 0$), are r-generalized Fibonacci sequences, whose characteristic polynomial owns r distinct roots $z_1 = x_1 + iy_1, \cdots, z_r = x_r + iy_r$ in $\mathbb{C}$. $\square$

REMARK 1. It ensues from Proposition 2 and the result of Bayer-Teichmann (see [9]), on the truncated multivariable moment problem, that if the multisequence $\beta \equiv \beta^{(2d)} \equiv \{\beta_{i,j}\}_{\{(i,j)\in\mathbb{Z}_+^2,\ i+j\leqslant 2d\}}$ is a 2 variable truncated moment sequence, then β may be extended to a full moment problem, which is in turn a bi-indexed generalized Fibonacci sequence.

2.2. A combinatorial expression for the 2 variable moment sequence via Fibonacci sequence

In this subsection, we are going to exploit the combinatorial expression of generalized Fibonacci sequences. More precisely, we manage to obtain the combinatorial expression for each term of the 2-variable moment sequence, which admits a finitely atomic representing measure. Assume that $\beta \equiv \beta^{(2d)} \equiv \{\beta_{i,j}\}_{\{(i,j)\in\mathbb{Z}_+^2,\ i+j\leqslant 2d\}}$ is a truncated moment sequence. Let $k,\ell \in \mathbb{Z}_+$ and consider $A \in M_k(\mathbb{C})$ satisfying $A = A^*$, $B \in M_{k,\ell}(\mathbb{C})$ and $C \in M_\ell(\mathbb{C})$. Any matrix of the form,

$$\tilde{A} = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix},$$

will designate an *extension* of A. For $A \geqslant 0$, the following assertions are equivalent, $\tilde{A} \geqslant 0 \Leftrightarrow$ There exists $W \in M_{k,\ell}(\mathbb{C})$ such that $AW = B$ and $C \geqslant W^*AW$ (see [10, 11, 12]).Furthermore, if $A \geqslant 0$ then, any extension $\tilde{A}$ satisfying $rank(\tilde{A}) = rank(A)$ is necessarily positive (see [11]).

For every integer $d \geqslant 1$, the moment matrix $M_d(\beta)$ associated to the sequence β, admits the following decomposition $M_d = (B[i,j])_{0\leqslant i,j\leqslant d}$, where

$$B[i,j] = \begin{pmatrix} \beta_{i+j,0} & \cdots & \beta_{i,j} \\ \vdots & \vdots & \vdots \\ \beta_{j,i} & \cdots & \beta_{0,i+j} \end{pmatrix}.$$

Suppose that the characteristic polynomials P and Q, associated to the bi-indexed generalized Fibonacci sequence emanated from β, admit distinct roots $x_1,\cdots,x_r$ and $y_1,\cdots,y_s$ (respectively). This former hypothesis permits to construct an interpolating measure for the sequence β. Indeed, consider the measure μ defined as follows $\mu = \sum_{1\leqslant i\leqslant r,\ 1\leqslant j\leqslant s} \rho_{i,j}\delta_{(x_i,y_j)}$, where the coefficients $\rho_{i,j}$ are solution of the following system of $r \times s$ equations,

$$\sum_{1\leqslant i\leqslant r, 1\leqslant j\leqslant s} \rho_{i,j} x_i^n y_j^m = \beta_{n,m}, \ \ 0 \leqslant n \leqslant r-1, \ \ 0 \leqslant m \leqslant s-1. \tag{6}$$

Notice that the determinant of this system (of Vandermonde type) is nonzero, because we have $\prod_{1\leqslant i<j\leqslant r}(x_i - x_j) \neq 0$ and $\prod_{1\leqslant i<j\leqslant s}(y_i - y_j) \neq 0$. Consequently, the measure $\mu = \sum_{1\leqslant i\leqslant r,\ 1\leqslant j\leqslant s} \rho_{i,j}\delta_{(x_i,y_j)}$, completely interpolates the sequence β. Obviously, the polynomials P and Q can be written, respectively, under the forms $P(x) = x^r - a_0x^{r-1} - \cdots - a_{r-1}$ and $Q(y) = y^s - b_0y^{s-1} - \cdots - b_{s-1}$. Therefore, we may show that the moment matrix $M_d(\beta)$ satisfies the following column dependence properties $X^{n+1} = a_0X^n + \cdots + a_{r-1}X^{n-r+1} = X^{n-r+1}P(X)$ and $Y^{m+1} = b_0Y^m + \cdots + b_{s-1}Y^{m-s+1}$ $= Y^{m-s+1}Q(Y)$, for every $n \geqslant r-1$; $m \geqslant s-1$. Thus, we show that the powers X^n and Y^m satisfy the r-th and s-th linear relations: $X^{n+1} = a_0X^n + \cdots + a_{r-1}X^{n-r+1}$ (for $n \geqslant r-1$) and $Y^{m+1} = b_0Y^m + \cdots + b_{s-1}Y^{m-s+1}$ (for $m \geqslant s-1$). Employing the combinatorial expression of generalized Fibonacci sequences (see [7]), we get

$$\begin{cases} X^n = \sum_{k=0}^{r-1} \left(\sum_{i=0}^{k} a_{r-k+i-1}\rho(n-i,r)\right) X^k, \\ \text{and} \\ Y^m = \sum_{\ell=0}^{s-1} \left(\sum_{j=0}^{\ell} b_{s-\ell+j-1}\phi(m-j,s)\right) Y^\ell, \end{cases} \tag{7}$$

for any $n \geqslant r$ and $m \geqslant s$, where

$$\begin{cases} \rho(n,r) = \sum_{k_0+2k_1+\ldots+rk_{r-1}=n-r} \frac{(k_0+\ldots+k_{r-1})!}{k_0!k_1!\ldots k_{r-1}!} a_0^{k_0} a_1^{k_1} \ldots a_{r-1}^{k_{r-1}}, \\ \text{and} \\ \phi(m,s) = \sum_{\ell_0+2\ell_1+\ldots+s\ell_{s-1}=m-s} \frac{(\ell_0+\ldots+\ell_{s-1})!}{\ell_0!\ell_1!\ldots\ell_{s-1}!} b_0^{\ell_0} b_1^{\ell_1} \ldots b_{s-1}^{\ell_{s-1}}. \end{cases}$$

Since the roots of the polynomials P and Q are considered distinct, the expressions of $\rho(n,r)$ and $\phi(m,s)$, can be expressed in terms of the roots x_i ($1 \leqslant i \leqslant r$) and y_i ($1 \leqslant i \leqslant s$), under the following simpler forms $\rho(n,r) = \sum_{i=1}^{r} \frac{x_i^{n-1}}{P'(x_i)}$ and $\phi(m,s) = \sum_{j=1}^{s} \frac{y_j^{m-1}}{Q'(y_j)}$ (see [7]). Therefore, we show that every column of the form X^nY^m with $n \geqslant r$ or/and $m \geqslant s$), may be expressed, using formula (7), in terms of X^kY^ℓ with $k \leqslant r-1$ and $\ell \leqslant s-1$. More precisely, we have the following three situations. If $n \leqslant r-1$ and $m \geqslant s$, we have

$$X^nY^m = \sum_{\ell=0}^{s-1} \left(\sum_{j=0}^{\ell} b_{s-\ell+j-1}\phi(m-j,s) \right) X^nY^\ell,$$

if $n \geqslant r$ and $m \leqslant s-1$, we get

$$X^nY^m = \sum_{k=0}^{r-1} \left(\sum_{i=0}^{k} a_{r-k+i-1}\rho(n-i,r) \right) X^kY^m,$$

and if $n \geqslant r$ and $m \geqslant s$ then, we obtain

$$X^nY^m = \sum_{k=0}^{r-1} \sum_{\ell=0}^{s-1} \Delta_{k,\ell} X^kY^\ell,$$

where $\Delta_{k,\ell} = \sum_{i=0}^{k} \sum_{j=0}^{\ell} a_{r-k+i-1} b_{s-\ell+j-1} \rho(n-i,r)\phi(m-j,s)$, $\rho(n,r)$ and $\phi(m,s)$ are given as above.

Consider the moment matrix $\mathcal{M}_{r+s-2}(\beta)$ associated to the sequence β. We can observe that $\mathcal{M}_{r+s-1}(\beta)$ is a matrix extension of $\mathcal{M}_{r+s-2}(\beta)$ with the same *rank*. Indeed, the matrix $\mathcal{M}_{r+s-1}(\beta)$ is given by,

$$\mathcal{M}_{r+s-1}(\beta) = \left[\begin{array}{c|c} \mathcal{M}_{r+s-2} & B(r+s-1) \\ \hline B(r+s-1)^T & C(r+s-1) \end{array} \right],$$

where $B(r+s-1) = (B[i,r+s-1])_{0 \leqslant i \leqslant r+s-2}$ and $C = B[r+s-1,r+s-1]$. Clearly, the columns of $B(r+s-1)$ will be denoted by $X^{r+s-1}, \cdots, Y^{r+s-1}$. All the previous column, appearing in $B(r+s-1)$, may be expressed in terms of X^aY^b, with $a,b \in \mathbb{N}$ and $a+b \leqslant r+s-2$ as explained by the three cases above. In other words, the entries of $\mathcal{M}_{r+s-1}(\beta)$ are expressed in terms of those of $M_{r+s-2}(\beta)$.

For every matrix extension $\mathcal{M}_{r+s-2+k}(\beta)$ (with $k \in \mathbb{N}^*$) of the matrix $\mathcal{M}_{r+s-2}(\beta)$, using the three cases derived from (7), it can be easily shown that the columns X^aY^b

(with $a,b \in \mathbb{N}$ and $a+b \geqslant r+s-1$) are completely expressed in terms of the columns $1, X, Y, \cdots, X^{r+s-2}, \cdots, Y^{r+s-1}$. To sum up for all $k \geqslant 1$, we have $rank[\mathscr{M}_{r+s-2}(\beta)] = rank[\mathscr{M}_{r+s-2+k}(\beta)]$. For proving our next result the following lemma is required,

LEMMA 2. [12] *Let $\gamma \equiv \gamma^{(2d)}$ be a complex sequence, admitting an r-atomic interpolating measure ν, with $r \leqslant k+1$. If $M(k)(\gamma) \geqslant 0$, then $\nu \geqslant 0$.*

THEOREM 1. *Let $\{\beta_{i,j}\}_{(i,j)\in\mathbb{Z}_+^2}$ be a bi-indexed generalized Fibonacci sequence, of order (r,s), such that its associated characteristic polynomials P and Q admit distinct roots. If $\mathscr{M}_{r+s-2}(\beta) \geqslant 0$, then the full moment sequence $\beta \equiv \{\beta_{i,j}\}_{(i,j)\in\mathbb{Z}_+^2}$ has a representing measure and $rank[\mathscr{M}_{r+s-2}(\beta)] \leqslant \frac{(r+s-1)(r+s)}{2} - (r+s-2)$.*

Proof. Assume that the roots of the characteristic polynomials P and Q are given by $x_1, \cdots, x_r$ and $y_1, \cdots, y_s$ (respectively). Consider the truncated moment problem defined by the sequence $\beta^{(2(r+s-2))}$. Observe that $\beta^{(2(r+s-2))}$ is interpolated by at most the $r.s$-atomic measure of the form $\mu = \sum_{1\leqslant i\leqslant r,\ 1\leqslant j\leqslant s} \rho_{i,j}\delta_{(x_i,y_j)}$, where the coefficients $\rho_{i,j}$ are solution of the system (6). On the other hand, since we have rank$[\mathscr{M}_{r+s-2}(\beta)] =$ rank $[\mathscr{M}_{r\times s}(\beta)]$, it follows from Lemma 2 that $\mathscr{M}_{r\times s}(\beta) \geqslant 0$, which implies that $\mu \geqslant 0$. Finally, according to the hypothesis, the measure μ is also a representing measure of the full moment problem $\beta \equiv \{\beta_{i,j}\}_{(i,j)\in\mathbb{Z}_+^2}$. □

It is known that the real $\beta_{n,m}$ (n, $m \geqslant 0$) is precisely the entry in the row X^n and the column Y^m, therefore based on the three previous cases, we can give the explicit expression of the $\beta_{n,m}$, for every n and m, in terms of the $\{\beta_{k,l}\}_{0\leqslant k\leqslant r, 0\leqslant l\leqslant s}$. Indeed, we get the following result,

THEOREM 2. *Let $\{\beta_{i,j}\}_{(i,j)\in\mathbb{Z}_+^2}$ be a bi-indexed generalized Fibonacci sequence of order (r,s), such that its associated characteristic polynomials P and Q admit distinct roots. Then, we have*

$$\begin{cases} 1)\ \beta_{nm} = \langle X^n, Y^m \rangle = \sum_{l=0}^{s-1} \left(\sum_{j=0}^{l} b_{s-l+j-1}\phi(m-j,s) \right) \beta_{nl}, \\ for\ \ n \leqslant r-1 \ \ and \ \ m \geqslant s \\ 2)\ \beta_{nm} = \langle X^n, Y^m \rangle = \sum_{k=0}^{r-1} \left(\sum_{i=0}^{k} a_{r-k+i-1}\rho(n-i,r) \right) \beta_{km}, \\ for\ \ n \geqslant r \ \ and \ \ m \leqslant s-1 \\ 3)\ \beta_{nm} = \langle X^n, Y^m \rangle = \sum_{k=0}^{r-1} \sum_{l=0}^{s-1} \Delta_{k,l}\beta_{kl}, \\ for\ \ n \geqslant r \ \ and \ \ m \geqslant s, \end{cases} \tag{8}$$

where $\Delta_{k,l} = \sum_{i=0}^{k} \sum_{j=0}^{l} a_{r-k+i-1} b_{s-l+j-1} \rho(n-i,r)\phi(m-j,s)$, $\rho(n,r)$ and $\phi(m,s)$ are given as above.

We provide here an illustrative example, which allows us to show the application of Expression (8).

EXAMPLE 2. Let consider $\beta=\{\beta_{ij}\}_{i,j\geqslant 0}$ a bi-indexed generalized Fibonacci sequence, whose associated characteristic polynomials are $Q(y)=y^2-4y+3$ and $P(x)=x^3-2x^2-x+2$. It turns out from Expression (8) that the initial data $\{\beta_{00},\beta_{01},\ \beta_{10},\beta_{11},\ \beta_{20},\beta_{21}\}$ are sufficient for the determination of the explicit expression of the terms β_{ij}. That is, a direct calculation leads to have $\rho(n,3)=\frac{-1}{2}+\frac{(-1)^{n-1}}{6}+2^{n-2}$ for $n\geqslant 3$ and $\phi(m,2)=\frac{-1}{2}+\frac{3^{m-1}}{2}$, for $m\geqslant 2$. Assume that $\beta_{00}=1$, $\beta_{01}=0$, $\beta_{10}=0$, $\beta_{11}=1$, $\beta_{20}=1$, $\beta_{21}=0$. Thence, we obtain,

1) For $m\geqslant 2$, we have

$$\beta_{0,m}=\frac{3-3^m}{2},\ \ \beta_{1,m}=\frac{-1+3^m}{2}\quad\text{and}\quad\beta_{2,m}=\frac{3-3^m}{2}$$

2) For $n\geqslant 3$, we have

$$\beta_{n,0}=\frac{1}{2}+\frac{(-1)^{n-2}}{2}\quad\text{and}\quad\beta_{n,1}=\frac{1}{2}+\frac{(-1)^{n-1}}{2}$$

3) For $n\geqslant 3$ and $m\geqslant 2$, we have

$$\beta_{n,m}=\left(\frac{-3}{2}+3\frac{(-1)^{n-1}}{2}\right)\left(\frac{-1}{2}+\frac{3^{m-1}}{2}\right)+\left(\frac{1}{2}+\frac{(-1)^{n-1}}{2}\right)\left(\frac{-1}{2}+\frac{3^m}{2}\right).$$

A straightforward application of Theorem 1 shows that the positivity of the matrix $\mathcal{M}_3(\beta)$ implies that the sequence $\beta=\{\beta_{ij}\}_{i,j\geqslant 0}$ admits a representing measure. That is, using Expression (8) we get,

$$\mathcal{M}_3(\beta)=\begin{pmatrix}\mathbf{1} & X & Y & X^2 & XY & Y^2 & X^3 & X^2Y & XY^2 & Y^3\\ 1 & 0 & 0 & 1 & 1 & -3 & 0 & 0 & 4 & -12\\ 0 & 1 & 1 & 0 & 0 & 4 & 1 & 1 & -3 & 13\\ 0 & 1 & -3 & 0 & 4 & -12 & 1 & -3 & 13 & -39\\ 1 & 0 & 0 & 1 & 1 & -3 & 0 & 0 & 4 & -12\\ 1 & 0 & 4 & 1 & -3 & 13 & 0 & 4 & -12 & 40\\ -3 & 4 & -12 & -3 & 13 & -39 & 4 & -12 & 40 & -120\\ 0 & 1 & 1 & 0 & 0 & 4 & 1 & 1 & -3 & 13\\ 0 & 1 & -3 & 0 & 4 & -12 & 1 & -3 & 13 & -39\\ 4 & -3 & 13 & 4 & -12 & 40 & -3 & 13 & -39 & 121\\ -12 & 13 & -39 & -12 & 40 & -120 & 13 & -39 & 121 & -363\end{pmatrix}.$$

A calculation via FreeMat permits to infer that $rank(\mathcal{M}_3(\beta))=3$ and its eigenvalues are -456.9186, -0.0000, -0.0000, -0.0000, -0.0000, 0.0000, 0.0000, 0.0000, 4.5121, 6.4065. Thereby, the sequence $\beta=\{\beta_{ij}\}_{i,j\geqslant 0}$ does not admit a representing measure.

In view of Proposition 2 and Theorem 1, we can now state and prove a result concerning two sequences of measures $\{\mu_n\}_{n\geqslant 0}$ and $\{\nu_m\}_{m\geqslant 0}$, where μ_n is the representing measure of the subsequence $\{\beta_{n,j}\}_{j\geqslant 0}$ and ν_m is the representing measure of the subsequence $\{\beta_{i,m}\}_{i\geqslant 0}$. Thus, we have the following property.

PROPOSITION 3. Let $\{\beta_{i,j}\}_{i,\ j\geqslant 0}$ be a sequence of real numbers. Then, the two following affirmations are equivalent,

1. $\{\beta_{i,j}\}_{i,j\geqslant 0}$ is a moment sequence, whose representing measure μ is a finitely atomic.

2. Each of the two sequences of measures $\{\mu_n\}_{n\geqslant 0}$ and $\{\nu_m\}_{m\geqslant 0}$, representing measures of the two sequences $\{\beta_{n,j}\}_{j\geqslant 0}$ and $\{\beta_{i,m}\}_{i\geqslant 0}$, for fixed n, m (respectively), is a Fibonacci sequences of order r, s (respectively) such that the roots of their characteristic polynomial are distinct. In addition, we have $\mathscr{M}_{r+s-2}(\beta)\geqslant 0$.

Proof. We begin with $i)\Rightarrow ii)$. If $\{\beta_{i,j}\}_{i,j\geqslant 0}$ is a moment sequence, which owns a k-atomic representing measure μ whose atoms are $\{(x_i,y_i)\}_{i=1}^{k}$, then it ensues from Proposition 2.3 that we may write $\beta_{nm}=\sum_{i=1}^{k}\rho_i x_i^n y_i^m$ with $\rho_i\geqslant 0$. By putting $\mu_n=\sum_{i=1}^{k}\rho_i x_i^n\delta_{y_i}$ and $\nu_m=\sum_{i=1}^{k}\rho_i\delta_{x_i}y_i^m$, we infer that each of the two sequences of measures $\{\mu_n\}_{n\geqslant 0}$ and $\{\nu_m\}_{m\geqslant 0}$, representing respectively $\{\beta_{n,j}\}_{j\geqslant 0}$ and $\{\beta_{i,m}\}_{i\geqslant 0}$ (n, m fixed), is a Fibonacci sequence of order r, s (respectively) less than k; such that the roots of their characteristic polynomials are distinct. In addition, it follows from Theorem 1 that $\mathscr{M}_{r+s-2}(\beta)\geqslant 0$.

To prove $ii)\Rightarrow i)$, we suppose that the sequence of measures $\{\mu_n\}_{n\geqslant 0}$, where μ_n is a representing measure of the sequence $\{\beta_{n,j}\}_{j\geqslant 0}$, is a Fibonacci sequence of characteristic polynomial $P(x)=(x-x_1)\ldots(x-x_r)$ such that $x_i\neq x_j$, for $i\neq j$. Similarly, suppose that $\{\nu_m\}_{m\geqslant 0}$, where ν_m is the representing measure of the sequences $\{\beta_{i,m}\}_{i\geqslant 0}$, is a Fibonacci sequence of characteristic polynomial $Q(y)=(y-y_1)\ldots(y-y_s)$ such that $y_i\neq y_j$, for $i\neq j$. Thereafter, for i fixed, we get $\beta_{i,m}=\int x^i d\nu_m$. Thus, the sequence $\{\beta_{i,m}\}_{m\geqslant 0}$ is also a Fibonacci sequence of order s. The same reasoning, allows us to infer that $\{\beta_{n,j}\}_{n\geqslant 0}$ is a Fibonacci sequence of order r. In addition, we have $\mathscr{M}_{r+s-2}(\beta)\geqslant 0$, it turns out from Theorem 1 that the sequence $\{\beta_{i,j}\}_{i,j\geqslant 0}$ is a moment sequence, whose representing measure μ is a finitely atomic measure. $\square$

Proposition 3 gives an equivalent formulation of Proposition 2 in terms of sequences of representing measures, satisfying a recursive relation of Fibonacci type.

3. Application to the quartic moment problem

Let $\beta=\{\beta_{i,j}\}_{i,j\geqslant 0}$ be a bi-indexed generalized Fibonacci sequence, such that the characteristic polynomial $Q(y)$ associated to the family of sequences $\{\beta_{i,j}\}_{j\geqslant 0}$ ($i\geqslant 0$ fixed) owns two distinct roots y_1,y_2 in $\mathbb{R}$ i.e $Q(y)=(y-y_1)(y-y_2)=y^2-b_0y-b_1$. And assume that $P(x)$, the characteristic polynomial associated to the family of sequences $\{\beta_{i,j}\}_{i\geqslant 0}$ ($j\geqslant 0$ fixed), owns also two distinct roots x_1,x_2 in $\mathbb{R}$ i.e $P(x)=(x-x_1)(x-x_2)=x^2-a_0x-a_1$. It ensues from Theorem 2.6 that if $M_2(\beta)\geqslant 0$ then $\beta=\{\beta_{i,j}\}_{i,j\geqslant 0}$ is a moment sequence whose representing measure μ is a k-atomic measure with $k\leqslant 4$. The columns of $\mathscr{M}_2(\beta)$ are labeled $\mathbf{1},X,Y,X^2,XY,Y^2$, we observe that the columns X^2 and Y^2 are given by $X^2=a_0X+a_1 1$ and $Y^2=b_0Y+b_1 1$. Thus, we have $rank[\mathscr{M}_2(\beta)]\leqslant 4$.

Once $rank[\mathscr{M}_2(\beta)]=4$, we show that the family of columns $\{\mathbf{1},X,Y,XY\}$ is linearly independent. Since $4=rank[\mathscr{M}_2(\beta)]\leqslant card[supp(\mu)]\leqslant 4$ (see [12, Proposition 3.1 and corollary 3.7]). Therefore, we obtain $card[supp(\mu)]=4$.

PROPOSITION 4. Assume that the family of columns $\{\mathbf{1}, X, Y, XY\}$ is linearly independent. Then, we have $rank[\mathscr{M}_2(\beta)] = 4$, and $\mathscr{M}_2(\beta) \geqslant 0$ if and only if $\beta = \{\beta_{i,j}\}_{i,j\geqslant 0}$ admits a representing measure owning exactly 4 atoms, with

$$supp(\mu) = \{(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2)\},$$

where x_1, x_2 and y_1, y_2 are (respectively) the roots of the characteristic polynomials P, Q of the bi-indexed generalized Fibonacci sequence $\beta = \{\beta_{i,j}\}_{i,j\geqslant 0}$.

If $rank[\mathscr{M}_2(\beta)] < 4$ then the family of columns $\{\mathbf{1}, X, Y, XY\}$ is linearly dependent. Thus, if $\mathscr{M}_1(\beta) > 0$ we have $rank[\mathscr{M}_2(\beta)] = rank[\mathscr{M}_1(\beta)]$. Therefore, we get,

PROPOSITION 5. Assume that the family of columns $\{\mathbf{1}, X, Y, XY\}$ is linearly dependent. Then, we have $\mathscr{M}_1(\beta) > 0 \iff \beta = \{\beta_{i,j}\}_{i,j\geqslant 0}$ admits a finitely atomic representing measure.

Now, we will succeed in weakening the condition of [14, Theorem 3.8]. That is, due to the fact that the sequence $\beta = \{\beta_{i,j}\}_{i,j\geqslant 0}$ is a bi-indexed generalized Fibonacci sequence, we show that the three cases studied by Fialkow *et al.* in [14] are reduced to one studied by our approach in our Proposition 4.

In what follows, we give in details the calculations when the columns of the set $\{\mathbf{1}, X, Y, XY\}$, are linearly dependent. We consider the moment matrix $\mathscr{M}_2(\beta)$

$$\mathscr{M}_2(\beta) = \begin{bmatrix} \beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} \\ \beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} \\ \beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} \\ \beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} \\ \beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} \\ \beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} \end{bmatrix}.$$

We may rewrite $\mathscr{M}_2(\beta)$ in terms of $\mathscr{M}_1(\beta)$ and the blocs $B(2)$ and $C(2)$, we have

$$\mathscr{M}_2 = \left[\begin{array}{c|c} \mathscr{M}_1(\beta) & B(2) \\ \hline B(2)^T & C(2) \end{array}\right],$$

where the matrices $\mathscr{M}_1(\beta)$, $B(2)$ and $C(2)$ are given by,

$$\mathscr{M}_1(\beta) = \begin{bmatrix} \beta_{00} & \beta_{10} & \beta_{01} \\ \beta_{10} & \beta_{20} & \beta_{11} \\ \beta_{01} & \beta_{11} & \beta_{02} \end{bmatrix}, \ B(2) = \begin{bmatrix} \beta_{20} & \beta_{11} & \beta_{02} \\ \beta_{30} & \beta_{21} & \beta_{12} \\ \beta_{21} & \beta_{12} & \beta_{03} \end{bmatrix}, \ C(2) = \begin{bmatrix} \beta_{40} & \beta_{30} & \beta_{21} \\ \beta_{31} & \beta_{22} & \beta_{13} \\ \beta_{22} & \beta_{13} & \beta_{04} \end{bmatrix}.$$

Suppose that the family $\{\mathbf{1}, X, Y, XY\}$ is linearly dependent. We release from the column dependence relation that there exists a, b and c in $\mathbb{R}$ such that

$$\begin{cases} \beta_{11} = a\beta_{00} + b\beta_{10} + c\beta_{01}, \\ \beta_{21} = a\beta_{10} + b\beta_{20} + c\beta_{11}, \\ \beta_{12} = a\beta_{01} + b\beta_{11} + c\beta_{02}. \end{cases}$$

If $\mathscr{M}_1(\beta) > 0$, we must have $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (\mathscr{M}_1(\beta))^{-1} \begin{pmatrix} \beta_{11} \\ \beta_{21} \\ \beta_{12} \end{pmatrix}$. Furthermore, the columns dependence $X^2 = a_0 X + a_1 1$ and $Y^2 = b_0 Y + b_1.\mathbf{1}$ lead to infer that $Ran[B(2)] \subseteq Ran[\mathscr{M}_1(\beta)]$. Thereby, using the recursive relation emanated from the fact that $\beta = \{\beta_{i,j}\}_{i,j\geqslant 0}$ are considered to be a bi-indexed generalized Fibonacci sequence, we verify that

$$\begin{cases} \beta_{31} = a\beta_{20} + b\beta_{30} + c\beta_{21}, \\ \beta_{13} = a\beta_{02} + b\beta_{12} + c\beta_{03}. \end{cases}$$

It is straightforward to check that the terms β_{22} appearing in $C(2)$ must satisfy the relation

$$\beta_{22} = \begin{pmatrix} \beta_{11} \\ \beta_{21} \\ \beta_{12} \end{pmatrix}^T \mathscr{M}_1(\beta)^{-1} \begin{pmatrix} \beta_{11} \\ \beta_{21} \\ \beta_{12} \end{pmatrix}. \tag{9}$$

Now, we assume that $rank[\mathscr{M}_1(\beta)] = 2$, thus we have a column dependence relation. Hence, there exists reals α, λ such that $Y = \alpha 1 + \lambda X$. This leads to get,

$$\begin{cases} \beta_{01} = \alpha\beta_{00} + \lambda\beta_{10} \\ \beta_{11} = \alpha\beta_{10} + \lambda\beta_{20} \\ \beta_{02} = \alpha\beta_{01} + \lambda\beta_{11} \end{cases}$$

Employing that $\beta = \{\beta_{i,j}\}_{i,j\geqslant 0}$ is a bi-indexed generalized Fibonacci sequence and the above relation, a straightforward computation yields,

$$\begin{cases} \beta_{11} = \alpha\beta_{10} + \lambda\beta_{20} \\ \beta_{21} = \alpha\beta_{20} + \lambda\beta_{30} \end{cases}$$

Whence, in order that the condition $\mathscr{M}_1(\beta) \geqslant 0$ may be a sufficient condition so that β has a representing measure, we must merely have the condition $\beta_{12} = \alpha\beta_{11} + \lambda\beta_{21}$. That is, we manage to obtain $XY = \alpha X + \lambda X^2$ in $\mathscr{M}_2(\beta)$, which also leads to $rank[\mathscr{M}_2(\beta)] = rank[\mathscr{M}_1(\beta)]$. We suppose now that $rank[\mathscr{M}_1(\beta)] = 1$, so there exists two scalars h, k such that

$$\begin{cases} \beta_{10} = h\beta_{00}\,, \quad \beta_{11} = h\beta_{01}, \quad \beta_{01} = k\beta_{00}, \\ \beta_{20} = h\beta_{10}\,, \quad \beta_{11} = k\beta_{10}\,, \quad \beta_{02} = k\beta_{01}. \end{cases}$$

A straightforward computation permits to get in $\mathscr{M}_2(\beta)$ the columns dependence relation

$$X^2 = (a_0 h + a_1).\mathbf{1}, \;\; XY = hk.\mathbf{1}, \;\; Y^2 = (b_0 k + b_1).\mathbf{1}.$$

Hence, it turns out that $rank\mathscr{M}_1(\beta) = rank\mathscr{M}_2(\beta)$. We summarize the preceding discussion as follows.

PROPOSITION 6. Assume that the family of columns $\{\mathbf{1}, X, Y, XY\}$ is linearly dependent and $\mathscr{M}_1(\beta) \geqslant 0$. Then, we have

1. $rank[\mathscr{M}_1(\beta)] = 3 \iff \beta = \{\beta_{i,j}\}_{i,j\geqslant 0}$ is a full moment sequence satisfying the compatibility condition (9) in other words $\beta_{22} = \begin{pmatrix} \beta_{11} \\ \beta_{21} \\ \beta_{12} \end{pmatrix}^T \mathscr{M}_1(\beta)^{-1} \begin{pmatrix} \beta_{11} \\ \beta_{21} \\ \beta_{12} \end{pmatrix}$.

2. $rank[\mathscr{M}_1(\beta)] = 2$, put $Y = \alpha 1 + \lambda X$. Then, the sequence $\beta = \{\beta_{i,j}\}_{i,j\geqslant 0}$ is a full moment sequence satisfying $rank[\mathscr{M}_1(\beta)] = rank[\mathscr{M}_2(\beta)]$ if and only if $\beta_{12} = (\alpha + \lambda a_0)\beta_{11} + \lambda a_1 \beta_{01}$.

3. $rank[\mathscr{M}_1(\beta)] = 1 \iff \beta = \{\beta_{i,j}\}_{i,j\geqslant 0}$ is a full moment sequence satisfying the following rank condition $rank[\mathscr{M}_2(\beta)] = 1$

In fact, as shown in the proof of Proposition 6 the compatibility condition (9) is necessary for assuring the coincidence of extension $\widehat{\beta} = \{\widehat{\beta}_{i,j}\}_{i,j\geqslant 0}$ of the system $S = \{\beta_{0,0}, \beta_{1,0}, \beta_{0,1}, \beta_{1,1}\}$ and the $\beta = \{\beta_{i,j}\}_{i,j\geqslant 0}$. For reason of clarity let consider the following particular situation.

PROPOSITION 7. Assume that the family of columns $\{\mathbf{1}, X, Y, XY\}$ is linearly dependent and consider the initial data $S = \{\beta_{0,0}, \beta_{1,0}, \beta_{0,1}, \beta_{1,1}\}$. Let $\beta = \{\beta_{i,j}\}_{i,j\geqslant 0}$ be the completion of S as a bi-indexed generalized Fibonacci sequence, whose characteristic polynomials are $P(z) = z^2 - a_0 z - a_1$ and $Q(z) = z^2 - b_0 z - b_1$. Let $R(z) = det[\mathscr{M}_1(\beta) - zI_3]$ be the characteristic polynomial of the matrix $\mathscr{M}_1(\beta)$ and $\mathscr{Z}(R)$ the set of roots of $R(z)$. Therefore,

1. If $\mathscr{Z}(R) \subset [0; +\infty[$, then we have $\mathscr{M}_1(\beta) \geqslant 0$, and the conditions of Proposition 6 may assure that $\beta = \{\beta_{i,j}\}_{i,j\geqslant 0}$ is a full moment sequence of a finite atomic representing measure.

2. If $\mathscr{Z}(R) \cap]-\infty, 0[= \emptyset$, then the 2-dimensional real multisequence $\beta = \{\beta_{i,j}\}_{i,j\geqslant 0}$ is not a full moment sequence.

Application of results of Propositions 6 and 7, allows us to formulated two algorithmic process. In the first algorithm we give the process for characterizing the full moment sequence and in the second one we consider the construction of the representing measure of the full moment sequence.

To illustrate the fallout of Propositions 5, 6, 7, we treat the following numerical examples.

EXAMPLE 3. We consider the set $S = \{\beta_{00}, \beta_{10}, \beta_{01}, \beta_{11}\}$ such that $\beta_{00} = t$, $\beta_{10} = a$, $\beta_{01} = b$ and $\beta_{11} = c$. Now we can construct $\beta = \{\beta_{i,j}\}_{i,j\geqslant 0}$, the completion of S as a bi-indexed generalized Fibonacci sequence, of characteristic polynomials $P(x) = x^2 - 3x + 1$ and $Q(y) = y^2 - 2$. We obtain $\beta_{20} = 3a - t$, $\beta_{02} = 2t$ and thus, we have

$$\mathscr{M}_1(\beta) = \begin{bmatrix} t & a & b \\ a & \alpha & c \\ b & c & \eta \end{bmatrix},$$

where $\alpha = 3a - t$ and $\eta = 2t$. Suppose that $rank[\mathscr{M}_1(\beta)] = 3$ and the family $\{\mathbf{1}, X, Y, XY\}$ is linearly dependent. Then, the polynomial $R(z) = det[\mathscr{M}_1(\beta) - zI_3]$ is given by

$$R(z) = -z^3 + (t+\alpha+\eta)z^2 + [a^2+b^2+c^2-(\alpha+\eta)t-\alpha\eta]z + \Lambda(a,b,c,t,\alpha,\eta),$$

where $\Lambda(a,b,c,t,\alpha,\eta) = \alpha\eta t - tc^2 - a^2\eta - b^2\alpha - 2abc$.

Following the numerical values of the initial data $S = \{\beta_{00}, \beta_{10}, \beta_{01}, \beta_{11}\}$, the set $\mathscr{Z}(R)$ of the eigenvalues of $\mathscr{M}_1(\beta)$ may be a subset of $[0, +\infty[$ or not.

That is, we have the following numerical situations.

1) For $t \in]\frac{3}{4}; \frac{5}{2}[$ and $a = b = c = 1$ or $b \in [-\frac{1}{4}; \frac{5}{4}]$ and $t = a = c = 1$ we can show that $\mathscr{M}_1(\beta) > 0$. Hence, it turns out that the sequence $\beta = \{\beta_{i,j}\}_{i,j\geqslant 0}$ is a full moment sequence.

2) For $t \in]-5; \frac{3}{4}[$ and $a = b = c = 1$ or $b \in]-5; -\frac{2}{5}[$ and $t = a = c = 1$ we can establish that $\mathscr{Z}(R) \cap]-\infty; 0[\neq \emptyset$. Therefore, the sequence $\beta = \{\beta_{i,j}\}_{i,j\geqslant 0}$ is not a full moment sequence.

Other numerical situations can be given.

The following example illustrate the case when $rank[\mathscr{M}_1(\beta)] = 2$.

EXAMPLE 4. Let $\mathscr{M}_1(\beta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. It's obvious that $\beta_{20} = 1$, $\beta_{02} = 1$ and $\beta_{12} = 4$, and we have $rank[\mathscr{M}_1(\beta)] = 2$, where $\mathscr{M}_1(\beta) \geqslant 0$, $\beta_{12} = \beta_{21}$. It follows from the second case in Proposition 6 that $\beta = \{\beta_{i,j}\}_{i,j\geqslant 0}$ is a full moment sequence.

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