Operators
and
Matrices
Volume 11, Number 4 (2017), 969–998
doi:10.7153/oam-2017-11-68

NON–ARCHIMEDEAN GNS CONSTRUCTION AND NON–ARCHIMEDEAN KREIN—MILMAN THEOREM

TOMOKI MIHARA

(*Communicated by N.-C. Wong*)

Abstract. We establish non-Archimedean analogues of the GNS construction and Krein–Milman theorem. For this purpose, we introduce notions of a state on a non-Archimedean algebra and of a convex subset of a non-Archimedean vector space. As an application, we construct two operator algebras associated to topological groups over which cyclic Banach left modules correspond to cyclic unitary representations approximated by finite dimensional cyclic semisimple unitary representations.

Introduction

This paper is devoted to two topics. One is a non-Archimedean analogue of the GNS construction (cf. [12] Proposition 6.7.4), and the other one is a non-Archimedean analogue of Krein–Milman theorem (cf. [11] Theorem 5.11.1). For this purpose, we introduce notions of a state and a convexity in the non-Archimedean setting. We note that there is an open question in [10] on a formulation of a convexity in the non-Archimedean setting. Two convexities called M-convexity and 0-convexity in [9] are ones of answers for it, and the convexity in this paper is also one of answers, because they satisfy non-Archimedean analogues of Krein–Milman theorem.

To begin with, we recall the GNS construction and Krein–Milman theorem in the Archimedean analysis. A state is a positive linear functional on a C^*-algebra $\mathscr{A}$ of norm 1, which automatically sends 1 to 1 as long as $\mathscr{A}$ is unital. For any Hilbert $\mathscr{A}$-module $\mathscr{H}$ admitting a cyclic vector ξ of norm 1, the map $\mathscr{A} \to \mathbb{C}$, $f \mapsto \langle \xi \mid f\xi \rangle$ forms a state on $\mathscr{A}$, and is called the vector state associated to $(\mathscr{H}, \xi)$. The classical theorem on the GNS construction states that every state on $\mathscr{A}$ is a vector state. The GNS construction played an important role in Tomita–Takesaki theory (cf. [14]), which gave a majour breakthrough to the study of type III factor von Neumann algebras.

Let V be an $\mathbb{R}$-vector space, and S a convex subset of V. A point $F \in S$ is said to be an *extreme point of* S if $S \setminus \{F\}$ is convex. Krein–Milman theorem states that every compact convex subset in a topological $\mathbb{R}$-vector space coincides with the closure of the convex hull of the subset of its extreme points. It is well-known that the set of states on $\mathscr{A}$ forms a compact convex subset of the continuous dual of $\mathscr{A}$

Mathematics subject classification (2010): 11S23, 47L55, 43A10.
Keywords and phrases: Non-Archimedean analysis, GNS construction, Krein–Milman theorem.

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equipped with the weak star topology by Banach–Alaoglu theorem (cf. [12] Theorem 1.5.4). A state is said to be pure if it is an extreme points of the compact convex subset of states. By Krein–Milman theorem, every state lies in the closure of the convex hull of the set of pure states. Namely, every state can be approximated by mixed states, i.e. states presented as $[0,1]$-linear combinations of pure states. This implies that every C^*-algebra admits many pure states, which are morally regarded as points in the context of non-commutative geometry. There is a well-known conjecture on a non-commutative extension of the Stone–Weierstrass theorem related to pure states (cf. [7]), and the study of pure states itself is significant in non-commutative geometry.

Now we explain non-Archimedean analogues of the GNS-construction and Krein–Milman theorem. We introduce a notion of a Banach topological algebra over a valuation field (cf. Definition 1.15) as a non-Archimedean analogue of a C^*-algebra, and define a notion of a state on a Banach topological algebra. As a counterpart of a pair of a Hilbert module over a C^*-algebra and a cyclic vector of norm 1, we introduce a notion of a GNS triad over a Banach topological algebra. Roughly speaking, a GNS triad is a triad of a Banach left module V (cf. Definition 1.25), a cyclic element of V, and its dual cyclic element with several conditions on norms. We give a one-to-one correspondence between states and equivalence classes of GNS triads in Theorem 2.5. We introduce a notion of a pure (resp. mixed) states, and compare it with the notion of a finite dimensional simple (resp. finite dimensional cyclic semisimple) Banach left module in Theorem 2.6 (resp. Theorem 2.8).

There are several works on non-Archimedean analogues of the convexity and Krein–Milman theorem such as [9] Theorem 3 and [3] 2.3 Theorem. We introduce another non-Archimedean analogue of the convexity, which generalises the M-convexity in [9], and extend [9] Theorem 3 for the M-convexity to a case where the fundamental technique in its original proof using closed hyperplanes is not necessarily applicable. For the precise statement, see Theorem 3.3. As a consequence, we show that if the base field is a local field or a finite field, then the compact convex set of states on a Banach topological algebra coincides with the closure of the convex hull of the subsets of non-Archimedean extreme points in Theorem 3.5.

As an application, we define two operator algebras associated to topological groups. They form Banach topological algebras, and the class of cyclic Banach left modules corresponds to the class of cyclic unitary representations which can be approximated by finite dimensional cyclic unitary representations. For the precise statement, see Theorem 3.9. We note that we define the notion of the purity of a state in terms of a left ideal of a Banach topological algebra, and the subset of pure states does not coincides with the subset of non-Archimedean extreme points. Therefore Theorem 3.5 does not imply that every state can be approximated by mixed states, and hence the construction of such two operator algebras is not trivial.

We explain the contents of this paper. First, §1 consists of three subsections. In §1.1, we introduce the notion of a convexity in the non-Archimedean setting. In §1.2, we give a non-commutative analogue of Chinese reminder theorem, which helps us to compute the kernel of a mixed state. We note that it is well-known that Chinese reminder theorem does not necessarily hold in non-commutative ring theory, and hence we restrict a class of ideals in a non-commutative ring. In §1.3, we introduce the notion of a Banach topological algebra, and study module theory over a Banach topological algebra. Next, §2 consists of three subsections. In §2.1, we introduce the notions of a state and a GNS triad, and formulate the non-Archimedean analogue of the GNS construction. In §2.2, we introduce the notions of a pure state and a mixed state, and study the relations between pure (resp. mixed) states and finite dimensional simple (resp. finite dimensional cyclic semisimple) Banach left modules. In §2.3, we study the integrality condition on the coefficients of mixed states associated to a given finite dimensional cyclic semisimple Banach left module. Finally, §3 consists of three subsections. In §3.1, we study the convexity associated to $\mathbb{Q} \cap [0,1]$ called the Archimedean convexity. In §3.2, we establish the non-Archimedean analogue of Krein–Milman theorem, and apply it to the compact convex set of states. In §3.3, we define and study two operator algebras associated to topological groups.

1. Preliminaries

Throughout this paper, a ring is assumed to be associative and unital, but is not assumed to be commutative. We denote by k a valuation field with a fixed valuation $|-|\colon k \to [0,\infty)$, by $O_k \subset k$ the valuation ring of k, and by $m_k \subset O_k$ the maximal ideal of O_k. We say that k is a *local field* if k is a complete discrete valuation field with $\# O_k/m_k < \infty$. We introduce several notions which play important roles in the formulations of the non-Archimedean GNS construction in §2 and the non-Archimedean Krein–Milman theorem in §3.

1.1. General convexity

To begin with, we recall the the notion of the M-convexity appeared in [9]. Let V be a k-vector space, and $S \subset V$ a subset. We say that S is *semiconvex* (cf. [9] Definition 1) if $(1-c)F_1 + cF_2 \in S$ for any $(c,(F_i)_{i=1}^2) \in (1+m_k) \times S^2$, is M-*convex* if $\sum_{i=1}^3 c_i F_i \in S$ for any $(c_i,F_i)_{i=1}^3 \in (O_k \times S)^3$ with $\sum_{i=1}^3 c_i = 1$, and is 0-convex if it is M-convex and contains $0 \in V$. Let C denote the formal symbol M or 0. Suppose that S is C-convex. A subset $S' \subset S$ is said to be an *extreme set* (cf. [9] Definition 2) with respect to the C-convexity if S' is a non-empty semiconvex subset and $S \setminus S'$ is C-convex. An $x \in S$ is said to be an *extreme point* with respect to the C-convexity if it lies in some minimal extreme set of S with respect to the C-convexity. We denote by $\mathrm{Ext}_C(M)$ the set of extreme points of V with respect to the C-convexity. We generalise these notions.

Let R be a ring. A subset $R_0 \subset R$ is said to be a *convexity in* R if R_0 forms a multiplicative subset, i.e. a submonoid of R with respect to the multiplication.

EXAMPLE 1.1. We have the following three typical examples of convexities in R:

(i) Every subring of R forms a convexity in R. In particular, R forms a convexity in R.

(ii) The intersection of any non-empty family of convexities in R again forms a convexity in R. In particular, for any $c \in R$, the intersection $\mathbb{F}_{R,c}$ of the non-empty set of convexities in R containing c forms the smallest convexity in R containing c.

(iii) If R is a $\mathbb{Q}$-algebra, then the image $\mathbb{F}_{R,\infty}$ of $\mathbb{Q} \cap [0,1]$ in R forms a convexity in R.

Let R_0 be a convexity in R, M a left R-module, and S a subset of M. We say that S is R_0-*convex* if $\sum_{i=1}^n c_i F_i \in S$ for any $(c_i, F_i)_{i=1}^n \in (R_0 \times S)^n$ with $n \in \mathbb{N} \setminus \{0\}$ and $\sum_{i=1}^n c_i = 1$. For example, M itself forms an R_0-convex subset of M, and the intersection of any non-empty family of R_0-convex subsets again forms an R_0-convex subset of M. We denote by $\mathrm{co}(S;R_0) \subset M$ the R_0-convex hull of S, i.e. the smallest R_0-convex subset of M containing S, which is given as the intersection of the non-empty set of R_0-convex subsets of M containing S. We have the presentation $\mathrm{co}(S;R_0) = \{\sum_{i=1}^n c_i F_i \mid n \in \mathbb{N} \setminus \{0\}, (c_i, F_i)_{i=1}^n \in (R_0 \times S)^n, \sum_{i=1}^n c_i = 1\}$. Indeed, the right hand side contains S by $1 \in R_0$, is contained in $\mathrm{co}(S;R_0)$ by the definition of the R_0-convexity, and is R_0-convex by $c_1 c_2 \in R_0$ for any $(c_1, c_2) \in R_0^2$. When R_0 is a subring of R, then a subset of M is R_0-convex if and only if it is of the form $m + M_0 \subset M$ for an $m \in M$ and a left R_0-submodule $M_0 \subset M$.

EXAMPLE 1.2. By Example 1.1 (i), O_k forms a convexity in k. Let V be a k-vector space, and S a subset of V. Then S is M-convex if and only if S is O_k-convex, and S is 0-convex if and only if S is an O_k-submodule. Indeed, for any $(c_i, F_i)_{i=1}^n \in (O_k \times S)^n$ with $n \in \mathbb{N} \cap [2, \infty)$ and $\sum_{i=1}^n c_i = 1$, $\sum_{i=1}^n c_i F_i$ is contained in the M-convex hull of S by $\sum_{i=1}^n c_i F_i = ((c_{n-1} + c_n) F_{n-1} + \sum_{i=1}^{n-2} c_i F_i) - c_n F_{n-1} + c_n F_n$.

A *semiconvexity in* R_0 is a subset $R_{00} \subset R_0$ with $1 \in R_{00}$. Let R_{00} be a semiconvexity in R_0. We say that S is R_{00}-*semiconvex* if $(1-c)F_1 + cF_2 \in S$ for any $(c, (F_i)_{i=1}^2) \in R_{00} \times S^2$. Suppose that S is an R_0-convex subset of M. A subset $S' \subset S$ is said to be an R_{00}-*face of* S if S' is a non-empty R_{00}-semiconvex subset of M and $S \setminus S'$ is an R_0-convex subset of M, and an R_{00}-face $S' \subset S$ is said to be *extreme* if there is no R_{00}-face S'' of S with $S'' \subsetneq S'$. For an R_0-convex subset $S \subset M$, we denote by $\mathrm{Ext}(S;R_0,R_{00})$ the set of extreme R_{00}-faces of S, and by $[S;R_0,R_{00}] \subset S$ the closure of $\mathrm{co}(\bigcup_{S' \in \mathrm{Ext}(S;R_0,R_{00})} S';R_0)$ in S as long as M is equipped with a topology.

EXAMPLE 1.3. The subset $1 + m_k \subset k$ forms a semiconvexity in the convexity O_k in k. Let V be a k-vector space, and S a subset of V. Then S is semiconvex if and only if S is a $(1 + m_k)$-semiconvex subset of the underlying O_k-module of V. Suppose that S is O_k-convex. A subset $S' \subset S$ is an extreme set of S if and only if S' is a $(1 + m_k)$-face of S by Example 1.2. In particular, a $v \in V$ is an extreme point of S if and only if v is contained in an extreme $(1 + m_k)$-face of S.

EXAMPLE 1.4. Let $\wp \subsetneq O_k$ be an ideal, M an O_k-module with $\wp M = \{0\}$, and $S \subset M$ an O_k-convex subset. Then $1 + \wp$ forms a semiconvexity in the convexity O_k in k, and every subset of M is $(1 + \wp)$-semiconvex. Therefore we have $\mathrm{Ext}(S;O_k, 1 + \wp) = \{\{m\} \mid m \in S\}$ and $[S;O_k, 1 + \wp] = S$ for any topology on M.

EXAMPLE 1.5. Let M be an O_k-module, and $S \subset M$ an O_k-convex subset. If $\# O_k/m_k = 2$, then a subset of M is O_k-semiconvex if and only if it is $(1+m_k)$-semiconvex, and hence the equality $\mathrm{Ext}(S; O_k, O_k) = \mathrm{Ext}(S; O_k, 1+m_k)$ holds. If $\# O_k/m_k \neq 2$, then a subset of M is O_k-semiconvex if and only if it is O_k-convex, and hence the equality $\mathrm{Ext}(S; O_k, O_k) = \emptyset$ holds. (cf. [10] III Remarque in pp. 28–29)

1.2. Non-commutative Chinese reminder theorem

Let R be a ring. For a $\mathbb{Z}$-submodule $M \subset R$, we denote by $R^{-1}M$ (resp. MR^{-1}, $R^{-1}MR^{-1}$) the largest left (resp. right, two-sided) ideal of R contained in M, which is given as $\{f \in R \mid {}^{\forall} f' \in R, f'f \in M\}$ (resp. $\{f \in R \mid {}^{\forall} f' \in R, ff' \in M\}$, $\{f \in R \mid {}^{\forall}(f', f'') \in R^2, f'ff'' \in M\}$). We say that R is *primitive* if there exists a faithful simple left R-module. A two-sided ideal $I \subset R$ is said to be a *primitive ideal of* R or *primitive* if R/I is primitive, or equivalently if $I = \mathrm{Ann}_R(M)$ for some simple left R-module M (cf. [1] Proposition 15.1). We denote by $\mathrm{Max}(R)$ the set of left maximal ideals of R, and by $\mathrm{Prim}(R)$ the set of primitive ideals. We note that for any $\wp \in \mathrm{Max}(R)$, $M/\wp$ forms a simple left R-module with $\mathrm{Ann}_R(R/\wp) = \wp R^{-1}$, and hence $\wp R^{-1}$ is primitive.

PROPOSITION 1.6. *Let S be a subset of $\mathrm{Max}(R)$ with $\# S \geqslant 2$ and $\wp + \bigcap_{\wp' \in S \setminus \{\wp\}} \wp' = R$ for any $\wp \in S$. If R is semisimple and left Artinian, then there is an $S' \subset \mathrm{Max}(R)$ with $S \subset S'$, $\wp + \bigcap_{\wp' \in S' \setminus \{\wp\}} \wp' = R$ for any $\wp \in S'$, and $\bigcap_{\wp \in S'} \wp = \{0\}$.*

Proof. Put $\Sigma := \{S' \subset \mathrm{Max}(R) \mid S \subset S', (\wp + \bigcap_{\wp' \in S' \setminus \{\wp\}} \wp')_{\wp \in S'} = (R)_{\wp \in S'}\}$. Then we have $S \in \Sigma$. Since R is left Artinian, there is an $S' \in \Sigma$ such that $\bigcap_{\wp \in S'} \wp$ is a minimal element of $\{\bigcap_{\wp \in S''} \wp \mid S'' \in \Sigma\}$. It suffices to show $\bigcap_{\wp \in S'} \wp = \{0\}$. Assume $\bigcap_{\wp \in S'} \wp \neq \{0\}$. Since R is semisimple, there is a left ideal $\wp_0 \subset R$ with $R = \wp_0 \oplus \bigcap_{\wp' \in S'} \wp'$. By the assumption, we have $\wp_0 \neq R$, and hence there is a $\wp \in \mathrm{Max}(A)$ with $\wp_0 \subset \wp$. By $\wp_0 \subset \wp \subsetneq R = \wp_0 \oplus \bigcap_{\wp' \in S'} \wp'$, we have $\wp \notin S'$ and $\bigcap_{\wp' \in S' \sqcup \{\wp\}} \wp' \subsetneq \bigcap_{\wp' \in S'} \wp'$. By $\wp_0 \oplus \bigcap_{\wp' \in S'} \wp' = R$, there is a unique $(a,b) \in \wp_0 \times \bigcap_{\wp' \in S'} \wp'$ with $a + b = 1$. Let $\wp' \in S'$. By $\wp' + \bigcap_{\wp'' \in S' \setminus \{\wp'\}} \wp'' = R$, there is a $(c,d) \in \wp' \times \bigcap_{\wp'' \in S' \setminus \{\wp'\}} \wp''$ with $c + d = 1$. We have $da \in \wp_0 \subset \wp$, $da = d(a+b) - db = d - db \in \bigcap_{\wp'' \in S' \setminus \{\wp'\}} \wp''$, and hence $da \in \bigcap_{\wp'' \in (S' \sqcup \{\wp\}) \setminus \{\wp'\}} \wp''$. On the other hand, we have $c \in \wp'$, $db \in \bigcap_{\wp'' \in S'} \wp''$, and hence $c + db \in \wp'$. We obtain $1 = c + d = c + d(a+b) = (c + db) + da \in \wp' + \bigcap_{\wp'' \in (S' \sqcup \{\wp\}) \setminus \{\wp'\}} \wp''$. It implies $S' \sqcup \{\wp\} \in \Sigma$. This contradicts the minimality of $\bigcap_{\wp' \in S'} \wp'$ by $\bigcap_{\wp' \in S' \sqcup \{\wp\}} \wp' \subsetneq \bigcap_{\wp' \in S'} \wp'$. We conclude $\bigcap_{\wp \in S'} \wp = \{0\}$. $\square$

Suppose that R is commutative. Let A be an R-algebra. A left ideal $\wp \subset A$ is said to be *of finite codimension* if $A/\wp$ is finitely generated as an R-module. We denote by $\mathrm{Max}_R(A) \subset \mathrm{Max}(A)$ the subset of left maximal ideals of finite codimension, and by $\mathrm{Bl}_R(A) \subset \mathrm{Prim}(A)$ the subset of primitive ideals of finite codimension. We show the structure of the residue ring of the finite intersection of finite dimensional blocks.

PROPOSITION 1.7. *Let* $(I_1,\ldots,I_n) \in \mathrm{Bl}_R(A)^n$ *with* $n \in \mathbb{N}\setminus\{0\}$. *If* $I_i \neq I_j$ *for any* $(i,j) \in (\mathbb{N}\cap[1,n])^2$ *with* $i \neq j$, *then the map* $A/\bigcap_{i=1}^n I_i \to \prod_{i=1}^n A/I_i$ *given as the direct product of the canonical projections is an* R*-algebra isomorphism.*

Proof. Let $i \in \mathbb{N}\cap[1,n]$. Take a $\wp_i \in \mathrm{Max}_R(A)$ with $I_i \subset \wp_i$. Since I_i is of finite codimension, A/I_i forms a simple R-algebra by Wedderburn's theorem (cf. [1] 13.4 Theorem), and $\wp_i A^{-1} = \mathrm{Ann}_A(A/\wp_i)$ coincides with I_i. Since $\wp_i$ is of finite codimension, M is finitely generated as a right $\mathrm{End}_A(A/\wp_i)$-module. Therefore the map $\varphi_i \colon A/I_i = A/\mathrm{Ann}_A(A/\wp_i) \hookrightarrow \mathrm{End}_{\mathrm{End}_A(A/\wp_i)}(A/\wp_i)$ induced by the scalar multiplication $A \times A/\wp_i \to A/\wp_i$ is an R-algebra isomorphism by the simplicity of $A/\wp_i$ and Jacobson density theorem (cf. [1] 14.5 Corollary).

Put $M := \bigoplus_{i=1}^n A/\wp_i$. By $I_i \neq I_j$, we have $\mathrm{Hom}_A(A/\wp_i, A/\wp_j) = \{0\}$ for any $(i,j) \in (\mathbb{N}\cap[1,n])^2$ with $i \neq j$. Therefore the embedding $\bigoplus_{i=1}^n \mathrm{End}_A(A/\wp_i) \hookrightarrow \mathrm{End}_A(M)$ associated to the presentation $M = \bigoplus_{i=1}^n A/\wp_i$ is an R-linear isomorphism, and induces an R-algebra isomorphism $\varphi \colon \mathrm{End}_{\mathrm{End}_A(M)}(M) \to \prod_{i=1}^n \mathrm{End}_{\mathrm{End}_A(A/\wp_i)}(A/\wp_i)$.

Since M is finitely generated as an R-module, M is finitely generated as a right $\mathrm{End}_A(M)$-module. Therefore the map $\psi \colon A/\mathrm{Ann}_A(M) \hookrightarrow \mathrm{End}_{\mathrm{End}_A(M)}(M)$ induced by the scalar multiplication $A \times M \to M$ is an R-algebra isomorphism by the semisimplicity of M and Jacobson–Bourbaki density theorem (cf. [6] D 2.2). Since the map $A/\bigcap_{i=1}^n I_i = A/\mathrm{Ann}_A(M) \to \prod_{i=1}^n A/I_i$ given as the direct product of the canonical projections coincides with the composite of the R-algebra isomorphisms ψ, φ, and $(\bigoplus_{i=1}^n \varphi_i)^{-1}$, it is an R-algebra isomorphism. □

COROLLARY 1.8. (Non-commutative Chinese reminder theorem)
Let $(\wp_i)_{i=1}^n \in \mathrm{Max}_R(A)^n$ *with* $n \in \mathbb{N}$. *If* $\wp_i A^{-1} \neq \wp_j A^{-1}$ *for any* $(i,j) \in (\mathbb{N}\cap[1,n])^2$ *with* $i \neq j$, *then the map* $A/\bigcap_{i=1}^n \wp_i \to \prod_{i=1}^n A/\wp_i$ *given as the direct product of the canonical projections is an* A*-linear isomorphism.*

Proof. The injectivity is obvious, and the surjectivity follows from Proposition 1.7 by $\wp_i A^{-1} \in \mathrm{Bl}_R(A)$ for any $i \in \mathbb{N}\cap[1,n]$. □

COROLLARY 1.9. *Let* $(I_0,\ldots,I_n) \in \mathrm{Bl}_R(A)^{n+1}$ *with* $n \in \mathbb{N}\setminus\{0\}$ *and* $\bigcap_{i=1}^n I_i \subset I_0$. *If* $I_i \neq I_j$ *for any* $(i,j) \in (\mathbb{N}\cap[1,n])^2$ *with* $i \neq j$, *then there is an* $i \in \mathbb{N}\cap[1,n]$ *with* $I_i = I_0$.

Proof. By $I_0 \neq A$, the canonical projection $\prod_{i=0}^n A/I_i \to \prod_{i=1}^n A/I_i$ is not an isomorphism. Therefore the assertion follows from Proposition 1.7 and $\bigcap_{i=0}^n I_i = \bigcap_{i=1}^n I_i$. □

1.3. Banach topological algebra

We introduce notions of a Banach topological k-algebra and a Banach left module over a Banach topological k-algebra. For this purpose, we recall Banach k-vector spaces. For a k-vector space V, a map $\|-\| \colon V \to [0,\infty)$ is said to be a *norm* if it satisfies $\|v - v'\| \leqslant \max\{\|v\|, \|v'\|\}$ for any $(v,v') \in V^2$, $\|v\| > 0$ for any $v \in V\setminus\{0\}$, and $\|cv\| = |c|\,\|v\|$ for any $(c,v) \in k \times V$. A *normed k-vector space* is a pair $(V, \|-\|)$ of a k-vector space V and a norm $\|-\| \colon V \to [0,\infty)$.

Let $(V, \|-\|)$ be a normed k-vector space. We abbreviate $(V, \|-\|)$ to V. We equip V with the ultrametric $V \times V \to [0,\infty)$, $(v,v') \mapsto \|v-v'\|$, and call the metric topology on V *the norm topology on V*. For a subset $S \subset V$, we put $S_{\leqslant 1} := \{v \in S \mid \|v\| \leqslant 1\}$. We say that V is a *Banach k-vector space* if the ultrametric on V is complete, is *finite dimensional* if the underlying k-vector space of V is finite dimensional, and is *unramified* if $\|V\| \subset [0,\infty)$ is contained in the closure of $|k| \subset [0,\infty)$. We prepare the terminology on a condition on k and V appearing frequently.

DEFINITION 1.10. We refer as *the hypothesis (I)* to the condition that the valuation of k is discrete or the norm of V is trivial.

Let V_1 and V_2 be normed k-vector spaces, and $f \colon V_1 \to V_2$ a k-linear homomorphism. We say that f is *bounded* if there is a $C \in [0,\infty)$ with $\|f(v)\| \leqslant C\|v\|$ for any $v \in V_1$, and is *submetric* if $\|f(v)\| \leqslant \|v\|$ for any $v \in V_1$. If f is bounded, then f is continuous. Conversely, if the valuation of k is non-trivial and f is continuous, or if the norms of V_1 and V_2 are trivial, then f is bounded. We denote by $\mathrm{Ban}(k)$ the category of Banach k-vector spaces and bounded k-linear homomorphisms, and by $\mathrm{Ban}_{\leqslant 1}(k) \subset \mathrm{Ban}(k)$ the subcategory of submetric k-linear homomorphisms. The correspondence $V \rightsquigarrow V_{\leqslant 1}$ gives a functor $\mathrm{Ban}_{\leqslant 1}(k) \to \mathrm{Set}$, which is faithful as long as the valuation of k is non-trivial.

Let $W \subset V$ be a closed k-vector subspace. Then W forms a normed k-vector space with respect to the restriction to W of the norm of V, and V/W forms a normed k-vector space with respect to the quotient norm (cf. [2] 1.1.6). If V is a Banach k-vector space, then so are W and V/W by [2] Proposition 1.1.6/1, [2] Proposition 1.1.7/3, and [2] 2.8.1. Since the preimage in V of the image in V/W of an open subset $U \subset V$ can be presented as $\bigcup_{w \in W}(w+U)$, we have the following:

PROPOSITION 1.11. *The canonical projection $V \twoheadrightarrow V/W$ is an open submetric k-linear homomorphism.*

If f is continuous, then $\ker(f) \subset V_1$ forms a closed k-vector subspace because f is continuous and V_2 is T_1. In the case where V_1 and V_2 are Banach k-vector spaces, we say that f is *admissible* if $\ker(f)$ and $im(f)$ are closed and the induced map $V_1/\ker(f) \to \mathrm{im}(f)$ is an isomorphism in $\mathrm{Ban}(k)$. By Proposition 1.11, every admissible k-linear homomorphism between Banach k-vector spaces is bounded.

We denote by $\mathscr{B}(V_1,V_2)$ the k-vector space of bounded k-linear homomorphisms $V_1 \to V_2$, which forms a normed k-vector space with respect to the operator norm $\|-\| \colon \mathscr{B}(V_1,V_2) \to [0,\infty)$, $f \mapsto \|f\| := \inf\{C \in [0,\infty) \mid {}^{\forall}v \in V_1, \|f(v)\| \leqslant C\|v\|\}$. If V_2 is a Banach k-vector space, then so is $\mathscr{B}(V_1,V_2)$. For a k-vector subspace $W \subset \mathscr{B}(V_1,V_2)$, we put $W^{\top} := \bigcap_{w \in W} \ker(w) \subset V_1$. For a normed k-vector space V, we abbreviate $\mathscr{B}(V,k)$ to V^{D} and $\mathscr{B}(V,V)$ to $\mathscr{B}(V)$. We have the following two fundamental properties on bounded k-linear homomorphisms:

PROPOSITION 1.12. *Suppose that k is complete. Let V be a normed k-vector space, and $W \subset V$ a k-vector subspace. Under the hypothesis (I), for any $w \in W^{\mathrm{D}}$, there is a $\tilde{w} \in V^{\mathrm{D}}$ with $\tilde{w}|_W = w|_W$ and $\|\tilde{w}\| = \|w\|$.*

PROPOSITION 1.13. *Suppose that k is complete. Let V_1 and V_2 be Banach k-vector spaces and $f \colon V_1 \to V_2$ a k-linear homomorphism. The the following hold:*

(i) If V_1 is finite dimensional, then f is admissible.

(ii) If the valuation of k is non-trivial and f is continuous and surjective, then f is admissible.

Proposition 1.12 for the case where the valuation of k is discrete immediately follows from Hahn–Banach theorem (cf. [8] Theorem 3). Proposition 1.12 for the case where the norm on V is trivial immediately follows from the semisimplicity of the underlying ring of k. Proposition 1.13 for the case where the valuation of k is non-trivial immediately follows from [2] Proposition 2.3.3/4, [2] Corollary 2.3.3/5, and Banach's open mapping theorem (cf. [4] Theorem I.3.3/1). Proposition 1.13 (i) for the case where the valuation of k is trivial is easily reduced to the following:

PROPOSITION 1.14. *Suppose that the valuation of k is trivial. For any finite dimensional normed k-vector space V, $\|V\| \subset [0,\infty)$ is a bounded subset, and the norm topology on V coincides with the discrete topology.*

Proof. When $V = \{0\}$, then the assertions are obvious. Assume $V \neq \{0\}$. Take a k-linear basis $S \subset V$. By $V \neq \{0\}$, we have $S \neq \emptyset$. For any $(c_v)_{v \in S} \in k^{\oplus S}$, we have $\|\sum_{v \in S} c_v v\| \leqslant \max_{v \in S} |c_v| \ \|v\| = \max_{v \in S} \|v\|$. Therefore $\|V\| \subset [0,\infty)$ is a bounded subset.

We denote by P the set of subsets S' of S such that there is a $((c_{i,v})_{v \in S'})_{i \in \mathbb{N}} \in (k^{\oplus S'})^{\mathbb{N}}$ with $0 < \|\sum_{v \in S'} c_{i+1,v} v\| < \|\sum_{v \in S'} c_{i,v} v\|$ for any $i \in \mathbb{N}$. Assume that the norm topology on V does not coincide with the discrete topology. Then $\{0\} \subset V$ is not open, and hence for any $\varepsilon > 0$, there is a $v \in V$ with $0 < \|v\| < \varepsilon$. By a recursion, we obtain a sequence $((c_{i,v})_{v \in S})_{i \in \mathbb{N}} \in (k^{\oplus S})^{\mathbb{N}}$ with $0 < \|\sum_{v \in S} c_{i+1,v} v\| < \|\sum_{v \in S} c_{i,v} v\|$ for any $i \in \mathbb{N}$. It implies $S \in P \neq \emptyset$. Since P forms a non-empty finite partially ordered set with respect to the inclusions, it admits a minimal element S'. Take a $((c_{i,v})_{v \in S'})_{i \in \mathbb{N}} \in (k^{\oplus S'})^{\mathbb{N}}$ with $0 < \|\sum_{v \in S'} c_{i+1,v} v\| < \|\sum_{v \in S'} c_{i,v} v\|$ for any $i \in \mathbb{N}$. By $\|\sum_{v \in S'} c_{i,v} v\| > 0$, we have $\{v \in S' \mid c_{i,v} \neq 0\} \neq \emptyset$ for any $i \in \mathbb{N}$. By the pigeonhole principle, there is a $v_0 \in S'$ with $\#\{i \in \mathbb{N} \mid c_{i,v_0} \neq 0\} = \infty$. We denote by $(i_n)_{n \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ the unique sequence with $\{i_n \mid n \in \mathbb{N}\} = \{i \in \mathbb{N} \mid c_{i,v_0} \neq 0\}$ and $i_n < i_{n+1}$ for any $n \in \mathbb{N}$. We have $\|v_0 + \sum_{v \in S' \setminus \{v_0\}} c_{i_n,v_0}^{-1} c_{i_n,v} v\| = |c_{i_n,v_0}^{-1}| \ \|\sum_{v \in S'} c_{i_n,v} v\| = \|\sum_{v \in S'} c_{i_n,v} v\|$ for any $n \in \mathbb{N}$, and hence $\|\sum_{v \in S' \setminus \{v_0\}} c_{i_n,v_0}^{-1} (c_{i_n,v} - c_{i_{n+1},v}) v\| = \|\sum_{v \in S'} c_{i_n,v} v\|$ for any $n \in \mathbb{N}$. It implies that the sequence $((c_{i_n,v_0}^{-1}(c_{i_n,v} - c_{i_{n+1},v}))_{v \in S' \setminus \{v_0\}})_{n \in \mathbb{N}} \in (k^{\oplus(S' \setminus \{v_0\})})^{\mathbb{N}}$ satisfies $0 < \|\sum_{v \in S' \setminus \{v_0\}} c_{i_{n+1},v_0}^{-1} (c_{i_{n+1},v} - c_{i_{n+2},v}) v\| < \|\sum_{v \in S' \setminus \{v_0\}} c_{i_n,v_0}^{-1} (c_{i_n,v} - c_{i_{n+1},v}) v\|$ for any $n \in \mathbb{N}$. This contradicts the minimality of S'. It implies that the norm topology on V coincides with the discrete topology. $\square$

Now we introduce the notion of a Banach topological k-algebra. A *normed k-algebra* is a pair $(A, \| - \|)$ of a k-algebra A and a norm $\| - \| \colon A \to [0, \infty)$ on the underlying k-vector space of A with $\|1\| \in \{0, 1\}$ and $\|ff'\| \leqslant \|f\| \, \|f'\|$ for any $(f, f') \in A^2$.

Let $(A, \| - \|)$ be a normed k-algebra. We abbreviate $(A, \| - \|)$ to A. We also regard A as a normed k-vector space. We say that A is a *Banach k-algebra* if A is a Banach k-vector space. We denote by $\mathrm{Alg}_{\leqslant 1}(k)$ the category of Banach k-algebras and submetric k-algebra homomorphisms.

DEFINITION 1.15. A *Banach topological k-algebra* is a pair (A, τ) of a Banach k-algebra A and a topology τ on $A_{\leqslant 1}$ which is weaker than or equal to the relative topology of the norm topology on A and for which $A_{\leqslant 1}$ forms a topological O_k-algebra.

Let $\mathscr{A} = (A, \tau)$ be a Banach topological k-algebra. We denote by $\mathscr{A}_{\leqslant 1}$ the topological O_k-algebra $(A_{\leqslant 1}, \tau)$, by $\mathscr{A}_\circ$ the Banach k-algebra A, and by $\mathscr{A}^{\mathrm{op}}$ the Banach topological k-algebra (A^{op}, τ). We note that the inclusion $\mathscr{A}_{\leqslant 1} \hookrightarrow \mathscr{A}_\circ$ is an open map by definition. We say that $\mathscr{A}$ is *unramified* if $\mathscr{A}_\circ$ is unramified, and is *finite dimensional* if $\mathscr{A}_\circ$ is finite dimensional.

For Banach topological k-algebras $\mathscr{A}_1$ and $\mathscr{A}_2$, a *submetric k-algebra homomorphism* $\mathscr{A}_1 \to \mathscr{A}_2$ means a submetric k-algebra homomorphism $\varphi \colon (\mathscr{A}_1)_\circ \to (\mathscr{A}_2)_\circ$ whose restriction $(\mathscr{A}_1)_{\leqslant 1} \to (\mathscr{A}_2)_{\leqslant 1}$ is continuous. We denote by $\mathscr{A}\mathrm{lg}_{\leqslant 1}(k)$ the category of Banach topological k-algebras and submetric k-algebra homomorphisms. The correspondence $\mathscr{A} \rightsquigarrow \mathscr{A}_\circ$ gives a faithful functor $\mathscr{A}\mathrm{lg}_{\leqslant 1}(k) \to \mathrm{Alg}_{\leqslant 1}(k)$, and the correspondence $\mathscr{A} \rightsquigarrow \mathscr{A}_{\leqslant 1}$ gives a functor $\mathscr{A}\mathrm{lg}_{\leqslant 1}(k) \to \mathrm{Set}$, which is faithful as long as the valuation of k is non-trivial.

For example, every Banach k-algebra A forms a Banach topological k-algebra A_{disc} with respect to the relative topology on $A_{\leqslant 1}$ of the norm topology on A. The correspondence $A \rightsquigarrow A_{\mathrm{disc}}$ gives a fully faithful functor $\mathrm{Alg}_{\leqslant 1}(k) \hookrightarrow \mathscr{A}\mathrm{lg}_{\leqslant 1}(k)$. Therefore the notion of a Banach topological k-algebra is a generalisation of that of a Banach k-algebra.

EXAMPLE 1.16. Let G be a discrete group. Then the completion $\mathrm{C}_0(G, k)$ of $k[G]$ with respect to the supremum norm of coefficients forms a Banach k-algebra such that the canonical embedding $k[G] \hookrightarrow \mathrm{C}_0(G, k)$ is a k-algebra homomorphism, and hence forms a Banach topological k-algebra $\mathrm{C}_0(G, k)_{\mathrm{disc}}$.

We uses similar conditions many times in this paper, and hence we introduce the terminology on conditions on k and $\mathscr{A}$ for convenience.

DEFINITION 1.17. We refer as *the hypothesis (II)* to the condition that $|k^\times| \subset (0, \infty)$ is discrete, as *the hypothesis (III)* to the condition that k is a local field or a finite field equipped with the trivial valuation, as *the hypothesis (IV)* to the condition that $\mathscr{A}$ is unramified, as *the hypothesis (V)* to the condition that $\mathscr{A}_{\leqslant 1}$ is Hausdorff, and as *the hypothesis (VI)* to the condition that $\mathscr{A}$ coincides with $(\mathscr{A}_\circ)_{\mathrm{disc}}$.

We have another important example of a Banach topological k-algebra. A topological O_k-module is said to be *linear* if the set of its open O_k-submodules forms a fundamental system of neighbourhoods of 0. Let K be a topological O_k-algebra. We say that K is *flat* if the underlying O_k-module of K is flat, and is *linear*) if the underlying topological O_k-module of K is linear. Assume the hypothesis (III). Suppose that K is a compact Hausdorff flat linear topological O_k-algebra. Then $k \otimes_{O_k} K$ forms an unramified Banach topological k-algebra K_{comp} with respect to the norm $\| - \| \colon k \otimes_{O_k} K \to [0,\infty)$, $f \mapsto \inf\{|c| \mid c \in k, {}^{\exists} f' \in K,\ c \otimes f' = f\}$ and the topology on $(k \otimes_{O_k} K, \| - \|)_{\leqslant 1}$ associated to the topology on K through the O_k-algebra isomorphism $K \to (k \otimes_{O_k} K, \| - \|)_{\leqslant 1}$, $f \mapsto 1 \otimes f$. We denote by $\mathrm{Alg}_{\leqslant 1}(O_k)$ the category of compact Hausdorff flat linear topological O_k-algebras and continuous O_k-algebra homomorphisms. The correspondence $K \rightsquigarrow K_{\mathrm{comp}}$ gives a fully faithful functor $\mathrm{Alg}_{\leqslant 1}(O_k) \hookrightarrow \mathscr{A}\mathrm{lg}_{\leqslant 1}(k)$. Therefore the notion of a Banach topological k-algebra is also a generalisation of that of a compact Hausdorff flat linear topological O_k-algebra.

EXAMPLE 1.18. Assume the hypothesis (III). Let G be a profinite group. Then the Iwasawa algebra $O_k[[G]]$ forms a compact Hausdorff flat linear topological O_k-algebra, and hence $k \otimes_{O_k} O_k[[G]]$ forms a Banach topological k-algebra $O_k[[G]]_{\mathrm{comp}}$.

We have a structure theorem of a finite dimensional Banach topological k-algebra.

PROPOSITION 1.19. *Assume the hypotheses (III) and (V). If $\mathscr{A}$ is finite dimensional, then the hypothesis (VI) holds and $\mathscr{A}_{\leqslant 1}$ forms a compact Hausdorff flat linear topological O_k-algebra. In addition, under the hypothesis (IV), $\mathscr{A}$ is isomorphic to $(\mathscr{A}_{\leqslant 1})_{\mathrm{comp}}$ in $\mathscr{A}\mathrm{lg}_{\leqslant 1}(k)$.*

Proof. We equip $(\mathscr{A}_\circ)_{\leqslant 1}$ the relative topology of the norm topology on $\mathscr{A}_\circ$. Suppose that $\mathscr{A}$ is finite dimensional. If k is a local field, then every bounded closed subset of $\mathscr{A}_\circ$ is compact by Proposition 1.13 (i). If k is a finite field equipped with the trivial valuation, then $\mathscr{A}_\circ$ is a finite discrete set by Proposition 1.14. Therefore $(\mathscr{A}_\circ)_{\leqslant 1}$ forms a compact Hausdorff flat linear topological O_k-algebra. The inclusion $\mathscr{A}_{\leqslant 1} \hookrightarrow \mathscr{A}_\circ$ is an open map from a Hausdorff topological space onto $(\mathscr{A}_\circ)_{\leqslant 1}$, and hence is a homeomorphism onto the image. It implies that $\mathscr{A}_{\leqslant 1}$ is a compact Hausdorff flat linear topological O_k-algebra, and $\mathscr{A}$ coincides with $(\mathscr{A}_\circ)_{\mathrm{disc}}$. In addition, suppose that $\mathscr{A}$ is unramified. Then the inclusion $(\mathscr{A}_\circ)_{\leqslant 1} \hookrightarrow \mathscr{A}_\circ$ induces an isomorphism $(\mathscr{A}_{\leqslant 1})_{\mathrm{comp}} = ((\mathscr{A}_\circ)_{\leqslant 1})_{\mathrm{comp}} \to \mathscr{A}_\circ$ in $\mathscr{A}\mathrm{lg}_{\leqslant 1}(k)$. $\square$

We formulate a Banach topological k-algebra obtained as a quotient of a given Banach topological k-algebra. A *closed ideal of $\mathscr{A}$* is a closed two-sided ideal of $\mathscr{A}_\circ$, and a *strictly closed ideal of $\mathscr{A}$* is a strictly closed (cf. [2] Definition 1.1.5/1) two-sided ideal I of $\mathscr{A}_\circ$ such that $\mathscr{A}_{\leqslant 1} \cap I$ is closed in $\mathscr{A}_{\leqslant 1}$. We have a criterion for a two-sided ideal of $\mathscr{A}_\circ$ to be a strictly closed ideal of $\mathscr{A}$.

PROPOSITION 1.20. *Assume the hypotheses (II) and (IV). Then a two-sided ideal I of $\mathscr{A}_\circ$ is a strictly closed ideal of $\mathscr{A}$ if and only if $\mathscr{A}_{\leqslant 1} \cap I$ is closed in $\mathscr{A}_{\leqslant 1}$.*

Proof. The inverse implication is obvious. Suppose that $\mathscr{A}_{\leqslant 1} \cap I$ is closed in $\mathscr{A}_{\leqslant 1}$. We show that I is strictly closed in $\mathscr{A}_\circ$. For this purpose, we equip $(\mathscr{A}_\circ)_{\leqslant 1}$ with the relative topology of the norm topology on $\mathscr{A}_\circ$, and show that $(f + (\mathscr{A}_\circ)_{\leqslant 1}) \cap I$ is closed in $f + (\mathscr{A}_\circ)_{\leqslant 1}$ for any $f \in \mathscr{A}_\circ$. Let $f \in \mathscr{A}_\circ$. Since $\mathscr{A}_\circ$ is unramified, there is a $c \in O_k \setminus \{0\}$ with $cf \in \mathscr{A}_{\leqslant 1}$. By the continuity of the addition $\mathscr{A}_{\leqslant 1} \times \mathscr{A}_{\leqslant 1} \to \mathscr{A}_{\leqslant 1}$, the map $\mathscr{A}_{\leqslant 1} \to \mathscr{A}_{\leqslant 1}$, $f' \mapsto cf + f'$ is continuous, and hence $\mathscr{A}_{\leqslant 1} \cap (-cf + I) = -cf + (\mathscr{A}_{\leqslant 1} \cap I)$ is closed in $\mathscr{A}_{\leqslant 1}$. By the continuity of the scalar multiplication $O_k \times \mathscr{A}_{\leqslant 1} \to \mathscr{A}_{\leqslant 1}$, the map $\mathscr{A}_{\leqslant 1} \to \mathscr{A}_{\leqslant 1}$, $f' \mapsto cf'$ is continuous, and hence $\mathscr{A}_{\leqslant 1} \cap (-f + I) = \mathscr{A}_{\leqslant 1} \cap c^{-1}(\mathscr{A}_{\leqslant 1} \cap (-cf + I))$ is closed in $\mathscr{A}_{\leqslant 1}$. By the continuity of the identity map $(\mathscr{A}_\circ)_{\leqslant 1} \to \mathscr{A}_{\leqslant 1}$, $(\mathscr{A}_\circ)_{\leqslant 1} \cap (-f + I)$ is closed in $(\mathscr{A}_\circ)_{\leqslant 1}$. By the continuity of the addition $\mathscr{A}_\circ \times \mathscr{A}_\circ \to \mathscr{A}_\circ$, the map $\mathscr{A}_\circ \to \mathscr{A}_\circ$, $f' \mapsto -f + f'$ is continuous, and hence $(f + (\mathscr{A}_\circ)_{\leqslant 1}) \cap I = -(-f) + ((\mathscr{A}_\circ)_{\leqslant 1} \cap (-f + I))$ is closed in $f + (\mathscr{A}_\circ)_{\leqslant 1}$. Since $\{f + (\mathscr{A}_\circ)_{\leqslant 1} \mid f \in \mathscr{A}_\circ\}$ forms an open covering of $\mathscr{A}_\circ$, I is closed in $\mathscr{A}_\circ$. By the assumption, $\|\mathscr{A}_\circ \setminus \{0\}\| \subset (0, \infty)$ is discrete. Therefore every closed ideal of $\mathscr{A}_\circ$ is strictly closed by [2] Proposition 1.1.5/4. In particular, I is strictly closed in $\mathscr{A}_\circ$. □

Let $I \subset \mathscr{A}_\circ$ be a closed ideal of $\mathscr{A}$. Then $(\mathscr{A}_\circ/I)_{\leqslant 1}$ admits the strongest topology which is weaker than or equal to the relative topology of the norm topology on $\mathscr{A}_\circ/I$, for which $(\mathscr{A}_\circ/I)_{\leqslant 1}$ forms a topological O_k-algebra, and for which the O_k-algebra homomorphism $\mathscr{A}_{\leqslant 1} \to (\mathscr{A}_\circ/I)_{\leqslant 1}$, $f \mapsto f + I$ is continuous. We state the existence as a proposition in order to refer later.

PROPOSITION 1.21. *We denote by P the set of topologies τ on $(\mathscr{A}_\circ/I)_{\leqslant 1}$ which is weaker than or equal to the relative topology of the norm topology on $\mathscr{A}_\circ/I$, for which $(\mathscr{A}_\circ/I)_{\leqslant 1}$ forms a topological O_k-algebra, and for which the O_k-algebra homomorphism $\mathscr{A}_{\leqslant 1} \to (\mathscr{A}_\circ/I)_{\leqslant 1}$, $f \mapsto f + I$ is continuous. Then P admits the strongest element.*

Proof. Since P admits the weakest element $\{\emptyset, (\mathscr{A}_\circ/I)_{\leqslant 1}\}$, P is not empty. The strongest element of P is given as the pull-back of the topology on the topological O_k-algebra $\prod_{\tau \in P}((\mathscr{A}_\circ/I)_{\leqslant 1}, \tau)$ through the diagonal embedding

$$(\mathscr{A}_\circ/I)_{\leqslant 1} \hookrightarrow \prod_{\tau \in P}((\mathscr{A}_\circ/I)_{\leqslant 1}, \tau). \quad \square$$

We denote by $\mathscr{A}/I$ the Banach topological k-algebra given as the pair of the Banach k-algebra $\mathscr{A}_\circ/I$ and the topology on $(\mathscr{A}_\circ/I)_{\leqslant 1}$ which is the strongest element of P in Proposition 1.21.

EXAMPLE 1.22. If I is a strictly closed ideal of $\mathscr{A}$, then the restriction $\mathscr{A}_{\leqslant 1} \to (\mathscr{A}_\circ/I)_{\leqslant 1}$ of the canonical projection $\mathscr{A}_\circ \twoheadrightarrow \mathscr{A}_\circ/I$ is surjective by the definition of the strict closedness, and hence the topology on $(\mathscr{A}/I)_{\leqslant 1}$ coincides with the topology associated to the quotient topology on $\mathscr{A}_{\leqslant 1}/(\mathscr{A}_{\leqslant 1} \cap I)$ through the O_k-algebra isomorphism $\mathscr{A}_{\leqslant 1}/(\mathscr{A}_{\leqslant 1} \cap I) \to (\mathscr{A}_\circ/I)_{\leqslant 1}$.

EXAMPLE 1.23. Under the hypothesis (VI), the topology on $(\mathscr{A}/I)_{\leqslant 1}$ coincides with the restriction of the norm topology on $\mathscr{A}_\circ/I$ by the continuity of the canonical projection $\mathscr{A}_\circ \twoheadrightarrow (\mathscr{A}/I)_\circ = \mathscr{A}_\circ/I$, and hence $\mathscr{A}/I$ coincides with $(\mathscr{A}_\circ/I)_{\mathrm{disc}}$.

We have a structure theorem of a finite dimensional quotient of a Banach topological k-algebra.

PROPOSITION 1.24. *Let $I \subset \mathscr{A}_\circ$ be a closed ideal of $\mathscr{A}$. Assume either one of the two conditions that the hypotheses (III), (IV), (V) hold and $\mathscr{A}_{\leqslant 1} \cap I$ is closed in $\mathscr{A}_{\leqslant 1}$, or that the hypothesis (VI) holds and k is complete. If $\mathscr{A}_\circ/I$ is finite dimensional, then $\mathscr{A}/I$ coincides with $(\mathscr{A}_\circ/I)_{\mathrm{disc}}$.*

Proof. If k is complete and $\mathscr{A}$ coincides with $(\mathscr{A}_\circ)_{\mathrm{disc}}$, then $\mathscr{A}/I$ coincides with $(\mathscr{A}_\circ/I)_{\mathrm{disc}}$ by Example 1.23. Suppose that k is a local field or a finite field equipped with the trivial valuation, $\mathscr{A}_\circ$ is unramified, $\mathscr{A}_{\leqslant 1}$ is Hausdorff, and $\mathscr{A}_{\leqslant 1} \cap I$ is closed in $\mathscr{A}_{\leqslant 1}$. Then I is a strictly closed ideal of $\mathscr{A}$ by Proposition 1.20, and hence the canonical projection $\mathscr{A}_\circ \twoheadrightarrow \mathscr{A}_\circ/I$ induces a homeomorphic O_k-algebra isomorphism $\mathscr{A}_{\leqslant 1}/(\mathscr{A}_{\leqslant 1} \cap I) \to (\mathscr{A}/I)_{\leqslant 1}$ by Example 1.22. In particular, $(\mathscr{A}/I)_{\leqslant 1}$ is Hausdorff, and $\mathscr{A}/I$ coincides with $((\mathscr{A}/I)_\circ)_{\mathrm{disc}}$ by Proposition 1.19. □

Now we introduce the notion of a Banach left $\mathscr{A}$-module. Let A be a normed k-algebra. A *normed left A-module* is a pair $(V, \|-\|)$ of a left A-module V and a norm $\|-\| \colon V \to [0,\infty)$ of the underlying k-vector space of V with $\|fv\| \leqslant \|f\|\,\|v\|$ for any $(f,v) \in A \times V$.

Let $(V, \|-\|)$ be a normed left A-module. We abbreviate $(V, \|-\|)$ to V. We also regard V as a normed k-vector space. We say that V is a *Banach left A-module* if V is a Banach k-vector space. We denote by $\mathrm{Ban}(A)$ the category of Banach left A-modules and bounded A-linear homomorphisms.

DEFINITION 1.25. A *Banach left $\mathscr{A}$-module* is a Banach left $\mathscr{A}_\circ$-module V such that the scalar multiplication $\mathscr{A}_{\leqslant 1} \times V \to V$ is continuous. A *Banach right $\mathscr{A}$-module* is a Banach left $\mathscr{A}^{\mathrm{op}}$-module. We denote by $\mathrm{Ban}(\mathscr{A}) \subset \mathrm{Ban}(\mathscr{A}_\circ)$ the full subcategory of Banach left $\mathscr{A}$-modules.

For example, every Banach left A-module forms a Banach left A_{disc}-module with respect to the action of $(A_{\mathrm{disc}})_\circ = A$, and hence the inclusion $\mathrm{Ban}(A_{\mathrm{disc}}) \hookrightarrow \mathrm{Ban}(A)$ is the identity of a category. Therefore the notion of a Banach left module over a Banach topological k-algebra is a generalisation of that of a Banach left module over a Banach k-algebra.

EXAMPLE 1.26. For a topological group G, a *unitary k-linear representation of G* is a pair (V,ρ) of an unramified Banach k-vector space V and a continuous map $\rho \colon G \times V \to V$ giving a k-linear action of G on V with $\|\rho(g,v)\| = \|v\|$ for any $(g,v) \in G \times V$. Let G be a discrete group, and (V,ρ) a unitary k-linear representation of G. Then there is a unique structure on V as a Banach left $\mathrm{C}_0(G,k)$-module with $gv = \rho(g,v)$ for any $(g,v) \in G \times V$ by the definition of the norm of $\mathrm{C}_0(G,k)$. In particular, V forms a Banach left $\mathrm{C}_0(G,k)_{\mathrm{disc}}$-module $\int_G(V,\rho)$.

We have another example of a Banach left module over a Banach topological k-algebra. Let K be a compact Hausdorff flat linear topological O_k-algebra. A *Banach left K-module* is a Banach k-vector space V equipped with a continuous O_k-bilinear homomorphism $K \times V \to V$ for which V forms a left K-module with $\|fv\| \leqslant \|v\|$ for any $(f,v) \in K \times V$. We denote by $\mathrm{Ban}(K)$ the category of Banach left K-modules and bounded K-linear homomorphisms. Every Banach left K-module V forms a Banach left K_{comp}-module V_{comp} with respect to the natural action of $k \otimes_{O_k} K$, and the correspondence $V \rightsquigarrow V_{\mathrm{comp}}$ gives an equivalence $\mathrm{Ban}(K) \to \mathrm{Ban}(K_{\mathrm{comp}})$ of categories. Therefore the notion of a Banach left module over a Banach topological k-algebra is also a generalisation of a Banach left module over a compact Hausdorff flat linear topological O_k-module.

EXAMPLE 1.27. Assume the hypothesis (III). Let G be a profinite group, and (V,ρ) a unitary k-linear representation of G. By the argument in [13] p. 11, there is a unique structure on V as a topological left $O_k[[G]]$-module with $gv = \rho(g,v)$ for any $(g,v) \in G \times V$. We note that k is assumed to be a local field of characteristic 0 in [13], but the corresponding argument is also valid under the hypothesis (III). In particular, V forms a Banach left $O_k[[G]]$-module, and hence forms a Banach left $O_k[[G]]_{\mathrm{comp}}$-module $\int_G(V,\rho)$. When G is a finite discrete group, then there is a unique G-equivariant k-algebra isomorphism $\mathrm{C}_0(G,k)_{\mathrm{disc}} \to O_k[[G]]_{\mathrm{comp}}$ in $\mathscr{A}\mathrm{lg}_{\leqslant 1}(k)$ by the definitions, through which the convention of $\int_G(V,\rho)$ is compatible with Example 1.26.

Let V be a Banach left $\mathscr{A}$-module. By the continuity of the scalar multiplication $\mathscr{A}_{\leqslant 1} \times V \to V$, the map $\mathscr{A}_{\leqslant 1} \to V$, $f \mapsto fv$ is continuous for any $v \in V$. Therefore $\mathscr{A}_{\leqslant 1} \cap \mathrm{Ann}_{\mathscr{A}_\circ}(V) = \bigcap_{v \in V} \mathrm{Ann}_{\mathscr{A}_{\leqslant 1}}(v)$ is closed in $\mathscr{A}_{\leqslant 1}$. The scalar multiplication $\mathscr{A}_\circ \times V \to V$ induces an injective k-algebra homomorphism $\mathscr{A}_\circ/\mathrm{Ann}_{\mathscr{A}_\circ}(V) \hookrightarrow \mathscr{B}(V)$, and hence we obtain the following by Proposition 1.24:

PROPOSITION 1.28. *Assume either one of the two conditions that the hypotheses (III), (IV), (V) hold, or that the hypothesis (VI) holds and k is complete. If V is finite dimensional, then $\mathscr{A}/\mathrm{Ann}_{\mathscr{A}_\circ}(V)$ coincides with $(\mathscr{A}_\circ/\mathrm{Ann}_{\mathscr{A}_\circ}(V))_{\mathrm{disc}}$.*

Let $v \in V$ and $w \in V^{\mathrm{D}}$. We denote by $v * w \colon \mathscr{A}_\circ \to k$ the map given by setting $(v*w)(f) := w(fv)$ for an $f \in \mathscr{A}_\circ$. Then we have $\|(v*w)(f)\| = \|w(fv)\| \leqslant \|w\|\,\|fv\| \leqslant \|w\|\,\|f\|\,\|v\|$ for any $f \in \mathscr{A}_\circ$, and hence $v*w$ forms a bounded k-linear homomorphism with $\|v*w\| \leqslant \|v\|\,\|w\|$. We will use the correspondence $(w,v) \rightsquigarrow v*w$ in order to formulate a non-Archimedean analogue of the GNS construction in §2.1.

A *Banach left $\mathscr{A}$-submodule of V* is a closed left $\mathscr{A}_\circ$-module of V, which forms a Banach left $\mathscr{A}$-submodule with respect to the restrictions of the norm and the scalar multiplication. We say that V is *simple* if V admits exactly two Banach left $\mathscr{A}_\circ$-modules, is *isotypic* if $\mathrm{Ann}_{\mathscr{A}_\circ}(V) \in \mathrm{Bl}_k(\mathscr{A}_\circ)$, and is *semisimple* if the underlying k-vector space of V is the direct sum of a family of the underlying k-vector spaces of simple Banach left $\mathscr{A}$-submodules. By [2] Proposition 2.3.3/4, we have the following:

PROPOSITION 1.29. *Suppose that k is complete. Let $((A,\|-\|),\tau)$ be a Banach topological k-algebra, and $(V,\|-\|)$ a finite dimensional Banach left $((A,\|-\|),\tau)$-module. Then $(V,\|-\|)$ is a simple (resp. semisimple) Banach left $((A,\|-\|),\tau)$-module if and only if V is a simple (resp. semisimple) left A-module.*

Let V be a Banach left $\mathscr{A}$-module. A $v \in V$ is said to be *cyclic* if $\mathscr{A}_\circ v$ is dense in V. We say that V is *cyclic* if V admits a cyclic element. In particular, if V is simple, then V is cyclic. We regard V^{D} as a Banach right $\mathscr{A}$-module in a natural way. A $w \in V^{\mathrm{D}}$ is said to be *cocyclic* if $(w\mathscr{A}_\circ)^\top = \{0\}$. By Proposition 1.13 (i), Proposition 1.14, and [2] Proposition 2.3.3/4, we have the following:

PROPOSITION 1.30. *Suppose that k is complete and V is finite dimensional. Then a $w \in V^{\mathrm{D}}$ is cocyclic if and only if w is cyclic.*

As a consequence, we obtain the following:

COROLLARY 1.31. *Suppose that k is a complete valuation field with $\#k \neq 2$ and V is a finite dimensional cyclic semisimple Banach left $\mathscr{A}$-module with $V \neq \{0\}$. Then for any cyclic element $v \in V$, there is a cocyclic element $w \in V^{\mathrm{D}}$ with $w(v) = 1$.*

Proof. By Proposition 1.13 (i) and Proposition 1.29, V^{D} forms a finite dimensional cyclic semisimple Banach right $\mathscr{A}$-module. Take a cyclic element $w' \in V^{\mathrm{D}}$ and a family S of simple Banach left $\mathscr{A}$-submodules of V such that the underlying k-vector space of V is the direct sum of the underlying k-vector spaces of elements of S. By $V \neq \{0\}$, we have $S \neq \emptyset$. For a $W \in S$, we denote by $w'_W \in W$ the composite of the projection $V \twoheadrightarrow W$ associated to S, the inclusion $W \hookrightarrow V$, and w'. Then we have $w' = \sum_{W \in S} w'_W$, and $\sum_{W \in S} c_W w'_W$ is cyclic for any $(c_W)_{W \in S} \in (k^\times)^S$. By $\#k \neq 2$, there is a $(c_W)_{W \in S} \in (k^\times)^S$ with $(\sum_{W \in S} w'_W)(v) = 1$, and $\sum_{W \in S} w'_W$ is cocyclic by Proposition 1.30. $\square$

2. Non-Archimedean states

In this section, we assume that k is complete so that a finite dimensional Banach k-vector space is not necessarily trivial. Let $\mathscr{A}$ denote a Banach topological k-algebra. We introduce a notion of a state on $\mathscr{A}$, and study a non-Archimedean analogue of the GNS construction.

2.1. Non-Archimedean GNS construction

Let F be a bounded k-linear homomorphism $\mathscr{A}_\circ \to k$. We put $\wp_F := \mathscr{A}_\circ^{-1}\ker(F)$ and $I_F := \mathscr{A}_\circ^{-1}\ker(F)\mathscr{A}_\circ^{-1}$. By the continuity of the multiplication $\mathscr{A}_\circ \times \mathscr{A}_\circ \to \mathscr{A}_\circ$, $\wp_F$ and I_F are closed in $\mathscr{A}_\circ$. The map $\mathscr{A}_\circ \to [0,\infty)$, $f \mapsto \inf\{C \in [0,\infty) \mid {}^\forall f' \in \mathscr{A}_\circ, |F(f'f)| \leqslant C\|f'\|\}$ induces a well-defined norm $\|-\|_F$ of the underlying k-vector space of $\mathscr{A}_\circ/\wp_F$, which satisfies $\|v\|_F \leqslant \|F\|\ \|v\|$ for any $v \in \mathscr{A}_\circ/\wp_F$, $\|fv\|_F \leqslant \|f\|\ \|v\|_F$ for any $(f,v) \in \mathscr{A}_\circ \times \mathscr{A}_\circ/\wp_F$, and $|F(f)| \leqslant \|f + \wp_F\|_F$ for any $f \in \mathscr{A}_\circ$.

We denote by V_F the completion of the underlying left $\mathscr{A}_\circ$-module of $\mathscr{A}_\circ/\wp_F$ with respect to $\|-\|_F$, and regard it as a Banach left $\mathscr{A}_\circ$-module. We also denote by $\|-\|_F$ the extension of $\|-\|_F \colon \mathscr{A}_\circ/\wp_F \to [0,\infty)$ to V_F. Then the image $v_F \in V_F$ of $1+\wp_F \in \mathscr{A}_\circ/\wp_F$ forms a cyclic element, and F induces a well-defined bounded k-linear homomorphism $w_F \colon V_F \to k$. We put $T_F := (V_F, v_F, w_F)$, and study the correspondence $F \rightsquigarrow T_F$, which is a non-Archimedean analogue of the GNS construction.

PROPOSITION 2.1. *The equality $\|v\|_F = \|v * w_F\|$ holds for any $v \in V_F$, and w_F is submetric and cocyclic.*

Proof. Let $f \in \mathscr{A}_\circ$. We show $\|f+\wp_F\|_F = \|(f+\wp_F)*w_F\|$. For any $f' \in \mathscr{A}_\circ$, we have $|((f+\wp_F)*w_F)(f')| = |w_F(f'f+\wp_F)| = |F(f'f)| \leqslant \|f'\| \, \|f+\wp_F\|_F$. Since the image of $\mathscr{A}_\circ$ in V_F is dense, we obtain $\|(f+\wp_F)*w_F\| \leqslant \|f+\wp_F\|_F$. For any $C \in (0, \|f+\wp_F\|_F)$, there is an $f' \in \mathscr{A}_\circ$ with $C\|f'\| < |F(f'f)| = |((f+\wp_F)*w_F)(f')|$ by the definition of $\|-\|_F$. It ensures $\|f+\wp_F\|_F = \|(f+\wp_F)*w_F\|$. Since the image of $\mathscr{A}_\circ$ in V_F is dense, the map $V_F \to \mathscr{A}_\circ^{\mathrm{D}}$, $v \mapsto v*w_F$ is isometric. We have $|w_F(f+\wp_F)| = |F(f)| = |((f+\wp_F)*w_F)(1)| \leqslant \|f+\wp_F\|_F$ for any $f \in \mathscr{A}_\circ$. Since the image of $\mathscr{A}_\circ$ in V_F is dense, w_F is submetric. For any $v \in \bigcap_{f \in \mathscr{A}_\circ} \ker(w_F f)$, we have $v*w_F = 0$, and hence $\|v\|_F = \|v*w_F\| = 0$. It ensures that w_F is cocyclic. $\square$

We say that F is *positive* if the map $\mathscr{A}_{\leqslant 1} \to \mathscr{A}_\circ^{\mathrm{D}}$, $f \mapsto (ff')*F$ is continuous for any $f' \in \mathscr{A}_\circ$, is a *pre-state on $\mathscr{A}$* if F is positive and satisfies $F(1) = 1$, and is a *state on $\mathscr{A}$* if F is a submetric pre-state on $\mathscr{A}$.

PROPOSITION 2.2. *Suppose that F is positive. Then the following hold:*

(i) The map $\mathscr{A}_{\leqslant 1} \to \mathscr{A}_\circ^{\mathrm{D}}$, $f \mapsto (fv)*w_F$ is continuous for any $v \in V_F$.

(ii) The Banach left $\mathscr{A}_\circ$-module V_F forms a Banach left $\mathscr{A}$-module.

(iii) The equality $\mathrm{Ann}_{\mathscr{A}_\circ}(V_F) = I_F$ holds, and $\mathscr{A}_{\leqslant 1} \cap I_F$ is closed in $\mathscr{A}_{\leqslant 1}$.

(iv) Assume the hypotheses (II), (IV), and (V), or assume the hypothesis (VI). Then $(\mathscr{A}/I_F)_{\leqslant 1}$ is again Hausdorff.

(v) Assume the hypotheses (III), (IV) and (V), or assume the hypothesis (VI). If V_F is finite dimensional, then $\mathscr{A}/I_F$ coincides with $(\mathscr{A}_\circ/I_F)_{\mathrm{disc}}$.

Proof. We show the assertion (i). Let $v \in V_F$. For any $f \in \mathscr{A}_{\leqslant 1}$ and $\varepsilon \in (0,\infty)$, there is an $f' \in \mathscr{A}_\circ$ with $\|(f'+\wp_F) - v\|_F < \varepsilon$ by the definition of V_F, and we have $\{f'' \in \mathscr{A}_{\leqslant 1} \mid \|(f''f')*F - (ff')*F\| < \varepsilon\} = \{f'' \in \mathscr{A}_{\leqslant 1} \mid \|(f''v)*w_F - (fv)*w_F\| < \varepsilon\}$. It ensures the continuity of the map $\mathscr{A}_{\leqslant 1} \to \mathscr{A}_\circ^{\mathrm{D}}$, $f \mapsto (fv)*w_F$ by the positivity of F.

We show the assertion (ii). The map $V_F \to \mathscr{A}_\circ^{\mathrm{D}}$, $v \mapsto v*w_F$ is isometric by Proposition 2.1, and hence the map $\mathscr{A}_{\leqslant 1} \to V_F$, $f \mapsto fv$ is continuous for any $v \in V_F$ by the assertion (i). We show the continuity of the scalar multiplication $\mathscr{A}_{\leqslant 1} \times V_F \to V_F$. Let U_3 be an open subset of V_F with $fv \in U_3$ for an $(f,v) \in \mathscr{A}_{\leqslant 1} \times V_F$. Take an $\varepsilon \in (0,\infty)$

with $\{v' \in V_F \mid \|v' - fv\|_F < \varepsilon\} \subset U_3$. Put $U_1 := \{f' \in \mathscr{A}_{\leqslant 1} \mid \|f'v - fv\|_F < \varepsilon\}$ and $U_2 := \{v' \in V_F \mid \|v' - v\|_F < \varepsilon\}$. By the continuity of the map $\mathscr{A}_{\leqslant 1} \to V_F$, $f \mapsto fv$, U_1 forms an open neighbourhood of f in $\mathscr{A}_{\leqslant 1}$. For any $(f', v') \in U_1 \times U_2$, we have $\|f'v' - fv\|_F \leqslant \max\{\|f'(v' - v)\|_F, \|f'v - fv\|_F\} < \varepsilon$, and hence $f'v' \in U_3$. It implies the continuity of the scalar multiplication $\mathscr{A}_{\leqslant 1} \times V_F \to V_F$.

We show the assertion (iii). We have $\mathrm{Ann}_{\mathscr{A}_\circ}(V_F) \subset \mathrm{Ann}_{\mathscr{A}_\circ}(v_F) = I_F$. Since v_F is cyclic, we obtain $\mathrm{Ann}_{\mathscr{A}_\circ}(V_F) = I_F$. We show that $\mathscr{A}_{\leqslant 1} \cap I_F$ is closed in $\mathscr{A}_{\leqslant 1}$. For any $v \in V_F$, $\mathrm{Ann}_{\mathscr{A}_{\leqslant 1}}(v)$ is closed in $\mathscr{A}_{\leqslant 1}$, because the map $\mathscr{A}_{\leqslant 1} \to V_F$, $f \mapsto fv$ is continuous and V_F is T_1. We have $\mathscr{A}_{\leqslant 1} \cap I_F = \mathscr{A}_{\leqslant 1} \cap \mathrm{Ann}_{\mathscr{A}_\circ}(V_F) = \bigcap_{v \in V_F} \mathrm{Ann}_{\mathscr{A}_{\leqslant 1}}(v)$, and hence $\mathscr{A}_{\leqslant 1} \cap I_F$ is closed in $\mathscr{A}_{\leqslant 1}$. The assertion (v) follows from the assertion (iii) by Proposition 1.24.

We show the assertion (iv). If $\mathscr{A} = (\mathscr{A}_\circ)_{\mathrm{disc}}$, then we have $\mathscr{A}/I_F = (\mathscr{A}_\circ/I_F)_{\mathrm{disc}}$ by Example 1.23. Suppose that $|k^\times| \subset (0, \infty)$ is discrete and $\mathscr{A}_\circ$ is unramified. Then I_F is a strictly closed ideal of $\mathscr{A}$ by Proposition 1.20, and hence the canonical projection $\mathscr{A}_\circ \twoheadrightarrow \mathscr{A}_\circ/I_F$ induces a homeomorphic O_k-algebra isomorphism $\mathscr{A}_{\leqslant 1}/(\mathscr{A}_{\leqslant 1} \cap I_F) \to (\mathscr{A}/I_F)_{\leqslant 1}$ by Example 1.22. In particular, $(\mathscr{A}/I_F)_{\leqslant 1}$ is Hausdorff. $\square$

We denote by $\mathrm{S}(\mathscr{A}) \subset \mathscr{A}_\circ^{\mathrm{D}}$ the subset of pre-states on $\mathscr{A}$. Then $\mathrm{S}(\mathscr{A})_{\leqslant 1}$ coincides with the set of states on $\mathscr{A}$ by definition. By the continuity of the structure of $\mathscr{A}_\circ^{\mathrm{D}}$ as a k-vector space, the subset of $\mathscr{A}_\circ^{\mathrm{D}}$ consisting of positive functionals forms a k-vector subspace, and $\mathrm{S}(\mathscr{A})$ (resp. $\mathrm{S}(\mathscr{A})_{\leqslant 1}$) forms a k-convex subset of $\mathscr{A}_\circ^{\mathrm{D}}$ (resp. an O_k-convex subset of $(\mathscr{A}_\circ^{\mathrm{D}})_{\leqslant 1}$).

EXAMPLE 2.3. Let A be a Banach k-algebra. Since the inclusion $(A_{\mathrm{disc}})_{\leqslant 1} \hookrightarrow (A_{\mathrm{disc}})_\circ$ is a homeomorphism onto the image, an $F \in A^{\mathrm{D}}$ is a pre-state on A_{disc} if and only if $F(1) = 1$ by Proposition 2.1 and the continuity of the multiplication $\mathscr{A}_\circ \times \mathscr{A}_\circ \to \mathscr{A}_\circ$.

A *pre-GNS triad over* $\mathscr{A}$ is a triad (V, v, w) of a Banach left $\mathscr{A}$-module V, a cyclic element $v \in V$, and a cocyclic element $w \in V^{\mathrm{D}}$ with $w(v) = 1$ such that the k-linear homomorphism $V \to \mathscr{A}_\circ^{\mathrm{D}}$, $v \mapsto v * w$ is admissible. A *GNS triad over* $\mathscr{A}$ is a pre-GNS triad (V, v, w) over $\mathscr{A}$ with $\|v\| = 1$, $\|w\| = 1$, and $\|v'\| = \|v' * w\|$ for any $v' \in V$.

EXAMPLE 2.4. Let V be a Banach left $\mathscr{A}$-module, $v \in V$ a cyclic element, and $w \in V^{\mathrm{D}}$ a cocyclic element with $w(v) = 1$. If V is finite dimensional, then (V, v, w) forms a pre-GNS triad over $\mathscr{A}$ by Proposition 1.13 (i).

Let (V_1, v_1, w_1) and (V_2, v_2, w_2) be pre-GNS triads over $\mathscr{A}$. A *bounded* (resp. *submetric*) $\mathscr{A}$*-linear homomorphism* $(V_1, v_1, w_1) \to (V_2, v_2, w_2)$ means a bounded (resp. submetric) $\mathscr{A}_\circ$-linear homomorphism $f \colon V_1 \to V_2$ with $f(v_1) = v_2$ and $w_2 \circ f = w_1$. A bounded $\mathscr{A}$-linear homomorphism $(V_1, v_1, w_1) \to (V_2, v_2, w_2)$ is unique and has dense image, because v_1 and v_2 are cyclic. We denote by $\mathrm{GNS}(\mathscr{A})$ the category of pre-GNS triads over $\mathscr{A}$ and bounded $\mathscr{A}$-linear homomorphisms, and by $\mathrm{GNS}_{\leqslant 1}(\mathscr{A}) \subset \mathrm{GNS}(\mathscr{A})$ the subcategory of GNS triads over $\mathscr{A}$ and submetric $\mathscr{A}$-linear homomorphisms.

Let $T = (V, v, w)$ be a pre-GNS triad over $\mathscr{A}$. We denote by $F_T \colon \mathscr{A}_\circ \to k$ the map given by setting $F_T(f) := w(fv)$ for an $f \in \mathscr{A}_\circ$. We obtain a correspondence $T \rightsquigarrow F_T$, which is a non-Archimedean analogue of the construction of a vector state (cf. [12] (6.7.14)).

THEOREM 2.5. (Non-Archimedean GNS construction) *The following hold:*

(i) For any pre-state (resp. state) F on $\mathscr{A}$, T_F forms a pre-GNS (resp. GNS) triad over $\mathscr{A}$, and F_{T_F} coincides with F.

(ii) For any pre-GNS (resp. GNS) triad T over $\mathscr{A}$, F_T forms a pre-state (resp. state) on $\mathscr{A}$, and there is a unique isomorphism $T_{F_T} \to T$ in $\mathrm{GNS}(\mathscr{A})$ (resp. $\mathrm{GNS}_{\leqslant 1}(\mathscr{A})$).

Proof. Let F be a pre-state on $\mathscr{A}$. We have $w_F(v_F) = F(1) = 1$, and hence T_F forms a pre-GNS triad by Proposition 2.1 and Proposition 2.2 (ii). We obtain $F_{T_F}(f) = w_F(fv_F) = w_F(f + \wp_F) = F(f)$ for any $f \in \mathscr{A}_\circ$, and hence F_{T_F} coincides with F. The map $\mathscr{A}_\circ \to \mathscr{A}_\circ^{\mathrm{D}}$, $v \mapsto v * w_F$ is isometric by Proposition 2.1, and hence is admissible. Therefore T_F forms a pre-GNS state. Suppose that F is a state on $\mathscr{A}$. Then we have $1 = |w_F(v_F)| \leqslant \|w_F\| \, \|v_F\|_F \leqslant \|w_F\| \, \|v_F\| \leqslant \|w_F\| \, \|1\| = \|w_F\| \leqslant \|F\| \leqslant 1$, and hence $\|w_F\| = \|F\| = 1$ and $\|v_F\|_F = \|v_F\| = 1$. Therefore T_F forms a GNS triad over $\mathscr{A}$.

Let $T = (V, v, w)$ be a pre-GNS triad over $\mathscr{A}$. By the continuity of the scalar multiplication $\mathscr{A}_{\leqslant 1} \times V \to V$, the map $\mathscr{A}_{\leqslant 1} \to V$, $f \mapsto fv'$ is continuous for any $v' \in V$. Since w is cocyclic, the admissible k-linear homomorphism $\iota \colon V \to \mathscr{A}_\circ^{\mathrm{D}}$, $v \mapsto v * w$ is injective. For any $f \in \mathscr{A}_\circ$, we have $|w(fv)| \leqslant \|w\| \, \|fv\| \leqslant \|w\| \, \|f\| \, \|v\| \leqslant (\|w\| \, \|v\|)\|f\|$ and $f * F_T = f * (v * w) = (fv) * w$. Therefore F_T is a bounded k-linear homomorphism, and the map $\mathscr{A}_{\leqslant 1} \to \mathscr{A}_\circ^{\mathrm{D}}$, $f \mapsto (ff') * F_T$ coincides with the map given as the composite of the continuous maps $\mathscr{A}_{\leqslant 1} \to V$, $f \mapsto ff'v$ and $V \to \mathscr{A}_\circ^{\mathrm{D}}$, $v' \mapsto v' * w$. We have $F_T(1) = w(v) = 1$, and hence F_T forms a pre-state on $\mathscr{A}$.

By $f * F_T = f * (v * w) = (fv) * w$ for any $f \in \mathscr{A}_\circ$, the map $\mathscr{A}_\circ \to \mathscr{A}_\circ^{\mathrm{D}}$, $f \mapsto (fv) * w$ induces an isometric k-linear homomorphism $V_{F_T} \hookrightarrow \mathscr{A}_\circ^{\mathrm{D}}$ onto the closed image. It factors through ι, because $(fv) * w$ lies in the image of V for any $f \in \mathscr{A}_\circ$. The resulting map $\iota' \colon V_{F_T} \hookrightarrow V$ is an injective admissible $\mathscr{A}_\circ$-linear homomorphism with dense image, and hence is an isomorphism in $\mathrm{Ban}(\mathscr{A})$. Indeed, we have $\iota'(f + \wp_F) = fv$ for any $f \in \mathscr{A}_\circ$, and hence the image of V_{F_T} in V contains the dense subset $\mathscr{A}_\circ v$. By the assertion (i), T_{F_T} forms a pre-GNS triad over $\mathscr{A}$, and ι' gives an isomorphism $T_{F_T} \to T$ in $\mathrm{GNS}(\mathscr{A})$. Suppose that T is a GNS triad over $\mathscr{A}$. Then we have $\|F_T\| \leqslant \|w\| \, \|v\| = 1$, and hence F_T is a state on $\mathscr{A}$. By $\|v'\| = \|v' * w\|$ for any $v' \in V$, ι is isometry, and hence so is ι'. $\square$

2.2. Pure states and mixed states

Let F be a pre-state on $\mathscr{A}$. We say that F is *pure* if $\wp_F \in \mathrm{Max}_k(\mathscr{A}_\circ)$. We note that it is not good to formulate the notion of the purity of a state in a way similar to the Archimedean case, because there is no non-trivial extreme set with respect to the M-convexity consisting of a single point by [9] Theorem 4. We denote by $\mathrm{PS}(\mathscr{A}) \subset \mathrm{S}(\mathscr{A})$

the subset of pure pre-states on $\mathscr{A}$. We have a characterisation of the purity of a pre-state using the GNS construction, which is an analogue of [12] Theorem 6.8.11.

THEOREM 2.6. *The following hold:*

(i) Let $F \in \mathrm{PS}(\mathscr{A})$*. Then* V_F *forms a finite dimensional simple Banach left* $\mathscr{A}_\circ$*-module.*

(ii) Let V *be a finite dimensional simple Banach left* $\mathscr{A}$*-module. Then there is a pair* (F,ι) *of an* $F \in \mathrm{PS}(\mathscr{A})$ *and an isomorphism* $\iota \colon V_F \to V$ *in* $\mathrm{Ban}(\mathscr{A})$*.*

(iii) Let V *be a finite dimensional simple Banach left* $\mathscr{A}$*-module. If the hypothesis (II) holds and* V *is unramified, then there is a pair* (F,ι) *of an* $F \in \mathrm{PS}(\mathscr{A})_{\leqslant 1}$ *and an isomorphism* $\iota \colon V_F \to V$ *in* $\mathrm{Ban}(\mathscr{A})$*.*

Proof. Let $F \in \mathrm{PS}(\mathscr{A})$. Then the underlying left $\mathscr{A}_\circ$-module of V_F is simple, and hence V_F forms a finite dimensional simple Banach left $\mathscr{A}$-module by Proposition 1.29.

Let V be a finite dimensional simple Banach left $\mathscr{A}$-module. Since V admits exactly two Banach left $\mathscr{A}$-submodule, we have $V \neq \{0\}$. Take a $v \in V \setminus \{0\}$. By Proposition 1.13 (i), there is a $w \in V^{\mathrm{D}}$ with $w(v) = 1$. Every non-zero element of V or V^{D} is a cyclic element whose annihilator is a maximal one-sided ideal by Proposition 1.29. Therefore v is a cyclic element with $\wp_{F_T} = \mathrm{Ann}_{\mathscr{A}_\circ}(v) \in \mathrm{Max}_k(\mathscr{A}_\circ)$, and w is a cocyclic element by Proposition 1.30. By Proposition 1.11 and Proposition 1.13 (i), the k-linear homomorphism $\mathscr{A}_\circ \twoheadrightarrow V,\ f \mapsto fv$ is admissible. Therefore (V,v,w) forms a pre-GNS triad over $\mathscr{A}$.

Suppose that $|k^\times| \subset (0,\infty)$ is discrete and that V is unramified. Take a $c \in k^\times$ with $|c| = \|v\|$, and replace v by $c^{-1}v$ so that we obtain $\|v\| = 1$. Since the k-linear homomorphism $kv \to k,\ c'v \mapsto c'$ is of operator norm 1, there is a $w' \in V^{\mathrm{D}}$ with $w'(c^{-1}v) = 1$ and $\|w'\| = 1$ by Proposition 1.12. Replace w by w' so that we obtain $\|w\| = 1$. Then (V,v,w) forms a GNS triad over $\mathscr{A}$. □

We say that F is *isotypic* if $I_F \in \mathrm{Bl}_k(\mathscr{A})$. If F is pure, then F is isotypic because $\mathscr{A}_\circ/\wp_F$ is a faithful simple left $\mathscr{A}_\circ$-module with $I_F = \mathrm{Ann}_A(A/\wp_F)$. We denote by $\mathrm{IS}(\mathscr{A}) \subset \mathrm{S}(\mathscr{A})$ the subset of isotypic pre-states and by $\mathrm{IS}(\mathscr{A},I) \subset \mathrm{IS}(\mathscr{A})$ the subset of isotypic pre-states F with $I_F = I$, and put $\mathrm{PS}(\mathscr{A},I) := \mathrm{PS}(\mathscr{A}) \cap \mathrm{IS}(\mathscr{A},I)$ for an $I \in \mathrm{Bl}_k(\mathscr{A}_\circ)$. We show that isotypic pre-states are spanned by pure pre-states sharing blocks.

PROPOSITION 2.7. *Assume the hypotheses (III), (IV), and (V), or assume the hypothesis (VI). Let* $F \in \mathrm{S}(\mathscr{A})$*. Then the following are equivalent:*

(i) The pre-state F *is isotypic.*

(ii) There is a $(c_i,F_i)_{i=1}^n \in (k \times \mathrm{PS}(\mathscr{A}))^n$ *with* $n \in \mathbb{N} \setminus \{0\}$*,* $\sum_{i=1}^n c_i = 1$*,* $\sum_{i=1}^n c_i F_i = F$*, and* $I_{F_i} = I_F$ *for any* $i \in \mathbb{N} \cap [1,n]$*.*

(iii) There is an $I \in \mathrm{Bl}_k(\mathscr{A}_\circ)$ *with* $I \subset I_F$*.*

Proof. The implication from (ii) to (iii) follows from the fact that every two-sided ideal of $\mathscr{A}_\circ$ belonging to $\mathrm{Bl}_k(\mathscr{A}_\circ)$ is maximal by Wedderburn's theorem (cf. [1] 13.4 Theorem), and the implication (iii) to (i) follows from Jacobson density theorem (cf. [1] 14.5 Corollary). Suppose that F is isotypic. Replacing $\mathscr{A}$ by $\mathscr{A}/I_F$, we may assume that $\mathscr{A}$ coincides with $(\mathscr{A}_\circ)_{\mathrm{disc}}$ by Proposition 2.2 (v) and that there is a k-algebra isomorphism $\iota \colon \mathrm{M}_n(K) \to \mathscr{A}_\circ$ with $n \in \mathbb{N} \setminus \{0\}$ for a division k-algebra K finite dimensional as a k-vector space by Wedderburn's theorem. Then an $F \in \mathscr{A}_\circ^{\mathrm{D}}$ is a pre-state on $\mathscr{A}$ if and only if $F(1) = 1$ by Example 2.3.

Since $\mathscr{A}_\circ$ is a simple k-algebra with $\dim_k \mathscr{A}_\circ < \infty$, we have $I_F = \{0\}$ and $\mathrm{S}(\mathscr{A}) = \mathrm{PS}(\mathscr{A}) = \mathrm{PS}(\mathscr{A}, I_F)$. We denote by $E_{i,j} \in \mathrm{M}_n(K)$ the matrix whose (i,j)-th entry is 1 and whose other entries are 0 for an $(i,j) \in (\mathbb{N} \cap [1,n])^n$. We identify K with the image of the embedding $K \hookrightarrow \mathscr{A}_\circ$, $c \mapsto \sum_{i=1}^n \iota(cE_{i,i})$. Then K forms a Banach k-algebra by [2] Proposition 2.3.3/4. Moreover, the embedding $\ell_{i,j} \colon K \hookrightarrow \mathscr{A}_\circ$, $c \mapsto \iota(cE_{i,j})$ is an isomorphism in $\mathrm{Ban}(K)$ for any $(i,j) \in (\mathbb{N} \cap [1,n])^n$, and the bijective map $\ell \colon K^{n^2} \to \mathscr{A}_\circ$, $(c_{i,j})_{i,j=1}^n \mapsto \sum_{i,j=1}^n \iota(c_{i,j}E_{i,j})$ is an isomorphism in $\mathrm{Ban}(K)$ by Proposition 1.13 (i). We denote by $p_{i,j} \colon \mathscr{A}_\circ \to K$ the continuous K-linear homomorphism given as the composite of ℓ^{-1} and the (i,j)-th projection $K^{n^2} \to K$ for a $(i,j) \in (\mathbb{N} \cap [1,n])^2$ for an $(i,j) \in (\mathbb{N} \cap [1,n])^2$. By Proposition 1.13 (i), K_{disc} admits a pre-state $\varphi \colon K \to k$. We put $S_0 := \{i \in \mathbb{N} \cap [0,1] \mid F(\iota(E_{i,i})) = 0\}$, $S_1 := (\mathbb{N} \cap [1,n]) \setminus S_0$, $F_{i,1} := \varphi \circ p_{i,i} + \sum_{j=1}^{i-1} F \circ p_{i,j} \in \mathrm{PS}(\mathscr{A}, I_F)$ for an $i \in \mathbb{N} \cap [1,n]$, $F_{i,-1} := \varphi \circ p_{i,i} - \sum_{j=i}^n F \circ p_{i,j}$ for an $i \in \mathbb{N} \cap [1,n]$, $F_{i,(0,1)} := \varphi \circ p_{i,i} + F \circ p_{i,i} \in \mathrm{PS}(\mathscr{A}, I_F)$ for an $i \in S_0$, $F_{i,(0,-1)} := \varphi \circ p_{i,i}$ for an $i \in S_0$, $c_i := F(\iota(E_{i,i})) \in k^\times$ for an $i \in S_1$, and $F_{i,0} := c_i^{-1} F \circ p_{i,i}$ for an $i \in S_1$. Then we have $\sum_{i \in S_1} c_i = 1$, $F = \sum_{i=1}^n (F_{i,1} - F_{i,-1}) + \sum_{i \in S_0} (F_{i,(0,1)} - F_{i,(0,-1)}) + \sum_{i \in S_1} c_i F_{i,0}$, and $F_{i,j} \in \mathrm{PS}(\mathscr{A}, I_F)$ for any $(i,j) \in (\mathbb{N} \cap [1,n]) \times \{-1, 0, 1, (0,-1), (0,1)\}$. $\square$

We study the relation between decompositions of semisimple cyclic Banach left $\mathscr{A}$-modules and of mixed states on $\mathscr{A}$. Let $k_0 \subset k$ be a convexity (cf. Example 1.1). We say that F is k_0*-mixed* if there is a $(c_i, F_i)_{i=1}^n \in (k_0 \times \mathrm{PS}(\mathscr{A}))^n$ with $n \in \mathbb{N}$, $\sum_{i=1}^n c_i = 1$, and $\sum_{i=1}^n c_i F_i = F$. We denote by $\mathrm{MS}(\mathscr{A}; k_0) \subset \mathrm{S}(A)$ the k_0-convex subset of k_0-mixed pre-states on $\mathscr{A}$, which coincides with $\mathrm{co}(\mathrm{PS}(\mathscr{A}); k_0)$ by definition. We will deal with several convexities in k in §2.3, but only deal with the full-convexity k in k in this subsection. We state our main theorem on mixed pre-states.

THEOREM 2.8. *Assume the hypotheses (III), (IV), and (V), or assume the hypothesis (VI). Let $F \in \mathrm{S}(\mathscr{A})$. Then V_F is a finite dimensional semisimple Banach left $\mathscr{A}$-module if and only if $F \in \mathrm{MS}(\mathscr{A}; k)$.*

In order to verify Theorem 2.8, we study a linear combination of isotypic pre-states, and show the relation between the annihilator and the decomposition.

LEMMA 2.9. *Let $F \in \mathrm{S}(\mathscr{A})$. For any pair $((c_i)_{i=1}^n, (F_i)_{i=1}^n)$ of a $(c_i)_{i=1}^n \in (k^\times)^n$ and an $(F_i)_{i=1}^n \in \mathrm{IS}(\mathscr{A})^n$ with $n \in \mathbb{N}$ and $\sum_{i=1}^n c_i F_i = F$, if $I_{F_i} \neq I_{F_j}$ for any $(i,j) \in (\mathbb{N} \cap [1,n])^2$ with $i \neq j$, then the equality $\wp_F = \bigcap_{i=1}^n \wp_{F_i}$ holds.*

Proof. We have $\bigcap_{i=1}^n \ker F_i \subset \ker F$ and hence $\bigcap_{i=1}^n \wp_{F_i} \subset \wp_F$. Let $f \in \wp_F$. By Proposition 1.7, there is an $(e_i)_{i=1}^n \in \mathscr{A}_\circ^n$ with $\sum_{i=1}^n e_i = 1$ and $e_i \mathscr{A}_\circ \subset I_{F_j}$ for any

$(i,j) \in (\mathbb{N} \cap [1,n])^2$ with $i \neq j$. Let $i \in \mathbb{N} \cap [1,n]$. For any $f' \in \mathscr{A}_\circ$, we have $F_i(f'f) = F_i(e_i f'f) = c_i^{-1} F(e_i f'f) = 0$. It implies $f \in \wp_{F_i}$. We obtain $f \in \bigcap_{i=1}^n \wp_{F_i}$. $\square$

Proof of Theorem 2.8. First, suppose $F \in \mathrm{MS}(\mathscr{A};k)$. Put $F = \sum_{i=1}^n c_i F_i \in \mathrm{S}(\mathscr{A})$ with $n \in \mathbb{N}$, $((c_i)_{i=1}^n, (F_i)_{i=1}^n) \in (k^\times \times \mathrm{PS}(\mathscr{A}))^n$, $\sum_{i=1}^n c_i = 1$, and $\sum_{i=1}^n c_i F_i = F$. We have $\bigcap_{i=1}^n \ker F_i \subset \ker F$ and hence $\bigcap_{i=1}^n \wp_{F_i} \subset \wp_F$. The diagonal map $\mathscr{A}_\circ \to \prod_{i=1}^n V_{F_i}$ induces a bounded injective $\mathscr{A}_\circ$-linear homomorphism $\mathscr{A}_\circ / \bigcap_{i=1}^n \wp_{F_i} \hookrightarrow \prod_{i=1}^n V_{F_i}$. It implies that V_F is a subquotient of $\prod_{i=1}^n V_{F_i}$ by [2] Proposition 2.3.3/4. Since V_{F_i} is a finite dimensional simple Banach left $\mathscr{A}$-module for any $i \in \mathbb{N} \cap [1,n]$ by Theorem 2.6, V_F is a finite dimensional semisimple Banach $\mathscr{A}$-module by Proposition 1.29.

Next, suppose that V_F is a finite dimensional semisimple Banach left $\mathscr{A}$-module. By $F(1) = 1$, we have $\wp_F \subsetneq \mathscr{A}_\circ$, and hence $V_F \neq \{0\}$. By Proposition 1.29, V_F admits a family $(W_h)_{h=1}^m$ of simple Banach left $\mathscr{A}$-submodules with $m \in \mathbb{N} \setminus \{0\}$ such that the underlying k-vector space of V_F is presented as $\bigoplus_{h=1}^m W_h$. For an $h \in \mathbb{N} \cap [1,m]$, we denote by $v_h \in W_h$ the image of $1 + \wp_F$ by the h-th projection $V_F \twoheadrightarrow W_h$, and put $I_h := \mathrm{Ann}_{\mathscr{A}_\circ}(v_h) \mathscr{A}_\circ^{-1}$. We have $\wp_F = \mathrm{Ann}_{\mathscr{A}_\circ}(1 + \wp_F) = \bigcap_{h=1}^m \mathrm{Ann}_{\mathscr{A}_\circ}(v_h)$ and hence $I_F = \bigcap_{h=1}^n I_h$. Put $S := \{I_h \mid h \in \mathbb{N} \cap [1,m]\}$. By Proposition 1.7, the diagonal map $\mathscr{A}_\circ \to \prod_{I \in S} \mathscr{A}_\circ / I$ induces an isomorphism $\mathscr{A}_\circ / I_F \to \prod_{I \in S} \mathscr{A}_\circ / I$ in $\mathrm{Ban}(\mathscr{A}_\circ)$. Therefore there is an $(e_I)_{I \in S} \in \mathscr{A}_\circ^S$ with $\sum_{I \in S} e_I = 1$ and $e_I \mathscr{A}_\circ \subset \bigcap_{I' \in S \setminus \{I\}} I'$ for any $I \in S$.

Let $I \in S$. Then $\mathscr{A}/I$ coincides with $(\mathscr{A}_\circ / I)_{\mathrm{disc}}$ by Proposition 1.28, and $\mathscr{A}/I$ admits a pre-state by Proposition 1.13 (i). Composing the canonical projection $\mathscr{A} \twoheadrightarrow \mathscr{A}/I$, we obtain a pre-state φ_I on $\mathscr{A}$ with $I_{\varphi_I} = I$. We denote by $F_I \colon \mathscr{A} \to k$ the pre-state given by setting $F_I(f) := (1 - F(e_I))\varphi_I(f) + F(e_I f)$ for an $f \in \mathscr{A}_\circ$. We have $F_I \in \mathrm{IS}(\mathscr{A}, I)$, and hence there is a $(c_{i,I}, F_{i,I})_{i=1}^{n_I} \in (k \times \mathrm{PS}(\mathscr{A}, I))^{n_I}$ with $n_I \in \mathbb{N}$, $\sum_{i=1}^{n_I} c_{i,I} = 1$, and $\sum_{i=1}^{n_I} c_{i,I} F_{i,I} = F_I$ by Proposition 2.7. We obtain a presentation $F = \sum_{I \in S}(F_I - (1 - F(e_I))\varphi_I) = \sum_{I \in S}(-(1 - F(e_I))\varphi_I + \sum_{i=1}^{n_I} c_{i,I} F_{i,I})$ with $\sum_{I \in S}(-(1 - F(e_I)) + \sum_{i=1}^{n_I} c_{i,I}) = \sum_{I \in S} F(e_I) = F(1) = 1$. $\square$

As a consequence, we obtain the following:

COROLLARY 2.10. *Let V be a finite dimensional cyclic semisimple Banach left $\mathscr{A}$-module with $V \neq \{0\}$. Suppose $\#k \neq 2$. Assume the hypotheses (III), (IV), and (V), or assume the hypothesis (VI). Then there is a pair (F, ι) of an $F \in \mathrm{MS}(\mathscr{A};k)$ and an isomorphism $\iota \colon V_F \to V$ in* $\mathrm{Ban}(\mathscr{A})$.

Proof. Take a cyclic element $v \in V$. By Corollary 1.31, there is a cocyclic element $w \in V^{\mathrm{D}}$ with $w(v) = 1$. The k-linear homomorphism $V \to \mathscr{A}_\circ^{\mathrm{D}}$, $v' \mapsto v' * w$ is admissible by Proposition 1.13, and hence (V, v, w) forms a pre-GNS triad. Therefore the assertion follows from Theorem 2.5 (ii) and Theorem 2.8. $\square$

2.3. Integrality of coefficients

We consider several convexities in k for coefficients of mixed pre-states. Let V be a Banach left $\mathscr{A}$-module. We prepare the terminology on a condition on k, $\mathscr{A}$, and V appearing frequently.

DEFINITION 2.11. We refer as *the hypothesis (VII)* to the condition that the hypotheses (III), (IV), and (V) hold or the hypotheses (I) and (VI) hold.

We show an integral variant of Corollary 2.10.

THEOREM 2.12. *Let V be a finite dimensional cyclic semisimple Banach left $\mathscr{A}$-module with $V \neq \{0\}$. Suppose $\# k \neq 2$. Under the hypothesis (VII), there is a pair (F,ι) of an $F \in \mathrm{MS}(\mathscr{A};O_k)$ and an isomorphism $\iota \colon V_F \to V$ in $\mathrm{Ban}(\mathscr{A})$.*

When $\# k \neq 2$, we have $\# O_k^{\times} \neq 1$. Therefore in order to verify Theorem 2.12, it suffices to show the following:

LEMMA 2.13. *Let V be a finite dimensional cyclic semisimple Banach left $\mathscr{A}$-module with $V \neq \{0\}$. Under the hypothesis (VII), for any $c \in k \setminus \{0,1\}$, there is a pair (F,ι) of an $F \in \mathrm{MS}(\mathscr{A};\mathbb{F}_{k,c})$ and an isomorphism $\iota \colon V_F \to V$ in $\mathrm{Ban}(\mathscr{A})$.*

In order to verify Lemma 2.13, we compare isotypic pre-states and cyclic isotypic Banach left $\mathscr{A}$-modules. Let k_0 be a convexity in k. We put $\mathrm{IMS}(\mathscr{A};k_0) := \mathrm{IS}(\mathscr{A}) \cap \mathrm{MS}(\mathscr{A};k_0)$ and $\mathrm{IMS}(\mathscr{A},I;k_0) := \mathrm{IS}(\mathscr{A},I) \cap \mathrm{MS}(\mathscr{A};k_0)$ for an $I \in \mathrm{Bl}_k(\mathscr{A})$.

LEMMA 2.14. *The following hold:*

(i) Let $F \in \mathrm{IMS}(\mathscr{A};k_0)$. Then V_F is a finite dimensional cyclic isotypic semisimple Banach left $\mathscr{A}$-module.

(ii) Let V be a finite dimensional cyclic isotypic semisimple Banach left $\mathscr{A}$-module. Under the hypothesis (VII), for any $c \in k \setminus \{0,1\}$, there is a pair (F,ι) of an $F \in \mathrm{IMS}(\mathscr{A};\mathbb{F}_{k,c})$ and an isomorphism $\iota \colon V_F \to V$ in $\mathrm{Ban}(\mathscr{A})$.

Proof. The assertion (i) follows from Proposition 2.2 (iii) and Theorem 2.8. We show the assertion (ii). Replacing $\mathscr{A}$ by $\mathscr{A}/\mathrm{Ann}_{\mathscr{A}_\circ}(V)$, we may assume that $\mathscr{A}$ coincides with $(\mathscr{A}_\circ)_{\mathrm{disc}}$ by Proposition 1.28 and $\mathscr{A}_\circ$ is simple and left Artinian by Jacobson density theorem (cf. [1] 14.5 Corollary). Then an $F \in \mathscr{A}_\circ^{\mathrm{D}}$ is a pre-state on $\mathscr{A}$ if and only if $F(1) = 1$ by Example 2.3.

Take a cyclic element $v \in V$. By Proposition 1.29, V admits a family $(W_i)_{i=1}^n$ of simple Banach left $\mathscr{A}$-submodules with $n \in \mathbb{N} \setminus \{0\}$ and $V = \bigoplus_{i=1}^n W_i$. We denote by $v_i \in W_i$ the image of v by the projection $V \twoheadrightarrow W_i$ associated to the direct sum decomposition, and put $\wp_i := \mathrm{Ann}_{\mathscr{A}_\circ}(v_i)$ for an $i \in \mathbb{N} \cap [1,n]$. By Proposition 1.29, $\wp_i$ is a left maximal ideal for any $i \in \mathbb{N} \cap [1,n]$. Since $v = \sum_{i=1}^n v_i$ is cyclic, we have $\bigcap_{i=1}^n \wp_i = \mathrm{Ann}_{\mathscr{A}_\circ}(v)$ and $\wp_i + \bigcap_{j \in (\mathbb{N} \cap [1,n]) \setminus \{i\}} \wp_j = \mathscr{A}_\circ$ for any $i \in \mathbb{N} \cap [1,n]$. By Proposition 1.6, there is an $S \subset \mathrm{Max}(\mathscr{A}_\circ)$ with $\{\wp_i \mid i \in \mathbb{N} \cap [1,n]\} \subset S$, $\wp + \bigcap_{\wp' \in S \setminus \{\wp\}} \wp' = \mathscr{A}_\circ$ for any $\wp \in S$, and $\bigcap_{\wp \in S} \wp = \{0\}$. By Wedderburn's theorem (cf. [1] 13.4 Theorem), there is a pair (K,ι) of a division k-algebra K and a k-algebra isomorphism $\mathscr{A}_\circ \to \mathrm{M}_d(K)$ with $d \in \mathbb{N}$. Let $\wp \in S$. We have $\dim_K(\mathrm{M}_d(K) \otimes_{\mathscr{A}_\circ} \wp) = d(d-1)$. Put $\wp^\vee := \bigcap_{\wp' \in S \setminus \{\wp\}} \wp'$. By $\bigcap_{\wp \in S} \wp = \{0\}$ and $\wp + \wp^\vee = \mathscr{A}_\circ$, we obtain $\dim_K(\mathrm{M}_d(K) \otimes_{\mathscr{A}_\circ} \wp^\vee) = d$, and hence $\wp^\vee$ is a minimal left ideal of $\mathscr{A}_\circ$ for any

$\wp \in S$. In particular, $\#S$ coincides with d, and the underlying k-vector space of $\mathscr{A}_\circ$ is presented as the direct sum $\bigoplus_{\wp \in S} \wp^\vee$.

By [1] 7.2 Proposition, there is an orthogonal system $(e_\wp)_{\wp \in S} \in \mathscr{A}_\circ^S$ of primitive idempotents with $\sum_{\wp \in S} e_\wp = 1$ and $\mathscr{A}_\circ e_\wp = \wp^\vee$. We obtain a decomposition of the underlying k-vector space of $\mathscr{A}_\circ$ as the direct sum $\bigoplus_{(\wp,\wp') \in S^2} e_\wp \mathscr{A}_\circ e_{\wp'}$. The natural map $\mathscr{A}_\circ \to \prod_{(\wp,\wp') \in S^2} e_\wp \mathscr{A}_\circ e_{\wp'}$ is an isomorphism in $\mathrm{Ban}(k)$ by [2] Corollary 2.3.3/5 for the case where the valuation of k is non-trivial and by Proposition 1.14 for the case where the valuation of k is trivial. For each $i \in \mathbb{N} \cap [1,n]$, there is a bounded k-linear homomorphism $\varphi_i \colon e_{\wp_i} \mathscr{A}_\circ e_{\wp_i} \to k$ with $\varphi(e_{\wp_i}) = 1$ by Proposition 1.13 (i). Let $i \in \mathbb{N} \cap [1,n]$. We define a bounded k-linear homomorphism $F_i \colon \mathscr{A}_\circ \to k$ by setting $F_i(f) := \varphi_i(e_{\wp_i} f e_{\wp_i})$ for an $f \in \mathscr{A}_\circ$. Then we have $F_i(1) = \varphi_i(e_{\wp_i}) = 1$ and $\wp_i = \bigoplus_{\wp \in S} \bigoplus_{\wp' \in S \setminus \{\wp_i\}} e_\wp \mathscr{A}_\circ e_{\wp'} \subset \wp_{F_i}$. It implies $\wp_{F_i} = \wp_i$ and $F_i \in \mathrm{PS}(\mathscr{A})$. Put $F := c^{n-1} F_n + \sum_{i=1}^{n-1} (1-c) c^{i-1} F_i \in \mathrm{IMS}(\mathscr{A}; \mathbb{F}_{k,c})$. We have $\bigcap_{i=1}^n \ker(F_i) \subset \ker(F)$ and hence $\mathrm{Ann}_{\mathscr{A}_\circ}(v) = \bigcap_{i=1}^n \wp_i = \bigcap_{i=1}^n \wp_{F_i} \subset \wp_F$. Let $f \in \wp_F$. We show $f \in \mathrm{Ann}_{\mathscr{A}_\circ}(v)$. Assume $f \notin \wp_i$ for an $i \in \mathbb{N} \cap [1,n]$. By $\wp_i \in \mathrm{Max}(\mathscr{A}_\circ)$, there is an $f' \in \mathscr{A}_\circ$ with $1 - f'f \in \wp_i$. We obtain $F(e_{\wp_i} f' f) = u F_i(e_{\wp_i} f' f e_{\wp_i}) = u F_i(e_{\wp_i}) = u$ for a $u \in \{c^{n-1}\} \cup \{(1-c)c^{i-1} \mid i \in \mathbb{N} \cap [1, n-1]\} \subset \mathbb{F}_{k,c} \setminus \{0\}$. This contradicts $f \in \wp_F$. It implies $f \in \mathrm{Ann}_{\mathscr{A}_\circ}(v)$. We conclude $\wp_F = \bigcap_{i=1}^n \wp_i$, and the map $\mathscr{A}_\circ \to V$, $f \mapsto fv$ induces an isomorphism $V_F \to V$ in $\mathrm{Ban}(\mathscr{A})$. $\square$

Proof of Lemma 2.13. By Proposition 1.29, V admits a family $(W_i)_{i=1}^n$ of isotypic Banach left $\mathscr{A}_\circ$-submodules with $n \in \mathbb{N} \setminus \{0\}$ and $V = \bigoplus_{i=1}^n W_i$ such that $\mathrm{Ann}_{\mathscr{A}_\circ}(W_i) \neq \mathrm{Ann}_{\mathscr{A}_\circ}(W_j)$ for any $(i,j) \in (\mathbb{N} \cap [1,n])^2$ with $i \neq j$. For each $i \in \mathbb{N} \cap [1,n]$, there is a pair (F_i, ι_i) of an $F_i \in \mathrm{IMS}(\mathscr{A}; \mathbb{F}_{k,c})$ and an isomorphism $\iota \colon V_{F_i} \to W_i$ in $\mathrm{Ban}(\mathscr{A})$ by Lemma 2.14 (ii). If $n = 1$, then we have $V = W_1$, and (F_1, ι_1) is a pair of an $F \in \mathrm{MS}(\mathscr{A}; \mathbb{F}_{k,c})$ and an isomorphism $\iota \colon V_{F_1} \to V$ in $\mathrm{Ban}(\mathscr{A})$. Assume that $n \neq 1$. Put $F := c^{n-1} F_n + \sum_{i=1}^{n-1} (1-c) c^{i-1} F_i \in \mathrm{MS}(\mathscr{A}; \mathbb{F}_{k,c})$. By Lemma 2.9, we have $\wp_F = \bigcap_{i=1}^n \wp_{F_i}$. By Proposition 1.13 (i) and Corollary 1.8, $(\iota_i)_{i=1}^n$ induces an isomorphism $V_F \to V$ in $\mathrm{Ban}(\mathscr{A})$. $\square$

In the following in this subsection, suppose $\mathrm{ch}(k) = 0$ so that the Archimedean convexity $\mathbb{F}_{k,\infty}$ in k makes sense. For an $F \in \mathrm{MS}(\mathscr{A}; k)$, we put $\Sigma_F := \{I \in \mathrm{Bl}_k(\mathscr{A}_\circ) \mid I_F \subset I\}$, and call it *the support of* F. Then Σ_F is a finite subset by Corollary 1.9 and Lemma 2.9. We show the unique existence of an isotypic decomposition of an $\mathbb{F}_{k,\infty}$-mixed pre-state on $\mathscr{A}$.

PROPOSITION 2.15. *For any $F \in \mathrm{MS}(\mathscr{A}; \mathbb{F}_{k,\infty})$, there is a unique $(c_I, F_I)_{I \in \Sigma_F} \in \prod_{I \in \Sigma_F} ((\mathbb{F}_{k,\infty} \setminus \{0\}) \times \mathrm{IMS}(\mathscr{A}, I; \mathbb{F}_{k,\infty}))$ with $\sum_{I \in \Sigma_F} c_I = 1$ and $\sum_{I \in \Sigma_F} c_I F_I = F$.*

In order to verify Proposition 2.15, we study the uniqueness of such a decomposition first. For an $I \in \mathrm{Bl}_k(\mathscr{A}_\circ)$, we denote by $k \otimes \mathrm{IMS}(\mathscr{A}, I; k) \subset \mathscr{A}_\circ^{\mathrm{D}}$ the image of the map $k \times \mathrm{IMS}(\mathscr{A}, I; k)) \to \mathscr{A}_\circ^{\mathrm{D}}$, $(c, F) \mapsto cF$.

LEMMA 2.16. *For any finite subset $\Sigma \subset \mathrm{Bl}_k(\mathscr{A}_\circ)$, the map $\prod_{I \in \Sigma} (k \otimes \mathrm{IMS}(\mathscr{A}, I; k)) \to \mathscr{A}_\circ^{\mathrm{D}}$, $(G_I)_{I \in \Sigma} \mapsto \sum_{I \in \Sigma} G_I$ is injective.*

Proof. Let $(G_{i,I})_{I \in \Sigma})_{i=1}^{2} \in (\prod_{I \in \Sigma}(k \otimes \mathrm{IMS}(\mathscr{A}, I; k)))^2$ with $\sum_{I \in \Sigma} G_{1,I} = \sum_{I \in \Sigma} G_{2,I}$. We show $(G_{1,I})_{I \in \Sigma} = (G_{2,I})_{I \in \Sigma}$. Put $G := \sum_{I \in \Sigma} G_{1,I}$. Let $I \in \Sigma$. There is an $e_I \in \bigcap_{I' \in \Sigma \setminus \{I\}} I'$ with $1 - e_I \in I$ by Proposition 1.7. We obtain $G_{1,I}(f) = G_{1,I}(fe_I) = F(fe_I) = G_{2,I}(fe_I) = G_{2,I}(f)$ for any $f \in \mathscr{A}_\circ$. □

Proof of Proposition 2.15. By Lemma 2.16, it suffices to verify the existence of the isotypic decomposition. Take a presentation $F = \sum_{j=1}^{m} c'_j F'_j$ with $m \in \mathbb{N} \setminus \{0\}$, $(c'_j)_{j=1}^{m} \in \mathbb{F}_{k,\infty}^{m}$, $(F'_j)_{j=1}^{m} \in \mathrm{PS}(\mathscr{A})^m$, and $\sum_{j=1}^{m} c'_j = 1$. Replacing $(c'_j)_{j=1}^{m}$ by a subsequence, we may assume $(c'_j)_{j=1}^{m} \in (\mathbb{F}_{k,\infty} \setminus \{0\})^m$. Put $\Sigma := \{I_{F'_j} \mid j \in \mathbb{N} \cap [1,m]\} \subset \mathrm{Bl}_k(\mathscr{A}_\circ)$, and $S_I := \{j \in \mathbb{N} \cap [1,m] \mid I_{F'_j} = I\}$ for an $I \in \Sigma$. Then $c_I := \sum_{j \in S_I} c'_j$ is a non-empty sum of elements in the image of $\mathbb{Q} \cap (0,1]$ for any $I \in \Sigma$, and we have $\sum_{I \in \Sigma} c_I = \sum_{j=1}^{m} c'_j = 1$. It implies $(c_I)_{I \in \Sigma} \in (\mathbb{F}_{k,\infty} \setminus \{0\})^{\Sigma}$. Putting $F_I := \sum_{j \in S_I} c_I^{-1} c'_j F'_j \in \mathrm{IMS}(\mathscr{A}, I; \mathbb{F}_{k,\infty})$ for an $I \in \Sigma$, we obtain $\sum_{I \in \Sigma} c_I F_I = \sum_{I \in \Sigma} \sum_{j \in S_I} c'_j F'_j = \sum_{j=1}^{m} c'_j F'_j = F$. By Lemma 2.9, we have $I_F = \bigcap_{I \in \Sigma} I_{F_I} = \bigcap_{I \in \Sigma} I$. It implies $\Sigma \subset \Sigma_F$. Let $I \in \Sigma_F$. By $\bigcap_{I' \in \Sigma} I' \subset I$, we have $I \in \Sigma$ by Corollary 1.9. We obtain $\Sigma = \Sigma_F$. □

We denote by $(c_{F,I}, F[I])_{I \in \Sigma_F} \in \prod_{I \in \Sigma_F}((\mathbb{F}_{k,\infty} \setminus \{0\}) \times \mathrm{IMS}(\mathscr{A}, I; \mathbb{F}_{k,\infty}))$ the unique element with $\sum_{I \in \Sigma} c_{F,I} = 1$ and $\sum_{I \in \Sigma_F} c_{F,I} F[I]$ for an $F \in \mathrm{MS}(\mathscr{A}; \mathbb{F}_{k,\infty})$. For a Banach left $\mathscr{A}$-module V, we put $\Sigma_V := \{I \in \mathrm{Bl}_k(\mathscr{A}_\circ) \mid \mathrm{Ann}_{\mathscr{A}_\circ}(V) \subset I\}$, and call it *the support of* V. We obtain the compatibility of the isotypic decompositions and the supports of a cyclic semisimple Banach left $\mathscr{A}$-module and an $\mathbb{F}_{k,\infty}$-mixed pre-state on $\mathscr{A}$.

THEOREM 2.17. *Let V be a Banach left $\mathscr{A}$-module with $V \neq \{0\}$. Under the hypothesis (VII), V is a finite dimensional cyclic semisimple Banach left $\mathscr{A}$-module if and only if there is an $F \in \mathrm{MS}(\mathscr{A}; \mathbb{F}_{k,\infty})$ with $\Sigma_F = \Sigma_V$ such that V_F is isomorphic to V in* $\mathrm{Ban}(\mathscr{A})$ *and $V_{F[I]}$ is isomorphic to the I-isotypic component $\{v \in V \mid Iv = \{0\}\} \subset V$ in* $\mathrm{Ban}(\mathscr{A})$ *for any $I \in \Sigma_V$.*

Proof. The direct implication follows from Theorem 2.8. Suppose that V is cyclic and semisimple. By Lemma 2.13, there is a pair (F, ι) of an $F \in \mathrm{MS}(\mathscr{A}; \mathbb{F}_{k,2^{-1}})$ and an isomorphism $\iota \colon V_F \to V$ in $\mathrm{Ban}(\mathscr{A})$. By Proposition 2.15 and $\mathrm{MS}(\mathscr{A}; \mathbb{F}_{k,2^{-1}}) \subset \mathrm{MS}(\mathscr{A}; \mathbb{F}_{k,\infty})$, $(c_{F,I}, F[I])_{I \in \Sigma_F} \in \prod_{I \in \Sigma_F}((\mathbb{F}_{k,\infty} \setminus \{0\}) \times \mathrm{IMS}(\mathscr{A}, I; \mathbb{F}_{k,\infty}))$ makes sense. For any $I \in \Sigma_F$, we have $I_F \subset I = I_{F[I]}$, and the canonical projection $V_F \twoheadrightarrow V_{F[I]}$ gives an isomorphism in $\mathrm{Ban}(\mathscr{A})$ between the I-isotypic component $\{v \in V_F \mid Iv = \{0\}\}$ and $V_{F[I]}$ by Lemma 2.9. We have $\Sigma_V = \Sigma_{V_F} = \Sigma_F$ by $\mathrm{Ann}_{\mathscr{A}_\circ}(V_F) = I_F$. □

3. Non-Archimedean Krein–Milman theorems

We study the relation between states on $\mathscr{A}$ and several convexities. It is pity that we could not expect that the notion of a pure state on $\mathscr{A}$ is described in terms of an extreme face. On the other hand, we have several results on faces of $\mathrm{MS}(\mathscr{A})$ and $\mathrm{S}(\mathscr{A})$.

3.1. Archimedean convexity on states

In this subsection, we only consider the case $\mathrm{ch}(k) = 0$ so that we can deal with the Archimedean convexity $\mathbb{F}_{k,\infty}$ in k. We note that $\mathbb{F}_{k,\infty}$ is not totally bounded, and hence a Hausdorff topological k-vector space admits no non-trivial compact $\mathbb{F}_{k,\infty}$-convex subset. Therefore we do not have a result on the existence of an extreme $\mathbb{F}_{k,\infty}$-face as in [9] Theorem 3. Instead, we given an explicit example of an $\mathbb{F}_{k,\infty}$-face of $\mathrm{MS}(\mathscr{A};\mathbb{F}_{k,\infty})$.

THEOREM 3.1. *Suppose that k is complete. Then $\mathrm{IMS}(\mathscr{A},I;\mathbb{F}_{k,\infty})$ forms an $\mathbb{F}_{k,\infty}$-face of $\mathrm{MS}(\mathscr{A};\mathbb{F}_{k,\infty})$ for any $I \in \mathrm{Bl}_k(A)$.*

Proof. Put $S := \mathrm{IMS}(\mathscr{A},I;\mathbb{F}_{k,\infty})$. Let $(F_i,c_i)_{i=1}^n \in ((\mathrm{MS}(\mathscr{A};\mathbb{F}_{k,\infty}) \setminus S) \times \mathbb{F}_{k,\infty})^n$ with $n \in \mathbb{N} \setminus \{0\}$ and $\sum_{i=1}^n c_i = 1$. Put $F := \sum_{i=1}^n c_i F_i \in \mathrm{MS}(\mathscr{A};\mathbb{F}_{k,\infty})$. We show $F \in \mathrm{MS}(\mathscr{A};\mathbb{F}_{k,\infty}) \setminus S$. For each $i \in \mathbb{N} \cap [1,n]$, there is a $(c_{i,I'},F_{i,I'})_{I' \in \Sigma_{F_i}} \in \prod_{I \in \Sigma_{F_i}} ((\mathbb{F}_{k,\infty} \setminus \{0\}) \times \mathrm{IMS}(\mathscr{A},I';\mathbb{F}_{k,\infty}))$ with $\sum_{I' \in \Sigma_{F_i}} c_{i,I'} = 1$, and $\sum_{I' \in \Sigma_{F_i}} c_{i,I'} F_{i,I'} = F_i$ by Proposition 2.15. Put $\Sigma := \bigcup_{i=1}^n \Sigma_{F_i}$, $c_{I'} := \sum_{i=1}^n \sum_{I'' \in \Sigma_{F_i} \cap \{I'\}} c_i c_{i,I''} \in \mathbb{F}_{k,\infty} \setminus \{0\}$ and $F_{I'} := \sum_{i=1}^n \sum_{I'' \in \Sigma_{F_i} \cap \{I'\}} c_{I'}^{-1} c_i c_{i,I''} F_{i,I''} \in \mathrm{IMS}(\mathscr{A},I';\mathbb{F}_{k,\infty})$ for an $I' \in \Sigma$. We have $\sum_{I' \in \Sigma} c_{I'} = \sum_{I' \in \Sigma} \sum_{i=1}^n \sum_{I'' \in \Sigma_{F_i} \cap \{I'\}} c_i c_{i,I''} = \sum_{i=1}^n \sum_{I' \in \Sigma_{F_i}} c_i c_{i,I'} = \sum_{i=1}^n c_i = 1$ and $\sum_{I' \in \Sigma} c_{I'} F_{I'} = \sum_{I' \in \Sigma} \sum_{i=1}^n \sum_{I'' \in \Sigma_{F_i} \cap \{I'\}} c_i c_{i,I''} F_{i,I''} = \sum_{i=1}^n \sum_{I' \in \Sigma_{F_i}} c_i c_{i,I''} F_{i,I''} = \sum_{i=1}^n c_i F_i = F$. By $(F_i)_{i=1}^n \in (\mathrm{MS}(\mathscr{A};\mathbb{F}_{k,\infty}) \setminus S)^n$, there is an $I' \in \Sigma$ with $I' \neq I$. Since $c_{I'}$ lies in the image of $\mathbb{Q} \cap (0,1]$, we obtain $F \in \mathrm{IMS}(\mathscr{A},\mathbb{F}_{k,\infty}) \setminus S$ by Proposition 2.15 and Lemma 2.16. It implies that S is an $\mathbb{F}_{k,\infty}$-face of $\mathrm{MS}(\mathscr{A},\mathbb{F}_{k,\infty})$. $\square$

Theorem 3.1 implies that an $\mathbb{F}_{k,\infty}$-linear combination of non-isotypic pre-states is never an isotypic pre-state. This is a phenomenon specific to the Archimedean convexity $\mathbb{F}_{k,\infty}$ in k. Indeed, there are obvious examples of O_k-linear combinations of non-isotypic states which are isotypic states.

3.2. Non-Archimedean convexity on states

We verify a variant of Krein–Milman theorem for the non-Archimedean convexity O_k in k. For this purpose, we study extreme faces. By an obvious imitation of the proof of [9] Theorem 2, we obtain the existence of an extreme face.

PROPOSITION 3.2. *Let R be a topological ring, R_0 a convexity in R, R_{00} a subconvexity of R_0, M a topological left R-module, and $S \subset M$ a non-empty compact R_0-convex subset. If there is a $c \in R^\times$ with $(c,-c,c^{-1}) \in R_0^3$, then every closed R_{00}-face S' with $S' \subsetneq S$ contains an extreme R_{00}-face.*

Proof. We denote by Σ the set of closed R_{00}-faces of S contained in S', which is non-empty by $S' \in \Sigma$ and which is directed by anti-inclusions. For any non-empty totally ordered subset $\Sigma_0 \subset \Sigma$, $\bigcap_{S'' \in \Sigma_0} S''$ is non-empty by the compactness of S, and forms a closed R_{00}-face of S contained in S'. By Zorn's lemma, Σ admits a minimal element S_0. We show $S_0 \in \mathrm{Ext}(S;R_0,R_{00})$. Let S'' be an R_{00}-face of S with $S'' \subset S_0$. Take an $F_0 \in S \setminus S'$. Since S' is closed, there is an open neighbourhood $U_0 \subset S$ of F_0

contained in $S \setminus S'$. Let $F \in S \setminus S''$. Put $U := \{F' \in S \mid cF' - cF + F_0 \in U_0\}$. By the continuity of the R-module structure of M, U is an open neighbourhood of $F \in S$. For any $F' \in U$, we have $F' = c^{-1}(cF' - cF + F_0) + F - c^{-1}F_0 \in S \setminus S' \subset S \setminus S_0 \subset S \setminus S''$. It ensures that S'' is a closed R_{00}-face of S. By the minimality of S_0 in Σ, we obtain $S'' = S_0$. It implies $S_0 \in \mathrm{Ext}(S; R_0, R_{00})$. $\square$

Applying Proposition 3.2 to the case $(R, R_0, R_{00}) = (O_k, O_k, 1 + \wp)$ for a suitable ideal $\wp \subsetneq O_k$, we obtain a non-Archimedean variant of Krein–Milman theorem. We say that a topological space is *non-trivial* if it admits more than two open subsets.

THEOREM 3.3. (non-Archimedean Krein–Milman theorem) *Let $\wp \subset O_k$ be an ideal with $\wp \subset \varpi O_k$ for a $\varpi \in m_k$, M a linear topological O_k-module, and $S \subset M$ a non-trivial compact closed O_k-convex subset. Then the equality $[S; O_k, 1 + \wp] = S$ holds.*

Before proving Theorem 3.3, we note that through the equivalence in Example 1.3, Theorem 3.3 gives a wide generalisation of [9] Theorem 3 for the M-convexity, in which it was assumed that k is locally compact, $\wp = m_k$, M is Hausdorff, and M admits a structure of a topological k-vector space. As is written in [9], there is no non-trivial O_k-convex subset in a topological k-vector space unless k is locally compact, but there are many non-trivial O_k-convex subsets in several topological O_k-modules as long as $\# O_k / m_k < \infty$. It is remarkable that the fundamental technique in the proof of [9] Theorem 3 using the result on closed hyperplanes in [5] Proposition 69 is not applicable to Theorem 3.3 because we are free from many assumptions. In order to verify Theorem 3.3, we prepare the following:

LEMMA 3.4. *Let $\wp \subset O_k$ be an ideal with $\wp \subset \varpi O_k$ for a $\varpi \in m_k$, M a linear topological O_k-module, $S \subset M$ a compact O_k-convex subset, and $S' \subsetneq S$ a non-empty closed O_k-convex subset. Then S admits a closed $(1 + \wp)$-face S'' with $S' \cap S'' = \emptyset$.*

Proof. If $\wp = \{0\}$, then the assertion follows from Example 1.4, because $(m - m') + S'$ is a closed subset of S contained in $S \setminus S'$ for any $(m, m') \in (S \setminus S') \times S'$. Assume $\wp \neq \{0\}$. Let $F_0 \in S \setminus S'$. Since M is linear and S' is closed in S, there is an open O_k-submodule $L \subset M$ with $(F_0 + L) \cap S \subset S \setminus S'$. Take an $F'_0 \in S'$ and a $\varpi \in m_k \setminus \{0\}$ with $\wp \subset \varpi O_k$. For an $n \in \mathbb{N}$, put $L_n := \{F \in M \mid F'_0 + \varpi^n F \in S' + L\}$. In particular, we have $F'_0 + L_0 = S' + L$. By the continuity of the O_k-module structure of M, L_n is an open O_k-submodule of M for any $n \in \mathbb{N}$, and $\bigcup_{n \in \mathbb{N}} (F'_0 + L_n)$ coincides with M. By the compactness of S, there is an $n \in \mathbb{N}$ with $S \subset F'_0 + L_n$. We denote by n_0 the minimum of such an $n \in \mathbb{N}$. By $(F'_0 + L_0) \cap S = (S' + L) \cap S \subset S \setminus (F_0 + L) \subsetneq S$, we have $n_0 \neq 0$. Put $S'' := S \setminus (F'_0 + L_{n_0 - 1})$. Then S'' is a non-empty closed subset of S. Let $(c, (F_i)_{i=1}^2) \in (1 + \wp) \times (S'')^2$. Put $F := (1 - c)F_1 + cF_2 \in S$. We have $\varpi^{n_0 - 1}(1 - c)(F_1 - F'_0) \in \varpi^{n_0}(L_{n_0} \setminus L_{n_0 - 1}) \subset L_0$ and $\varpi^{n_0 - 1} c(F_2 - F'_0) \in \varpi^{n_0 - 1}(L_{n_0} \setminus L_{n_0 - 1}) \subset L_1 \setminus L_0$. It implies $\varpi^{n_0 - 1}(F - F'_0) + F'_0 = \varpi^{n_0 - 1}(1 - c)(F_1 - F'_0) + \varpi^{n_0 - 1} c(F_2 - F'_0) \in L_1 \setminus L_0$ and hence $F \in S''$. Therefore S'' is $(1 + \wp)$-semiconvex. $\square$

Proof of Theorem 3.3. If $S = \emptyset$, then the assertion is obvious. Assume $S \neq \emptyset$. To begin with, we show $[S; O_k, 1 + \wp] \neq \emptyset$. Since S is non-trivial, there is an $F_0 \in S$ such that the closure S' of $\{F_0\}$ in S does not coincide with S. Then S' forms a non-empty closed O_k-convex subset of S with $S' \subsetneq S$, and hence S admits a closed $(1+\wp)$-face S'' with $S' \cap S'' = \emptyset$ by Lemma 3.4. Therefore S admits an extreme $(1+\wp)$-face by Proposition 3.2. It implies $[S;O_k,1+\wp] \neq \emptyset$.

We show $[S;O_k,1+\wp] = S$. Since S is closed, we have $[S;O_k,1+\wp] \subset S$. Assume $[S;O_k,1+\wp] \subsetneq S$. Since $[S;O_k,1+\wp]$ is a non-empty closed O_k-convex subset of S, S admits a closed $(1+\wp)$-face S' with $[S;O_k,1+\wp] \cap S' = \emptyset$ by Lemma 3.4. Therefore S admits an extreme $(1+\wp)$-face S'' with $[S;O_k,1+\wp] \cap S'' = \emptyset$ by Proposition 3.2. It contradicts $S'' \subset \mathrm{co}(\bigcup_{S''' \in \mathrm{Ext}(S;O_k,1+\wp)} S''';O_k) \subset [S;O_k,1+\wp]$. We conclude $[S;O_k,1+\wp] = S$. □

We apply Theorem 3.3 to $\mathrm{S}(\mathscr{A})_{\leqslant 1}$. Here we equip $\mathscr{A}_\circ^{\mathrm{D}}$ with the topology of pointwise convergence, and equip its subset with the relative topology.

THEOREM 3.5. *Under the hypotheses (III) and (VI), the equality* $[\mathrm{S}(\mathscr{A})_{\leqslant 1}, O_k, 1+m_k] = \mathrm{S}(\mathscr{A})_{\leqslant 1}$ *holds.*

Proof. If $\#\mathrm{S}(\mathscr{A})_{\leqslant 1} \leqslant 1$, then the assertion holds. Assume $\#\mathrm{S}(\mathscr{A})_{\leqslant 1} \geqslant 2$. The topological O_k-module $(\mathscr{A}_\circ^{\mathrm{D}})_{\leqslant 1}$ is compact, Hausdorff, and linear, because the evaluation map $(\mathscr{A}_\circ^{\mathrm{D}})_{\leqslant 1} \to O_k^{\mathscr{A}_{\leqslant 1}}$, $F \mapsto (F(f))_{f \in \mathscr{A}_{\leqslant 1}}$ is a homeomorphism onto the closed image. It implies that $\mathrm{S}(\mathscr{A})_{\leqslant 1}$ is a Hausdorff topological space with $\#\mathrm{S}(\mathscr{A})_{\leqslant 1} \geqslant 2$, and hence $\mathrm{S}(\mathscr{A})_{\leqslant 1}$ is non-trivial. We have $\mathrm{S}(\mathscr{A})_{\leqslant 1} = \{F \in (\mathscr{A}_\circ^{\mathrm{D}})_{\leqslant 1} \mid F(1) = 1\}$ by Example 2.3. Since k is Hausdorff, $\{F \in (\mathscr{A}_\circ^{\mathrm{D}})_{\leqslant 1} \mid F(1) = 1\}$ is closed in $(\mathscr{A}_\circ^{\mathrm{D}})_{\leqslant 1}$, and hence is compact. Therefore we obtain $[\mathrm{S}(\mathscr{A})_{\leqslant 1}, O_k, 1+m_k] = \mathrm{S}(\mathscr{A})_{\leqslant 1}$ by Theorem 3.3. □

3.3. States on groups

In this subsection, suppose that k is complete so that a finite dimensional Banach k-vector space is not necessarily trivial. We apply the non-Archimedean GNS construction to Banach topological k-algebras associated to groups. Let G be a topological group. We recalled the notion of a unitary k-linear representation of G in Example 1.26. Let (V,ρ) be a unitary k-linear representation of G. We say that (V,ρ) is *irreducible* if (V,ρ) admits exactly two closed G-stable k-vector subspaces, is *semisimple* if the underlying k-vector subspace of V is the direct sum of a family of the underlying k-vector spaces of closed G-stable k-vector subspaces which are irreducible unitary k-linear representations of G with respect to the restriction of ρ, and is *finite dimensional* if V is finite dimensional.

First, suppose that G is a discrete group. We introduced the Banach topological k-algebra $\mathrm{C}_0(G,k)_{\mathrm{disc}}$ in Example 1.26. We consider another Banach topological k-algebra associated to G. We denote by $\tilde{\rho} \colon k[G] \to \mathscr{B}(V)$ the k-algebra homomorphism given by setting $\tilde{\rho}([g])v := \rho(g,v)$ for a $(g,v) \in G \times V$. Let $f = \sum_{g \in G} f(g)[g] \in k[G]$. We put $\|f\|_\rho := \|\tilde{\rho}(f)\|$. Then we have $\|f\|_\rho \leqslant \max_{g \in G} |f(g)|$ by definition. We note that the right hand side is the norm of the image of f in $\mathrm{C}_0(G,k)$ through the canonical embedding $k[G] \hookrightarrow \mathrm{C}_0(G,k)$.

We denote by $\|f\|_{\mathrm{ms}} \in [0, \max_{g \in G} |f(g)|]$ the infimum of a $C \in [0,\infty)$ with $\|f\|_\rho \leqslant C$ for any finite dimensional irreducible unitary k-linear representation (V,ρ) of G. Then the map $\| - \|_{\mathrm{ms}} \colon k[G] \to [0,\infty)$, $f \mapsto \|f\|_{\mathrm{ms}}$ forms a non-Archimedean seminorm on $k[G]$. We denote by $\mathrm{C}^*(G,k)$ the completion of $(k[G], \| - \|_{\mathrm{ms}})$, and by $\delta_G \colon G \to \mathrm{C}^*(G,k)$ the composite of the natural embedding $G \hookrightarrow k[G]$ and the canonical k-algebra homomorphism $k[G] \to \mathrm{C}^*(G,k)$. Then $\mathrm{C}^*(G,k)$ forms a Banach k-algebra, and admits a unique submetric G-equivariant k-algebra homomorphism $\mathrm{C}_0(G,k) \to \mathrm{C}^*(G,k)$ with dense image. When G is a finite discrete group, then the underlying k-algebra of $\mathrm{C}^*(G,k)$ is isomorphic to the semisimplification of $k[G]$ by Proposition 1.29. We obtain a Banach topological k-algebra $\mathscr{C}^*(G,k) := \mathrm{C}^*(G,k)_{\mathrm{disc}}$.

Next, suppose that the hypothesis (III) holds and G is a profinite group. We introduced the Banach topological k-algebra $O_k[[G]]_{\mathrm{comp}}$ in Example 1.27. We consider another Banach topological k-algebra associated to G. Let (V,ρ) be a unitary k-linear representation of G. We denote by $\tilde{\rho} \colon k \otimes_{O_k} O_k[[G]] \to \mathscr{B}(V)$ the k-algebra homomorphism given by setting $\tilde{\rho}(c \otimes f)v := cfv$ for a $(c,f,v) \in k \times O_k[[G]] \times V$. Let $f \in k \otimes_{O_k} O_k[[G]]$. We put $\|f\|_\rho := \|\tilde{\rho}(f)\|$. Then we have $\|f\|_\rho \leqslant \inf\{|c| \mid c \in k, {}^\exists f' \in O_k[[G]],\ c \otimes f' = f\}$ by definition. We note that the right hand side is the norm of the image of f in $(O_k[[G]]_{\mathrm{comp}})_\circ$ through the identity map $k \otimes_{O_k} O_k[[G]] \to (O_k[[G]]_{\mathrm{comp}})_\circ$.

We denote by $\|f\|_{\mathrm{ms}} \in [0, \inf\{|c| \mid c \in k, {}^\exists f' \in O_k[[G]],\ c \otimes f' = f\}]$ the infimum of a $C \in [0,\infty)$ with $\|f\|_\rho \leqslant C$ for any finite dimensional irreducible unitary k-linear representation (V,ρ) of G. Then the map $\| - \|_{\mathrm{ms}} \colon k \otimes_{O_k} O_k[[G]] \to [0,\infty)$, $f \mapsto \|f\|_{\mathrm{ms}}$ forms a non-Archimedean seminorm on $k[G]$. We denote by $\mathrm{C}^*(G,k)$ the completion of $(k \otimes_{O_k} O_k[[G]], \| - \|_{\mathrm{ms}})$, and by $\delta_G \colon G \to \mathrm{C}^*(G,k)$ the composite of the natural embedding $G \hookrightarrow k \otimes_{O_k} O_k[G]$ and the canonical k-algebra homomorphism $k \otimes_{O_k} O_k[[G]] \to \mathrm{C}^*(G,k)$. Then $\mathrm{C}^*(G,k)$ forms a Banach k-algebra, and admits a unique submetric G-equivariant k-algebra homomorphism $(O_k[[G]]_{\mathrm{comp}})_\circ \to \mathrm{C}^*(G,k)$ with dense image. When G is a finite discrete group, then the convention of $\mathrm{C}^*(G,k)$ is compatible with the previous one by definition.

We equip $\mathrm{C}^*(G,k)_{\leqslant 1}$ with the strongest topology τ_G which is weaker than or equal to the relative topology of the norm topology on $\mathrm{C}^*(G,k)$, for which $\mathrm{C}^*(G,k)_{\leqslant 1}$ forms a topological O_k-algebra, and for which the O_k-algebra homomorphism $O_k[[G]] \to \mathrm{C}^*(G,k)_{\leqslant 1}$ given as the composite of the embedding $O_k[[G]] \hookrightarrow k \otimes_{O_k} O_k[[G]]$, $f \mapsto 1 \otimes f$ and the canonical k-algebra homomorphism $k \otimes_{O_k} O_k[[G]] \to \mathrm{C}^*(G,k)$ is continuous. The existence of τ_G can be verified in a completely similar way to the proof of Proposition 1.21. We obtain a Banach topological k-algebra $\mathscr{C}^*(G,k) := (\mathrm{C}^*(G,k), \tau_G)$. When G is a finite discrete group, then the convention of $\mathscr{C}^*(G,k)$ is compatible with the previous one by definition.

Suppose that G is a discrete group (resp. that the hypothesis (III) holds and G is a profinite group). A unitary k-linear representation (V,ρ) of G is said to be *GNS* if $\tilde{\rho}$ admits a unique bounded extension $\tilde{\rho}^* \colon \mathrm{C}^*(G,k) \to \mathscr{B}(V)$ and V admits a pair $(v,w) \in V \times V^{\mathrm{D}}$ for which (V,v,w) forms a pre-GNS triad over $\mathscr{C}^*(G,k)$ through $\tilde{\rho}^*$. For example, every finite dimensional irreducible unitary k-linear representation (V,ρ) of G is GNS by the definition and Theorem 2.6 (ii), and hence every finite dimensional cyclic semisimple unitary k-linear representation of G is GNS by Proposition 1.13 (i) and Corollary 2.10.

We have $\#(k[G]) \leqslant \#(\bigsqcup_{n\in\mathbb{N}}(k^n \times G^n)) = \max\{\aleph_0, \#k, \#G\}$, and hence $\#\mathrm{C}^*(G,k) \leqslant \#(k[G]^{\mathbb{N}}) \leqslant (\max\{\aleph_0, \#k, \#G\})^{\aleph_0} = \max\{2^{\aleph_0}, \#k, \#G\}$. Therefore the underlying k-vector space of a GNS unitary k-linear representation of G is of dimension $\leqslant \max\{2^{\aleph_0}, \#k, \#G\}$. We denote by $\mathrm{Rep}(G,k)$ the set of GNS unitary k-linear representations of G whose underlying k-vector space is a k-vector subspace of $k^{\oplus \max\{2^{\aleph_0}, \#k, \#G\}}$. Using $\mathrm{Rep}(G,k)$, we show the hypotheses (IV) and (V) for $\mathscr{C}^*(G,k)$.

PROPOSITION 3.6. *The Banach topological k-algebra $\mathscr{C}^*(G,k)$ is unramified and the topological O_k-algebra $\mathscr{C}^*(G,k)_{\leqslant 1}$ is Hausdorff.*

Proof. When G is a discrete group, then the assertion is obvious. Assume that the hypothesis (III) holds and G is a profinite group. For any unramified Banach k-vector space V, $\mathscr{B}(V)$ is an unramified Banach k-algebra. Therefore $\mathscr{C}^*(G,k)$ is unramified. We show that $\mathscr{C}^*(G,k)_{\leqslant 1}$ is Hausdorff. Let $(V,\rho) \in \mathrm{Rep}(G,k)$. The map $O_k[[G]] \to V$, $f \mapsto fv$ is continuous for any $v \in V$. It implies that the map $O_k[[G]] \to \mathscr{B}(V)$ given as the composite of the embedding $O_k[[G]] \hookrightarrow k \otimes_{O_k} O_k[[G]]$, $f \mapsto 1 \otimes f$ and $\tilde{\rho} \colon k \otimes_{O_k} O_k[[G]] \to \mathscr{B}(V)$ is continuous, where $\mathscr{B}(V)$ is equipped with the topology of pointwise convergence. Therefore the diagonal map $O_k[[G]] \to \prod_{(V,\rho)\in\mathrm{Rep}(G,k)} \mathscr{B}(V)$ is continuous. In particular, the topology τ on $\mathrm{C}^*(G,k)_{\leqslant 1}$ given as the pull-back of the topology of $\prod_{(V,\rho)\in\mathrm{Rep}(G,k)} \mathscr{B}(V)$ through the restriction of $\prod_{(V,\rho)\in\mathrm{Rep}(G,k)} \tilde{\rho}^*$ satisfies that the O_k-algebra homomorphism $O_k[[G]] \to (\mathrm{C}^*(G,k)_{\leqslant 1}, \tau)$ is continuous. Moreover, $\prod_{(V,\rho)\in\mathrm{Rep}(G,k)} \tilde{\rho}^*$ is injective, because every finite dimensional irreducible k-linear representation of G admits a homeomorphic G-equivariant k-linear isomorphism to some $(V,\rho) \in \mathrm{Rep}(G,k)$. Therefore τ is Hausdorff. Since $\mathscr{B}(V)_{\leqslant 1}$ is a topological O_k-algebra with respect to the topology of pointwise convergence by the equicontinuity of the natural action $\mathscr{B}(V)_{\leqslant 1} \times V \to V$ for any $(V,\rho) \in \mathrm{Rep}(G,k)$, $(\mathrm{C}^*(G,k)_{\leqslant 1}, \tau)$ forms a topological O_k-algebra. Since the topology on $\mathscr{B}(V)_{\leqslant 1}$ of the pointwise convergence is weaker than or equal to the relative topology of the norm topology for any $(V,\rho) \in \mathrm{Rep}(G,k)$, τ is weaker than or equal to the relative topology of the norm topology. As a consequence, $\mathscr{C}^*(G,k)_{\leqslant 1}$ is Hausdorff by the universality of τ_G. □

We abbreviate $\mathrm{S}(\mathscr{C}^*(G,k))$ (resp. $\mathrm{PS}(\mathscr{C}^*(G,k))$, $\mathrm{MS}(\mathscr{C}^*(G,k);k)$) to $\mathrm{S}(G,k)$ (resp. $\mathrm{PS}(G,k)$, $\mathrm{MS}(G,k)$). We note that for any $F \in \mathrm{S}(G,k)$, V_F forms a unitary k-linear representation of G with respect to the action $G \times V_F \to V_F$, $(g,v) \mapsto [g]v$. We equip $\mathrm{C}^*(G,k)^{\mathrm{D}}$ with the topology of pointwise convergence, and its subset with the relative topology.

In the following in this subsection, we assume that the hypothesis (III) holds and that G is a discrete group or a profinite group. We show the following non-Archimedean analogue of [12] Theorem 6.8.11:

THEOREM 3.7. *The k-convex subset* $\mathrm{MS}(G,k)$ *is dense in* $\mathrm{S}(G,k)$.

In order to verify Theorem 3.7, it suffices to show the following:

LEMMA 3.8. *Let* $F \in \mathrm{S}(G,k)$. *For any finite dimensional* k*-vector subspace* $E \subset \mathrm{C}^*(G,k)$, *there is an* $F' \in \mathrm{MS}(G,k)$ *with* $F|_E = F'|_E$.

Proof. By $F(1) = 1 \neq 0 = F(0)$, we have $\mathrm{C}^*(G,k) \neq \{0\}$. Replacing E by $E + k1$, we may assume $1 \in E$. Since k is a complete discrete valuation field (resp. the valuation of k is trivial), $E_{\leqslant 1}$ admits an O_k-linear basis S_1 with $1 \in S_1$ and $\|f\| = 1$ for any $f \in S_1$ by $\|1\| = 1$, Proposition 3.6, and [10] IV 3 Corollaire 1 (resp. by the semisimplicity of the underlying ring of k). Take a complete system $S_2 \subset O_k$ of representatives of the canonical projection $O_k \twoheadrightarrow O_k/m_k$ with $1 \in S_2$. Let $c = (c_f)_{f \in S_1} \in S_2^{S_1} \setminus \{(0)_{f \in S_1}\}$. By $\|\sum_{f \in S_1} c_f f\| = \max_{f \in S_1} |c_f| = 1$, there is a finite dimensional irreducible unitary k-linear representation (V_c, ρ_c) of G such that the unique bounded extension $\tilde{\rho}_c^* \colon \mathrm{C}^*(G,k) \to \mathscr{B}(V)$ of $\tilde{\rho}_c$ satisfies $\|\tilde{\rho}_c^*(\sum_{v \in S_1} c_f f)\| = 1$. Put $(V,\rho) := \prod_{c \in S_2^{S_1}} (V_c, \rho_c)$. Then (V,ρ) is GNS, and the unique bounded extension $\tilde{\rho}^* \colon \mathrm{C}^*(G,k) \to \mathscr{B}(V)$ of $\tilde{\rho}$ satisfies $\|\rho(\sum_{f \in S_1} c_f f)\| = 1$ for any $(c_f)_{f \in S_1} \in S_2^{S_1}$. By $\sum_{f \in S_1} kf = E$, we have $\ker(\tilde{\rho}^*) \cap E = \{0\}$. We denote by $A \subset \mathscr{B}(V)$ the image of $\tilde{\rho}^*$, which forms a finite dimensional Banach k-algebra whose underlying k-algebra is semisimple by Proposition 1.29 and Jacobson–Bourbaki density theorem (cf. [6] D 2.2). By Proposition 1.28 and Proposition 3.6, $\mathscr{C}^*(G,k)/\ker(\tilde{\rho}^*)$ is isomorphic to A_{disc} in $\mathscr{A}\mathrm{lg}_{\leqslant 1}(k)$. By Theorem 2.8, we have $\mathrm{MS}(A_{\mathrm{disc}};k) = \mathrm{S}(A_{\mathrm{disc}})$. By Proposition 1.12, Example 2.3, and $(1,1) \in S_1 \times S_2$, there is an $F_0 \in \mathrm{S}(A_{\mathrm{disc}})$ with $F_0 \circ \tilde{\rho}^*|_E = F|_E$. By the surjectivity of $\tilde{\rho}^* \colon \mathrm{C}^*(G,k) \to A$, we have $F_0 \circ \tilde{\rho}^* \in \mathrm{MS}(G,k)$. $\square$

We denote by $G_k^\vee$ the quotient of $\mathrm{Rep}(G,k)$ by the equivalence relation given by homeomorphic G-equivariant k-linear isomorphisms. By Proposition 1.12 and Theorem 2.5 (ii), the correspondence $F \rightsquigarrow V_F$ yields a surjective map $\mathrm{S}(G,k) \twoheadrightarrow G_k^\vee$. We equip $G_k^\vee$ with the quotient topology of $\mathrm{S}(G,k)$. By Theorem 2.8, Proposition 3.6, and Theorem 3.7, we obtain the following:

THEOREM 3.9. *The subset of* $G_k^\vee$ *consisting of equivalence classes of finite dimensional cyclic semisimple unitary* k*-linear representations of* G *is dense.*

In other words, every GNS unitary k-linear representation of G is approximated by finite dimensional cyclic semisimple unitary k-linear representations of G with respect to the topology analogous to the one of convergence of matrix coefficients.

Acknowledgements. I would like to express my gratitude to Shuji Saito and Yoichi Mieda for their opinions and advices. I am grateful to friends and administration staffs in universities for their frequent helps. I am thankful to my family for their deep affection.

I am a research fellow of Japan Society for the Promotion of Science.

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(Received October 13, 2016)

Tomoki Mihara
Tokyo Institute of Technology
2-12-1 Ookayama, Meguro-ku, Tokyo, 152-8550 Japan
e-mail: mihara.t.ab@m.titech.ac.jp