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SHAPIRO'S UNCERTAINTY PRINCIPLE RELATED TO THE WINDOWED FOURIER TRANSFORM ASSOCIATED WITH THE RIEMANN–LIOUVILLE OPERATOR

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Abstract. Quantitative Shapiro's dispersion uncertainty principle and umbrella theorem are proved for the windowed Fourier transform associated with the Riemann-Liouville operator.

1. Introduction

In [1], the authors have defined the Riemann-Liouville operator $\mathscr{R}_\alpha$; $\alpha \geqslant 0$, by

$$\mathscr{R}_\alpha(f)(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1} f(rs\sqrt{1-t^2}, x+rt)(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}dtds; \text{ if } \alpha > 0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1} f(r\sqrt{1-t^2}, x+rt)\dfrac{dt}{\sqrt{1-t^2}}, \text{ if } \alpha = 0; \end{cases} \tag{1.1}$$

where f is any continuous function on $\mathbb{R}^2$, even with respect to the first variable.

A convolution product $*$ and a Fourier transform $\mathscr{F}_\alpha$ connected with the Riemann-Liouville operator $\mathscr{R}_\alpha$ have been defined and many harmonic analysis results have been established [2, 18]. Also, many uncertainty principles related to the Fourier transform $\mathscr{F}_\alpha$ have been proved see for example [16, 17] and the references therein.

In [20], Shapiro has studied the localization for an orthonormal sequence $(\varphi_k)_{k\in\mathbb{N}}$. He showed that if the means and the dispersions of the orthonormal sequence $(\varphi_k)_{k\in\mathbb{N}}$ and their Fourier transforms $(\widehat{\varphi}_k)_{k\in\mathbb{N}}$ are uniformly bounded, then $(\varphi_k)_{k\in\mathbb{N}}$ is finite. In [12], the authors gave a quantitative version of the Shapiro's theorem, that is if $(\varphi_k)_{k\in\mathbb{N}}$ is an orthonormal sequence in $L^2(\mathbb{R})$, then for every $n\in\mathbb{N}$,

$$\sum_{k=0}^{n}\left(||x\varphi_k||_2^2 + ||y\widehat{\varphi}_k||_2^2\right) \geqslant (n+1)^2. \tag{1.2}$$

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Recently, in [15], the author obtains a quantitative multivariables version of Shapiro's theorem for generalized dispersion, in fact the author showed that if $(\widehat{\varphi}_k)_{k\in\mathbb{N}}$ is an orthonormal sequence in $L^2(\mathbb{R}^d)$; then for every positive real number s and for every $n\in\mathbb{N}^*$

$$\sum_{k=1}^{n}\left(||\,|x|^s\varphi_k\,||_2^2+||\,|y|^s\widehat{\varphi}_k\,||_2^2\right)\geqslant C\,n^{1+\frac{s}{d}}, \tag{1.3}$$

where C is a constant which does not depend on s. The author obtains also a strong uncertainty principle of the above theorem by showing that if $(\varphi_k)_{k\in\mathbb{N}}$ is an orthonormal sequence of $L^2(\mathbb{R}^d)$, then for every positive real number s,

$$\sup_{k\in\mathbb{N}}\left(||\,|x|^s\varphi_k\,||_2^2\;+||\,|y|^s\widehat{\varphi}_k\,||_2^2\right)=+\infty. \tag{1.4}$$

Time frequency analysis [11] plays an important role in harmonic analysis, in particular in signal theory. In this context, Dennis Gabor [7] has introduced the Gabor transform, in which he uses translation, convolution and modulation operators of a single Gaussian to represent one dimensional signal. The Fourier transform gives no more than what frequency components exist in the signal, to investigate the time localization of the spectral components, one has needed to introduce a time-frequency representation through many distribution as notably Weyl-Heisenberg transform, short time Fourier transform and windowed Fourier transform. The uncertainty principles play an important role in harmonic analysis. These principles state that a nonzero function f and its Fourier transform $\widehat{f}$ can not be simultaneously and sharply localized at the same time. Many mathematical formulations of this fact have been checked in the last decades [3, 4, 8]. Recently, new uncertainty principles involving these representations have been investigated with different approaches, we refer in particular to the papers of Ghobber, Omri and Lamouchi [9, 10, 13], (see also [12, 15, 20]).

Motivated by their impact in real-life signals, we define in this paper the windowed Fourier transform connected with the Riemann-Liouville operator, which was introduced in [5, 7, 8]. For this, we define the modulation operator for any function g in $L^2(d\nu_\alpha)$ by

$$M_{(\xi_1,\xi_2)}(g)=\widetilde{\mathscr{F}}_\alpha\left(\sqrt{\tau_{(\xi_1,\xi_2)}\left(|\widetilde{\mathscr{F}}_\alpha(g)|^2\right)}\right);\quad\forall(\xi_1,\xi_2)\in[0,+\infty[\times\mathbb{R}, \tag{1.5}$$

where

- $d\nu_\alpha$ the product measure defined on $[0,+\infty[\times\mathbb{R}$ by $d\nu_\alpha(r,x)=\dfrac{r^{2\alpha+1}\,dr\,dx}{2^\alpha\Gamma(\alpha+1)\sqrt{2\pi}}$,

then, $L^p(d\nu_\alpha)$; $p\in[1,+\infty]$, is the Lebesgue space on $[0,+\infty[\times\mathbb{R}$ with respect to the measure $d\nu_\alpha$ and with the L^p-norm denoted by $||.||_{p,\nu_\alpha}$.

- $\widetilde{\mathscr{F}}_\alpha$ is the so-called Fourier-Bessel transform defined on $L^1(d\nu_\alpha)$ by

$$\forall(\mu,\lambda)\in[0,+\infty[\times\mathbb{R};\ \widetilde{\mathscr{F}}_\alpha(f)(\mu,\lambda)=\int_0^\infty\int_{\mathbb{R}}f(r,x)\,j_\alpha(r\mu)\,e^{-i\lambda x}d\nu_\alpha(r,x). \tag{1.6}$$

- j_α [22] is the modified Bessel function defined by

$$\forall z\in\mathbb{C};\ j_\alpha(z)=\Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\left(\frac{z}{2}\right)^{2k}. \tag{1.7}$$

- $\tau_{(\xi_1,\xi_2)}$ is the translation operator associated with the Riemann-Liouville transform that will be defined in the second section.

Then, for any non-zero window function g in $L^2(d\nu_\alpha)$, the windowed Fourier transform associated with the Riemann-Liouville operator (WFTRL) of any signal $f \in L^2(d\nu_\alpha)$ with respect to the window g is given by:

$$\mathscr{V}_g(f)\big((r,x),(\xi_1,\xi_2)\big) = \int_0^\infty \int_{\mathbb{R}} f(s,y)\, \overline{\tau_{(r,x)}\big(M_{(\xi_1,\xi_2)}(g)\big)}\, d\nu_\alpha(s,y). \tag{1.8}$$

We prove for this transform the following Parseval's formula

$$\langle \mathscr{V}_g(f)|\mathscr{V}_g(h)\rangle_{\nu_\alpha\otimes\nu_\alpha} = ||g||^2_{2,\nu_\alpha}\, \langle f|h\rangle_{\nu_\alpha}; \quad f,h \in L^2(d\nu_\alpha), \tag{1.9}$$

where $\nu_\alpha \otimes \nu_\alpha$ is the product measure on $([0,+\infty[\times\mathbb{R})^2$ defined by

$$d(\nu_\alpha\otimes\nu_\alpha)\big((r,x),(s,y)\big) = d\nu_\alpha(r,x)\otimes d\nu_\alpha(s,y),$$

then $L^2(d\nu_\alpha\otimes d\nu_\alpha)$ is the Hilbert space of square integrable functions on $([0,+\infty[\times\mathbb{R})^2$ with respect to the measure $\nu_\alpha\otimes\nu_\alpha$ equipped with the inner product

$$\langle f|g\rangle_{\nu_\alpha\otimes\nu_\alpha} = \int\int_{\big([0,+\infty[\times\mathbb{R}\big)^2} f\big((r,x),(s,y)\big)\, \overline{g\big((r,x),(s,y)\big)}\, d\nu_\alpha(r,x)\, d\nu_\alpha(s,y)$$

and the norm $||f||_{2,\nu_\alpha\otimes\nu_\alpha} = \sqrt{\langle f|f\rangle_{\nu_\alpha\otimes\nu_\alpha}}$.

We conclude a resolution of identity when g is non-zero positive window, that is

$$\begin{aligned}\langle f|h\rangle_{\nu_\alpha} = \Big\langle \frac{1}{||g||^2_{2,\nu_\alpha}} \int\int_{([0,+\infty[\times\mathbb{R})^2} &\mathscr{V}_g(f)((r,x),(\xi_1,\xi_2))\tau_{(r,x)} \\ &\times(M_{(\xi_1,\xi_2)}(g))(.,.)d\nu_\alpha(r,x)d\nu_\alpha(\xi_1,\xi_2)|h\Big\rangle_{\nu_\alpha}.\end{aligned} \tag{1.10}$$

Therefore, the signal f can be recovered from its the WFTRL by

$$\begin{aligned}f(.,.) = \frac{1}{||g||^2_{2,\nu_\alpha}} \int\int_{([0,+\infty[\times\mathbb{R})^2} &\mathscr{V}_g(f)((r,x),(\xi_1,\xi_2))\tau_{(r,x)} \\ &\times(M_{(\xi_1,\xi_2)}(g))(.,.)d\nu_\alpha(r,x)d\nu_\alpha(\xi_1,\xi_2),\end{aligned} \tag{1.11}$$

in a weak sense.

In the second part of this work, based on the paper of Malinnikova [15], we will prove a quantitative Shapiro's dispersion uncertainty principle for the WFTRL. More precisely, we show the following result:

If $(\varphi_{m,n})_{(m,n)\in\mathbb{N}^2}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$, then for every positive real number s and for every nonempty finite subset $\mathscr{K}\subset\mathbb{N}^2$, we have

$$\begin{aligned}\sum_{(m,n)\in\mathscr{K}} (|| \,|(r,x)|^s\mathscr{V}_g(\varphi_{m,n})||^2_{2,\nu_\alpha\otimes\nu_\alpha} + || \,|(\xi_1,\xi_2)|^s\mathscr{V}_g(\varphi_{m,n})||^2_{2,\nu_\alpha\otimes\nu_\alpha}) \\ \geqslant c_{s,\alpha}\, \mathrm{card}(\mathscr{K})^{1+\frac{s}{2\alpha+3}},\end{aligned} \tag{1.12}$$

where the constant $c_{s,\alpha}$ depends only on s and α.

Using the pervious inequality, we obtain the following strong uncertainty principle for the WFTRL:

$$\sup_{(m,n)\in \mathbb{N}^2} \left(\| \, |(r,x)|^s \mathscr{V}_g(\varphi_{m,n}) \|^2_{2,\nu_\alpha\otimes\nu_\alpha} + \| \, |(\xi_1,\xi_2)|^s \mathscr{V}_g(\varphi_{m,n}) \|^2_{2,\nu_\alpha\otimes\nu_\alpha} \right) = +\infty. \tag{1.13}$$

Next, based on an idea of Malinnikova [15], we will show an analogue of Shapiro's Umbrella theorem for the WFTRL, we formulate the following theorem:

Let $\mathscr{K} \subset \mathbb{N}^2$ be a nonempty subset and $(\varphi_{m,n})_{(m,n)\in\mathscr{K}}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$, if there is a function $\psi \in L^2(d\nu_\alpha \otimes d\nu_\alpha)$ such that

$$| \, \mathscr{V}_g(\varphi_{m,n})\big((r,x),(\xi_1,\xi_2)\big) \, | \leqslant \psi\big((r,x),(\xi_1,\xi_2)\big),$$

for every $(m,n) \in \mathscr{K}$ and for almost every $\big((r,x),(\xi_1,\xi_2)\big) \in \big([0,+\infty[\times\mathbb{R}\big)^2$, then $\mathscr{K}$ is finite.

2. Harmonic analysis results related to the Fourier transform associated with the Riemann-Liouville operator

In this section, we recall some harmonic analysis results related to the Fourier transform associated with the Riemann-Liouville operator (see [1, 2]).

Let $\Delta_1 = \dfrac{\partial}{\partial x}$ and Δ_2 be the singular partial differential operator defined by

$$\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2}; \quad (r,x) \in \,]0,+\infty[\times\mathbb{R}, \quad \alpha \geqslant 0.$$

Then, for all $(\lambda_0,\lambda) \in \mathbb{C}^2$; the system

$$\begin{cases} \Delta_1 u(r,x) = -i\lambda u(r,x); \\ \Delta_2 u(r,x) = -\lambda_0^2 u(r,x); \\ u(0,0) = 1, \quad \dfrac{\partial u}{\partial r}(0,x) = 0; \quad \forall x \in \mathbb{R}, \end{cases}$$

admits a unique solution $\varphi_{\lambda_0,\lambda}$ given by

$$\forall (r,x) \in \, [0,+\infty[\times\mathbb{R}; \quad \varphi_{\lambda_0,\lambda}(r,x) = j_\alpha\Big(r\sqrt{\lambda_0^2+\lambda^2}\Big)\, e^{-i\lambda x}, \tag{2.1}$$

where j_α is the modified Bessel function defined by

$$\forall z \in \mathbb{C}; \quad j_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1)\sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\left(\frac{z}{2}\right)^{2k}, \tag{2.2}$$

and J_α is the Bessel function of first kind and index α. The Bessel functions $(J_\alpha)_{\alpha>-\frac{1}{2}}$ have been studied by many authors and from many points of view [6, 14]. In particular, the modified Bessel function j_α has the integral representation

$$j_\alpha(z) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{\alpha-\frac{1}{2}}\exp(-izt)dt. \tag{2.3}$$

Consequently, for every $k \in \mathbb{N}$ and $z \in \mathbb{C}$; we have $\left|j_\alpha^{(k)}(z)\right| \leqslant e^{|Im(z)|}$.

It is known that the eigenfunction $\varphi_{\lambda_0,\lambda}$ is bounded on $\mathbb{R}^2$ if and only if $(\lambda_0,\lambda) \in \Upsilon$, where Υ is the set given by

$$\Upsilon = \mathbb{R}^2 \cup \left\{(i\lambda_0,\lambda);\ (\lambda_0,\lambda) \in \mathbb{R}^2;\ |\lambda_0| \leqslant |\lambda|\right\}, \tag{2.4}$$

and in this case, $\sup\limits_{(r,x)\in\mathbb{R}^2}\left|\varphi_{\lambda_0,\lambda}(r,x)\right| = 1.$

• The function $\varphi_{\lambda_0,\lambda}$ has the following Mehler integral representation

$$\varphi_{\lambda_0,\lambda}(r,x) = \begin{cases} \dfrac{\alpha}{\pi}\displaystyle\int_{-1}^{1}\int_{-1}^{1}\cos\left(\lambda_0 rs\sqrt{1-t^2}\right)e^{-i\lambda(x+rt)}(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\alpha-1}dtds; \text{if } \alpha>0, \\ \dfrac{1}{\pi}\displaystyle\int_{-1}^{1}\cos\left(r\lambda_0\sqrt{1-t^2}\right)e^{-i\lambda(x+rt)}\frac{dt}{\sqrt{1-t^2}}, \text{if } \alpha=0. \end{cases} \tag{2.5}$$

• From the definition of the operator $\mathscr{R}_\alpha$ given in the first section and the relation (2.5), we deduce that $\varphi_{\lambda_0,\lambda}(r,x) = \mathscr{R}_\alpha\big(\cos(\lambda_0\cdot)e^{-i\lambda\cdot}\big)(r,x)$ which gives the mutual connexion between the functions $\varphi_{\lambda_0,\lambda}$ and $\cos(\lambda_0\cdot)e^{-i\lambda\cdot}$. For this reason, the operator $\mathscr{R}_\alpha$ is called the Riemann-Liouville transform associated with the operators Δ_1 and Δ_2.

The eigenfunction $\varphi_{\lambda_0,\lambda}$ satisfies the product formula

$$\varphi_{\lambda_0,\lambda}(r,x)\varphi_{\lambda_0,\lambda}(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^{\pi}\varphi_{\lambda_0,\lambda}\left(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\right)\sin^{2\alpha}\theta d\theta. \tag{2.6}$$

This formula allows us to define the translation operators and the convolution product.

DEFINITION 2.1. (i) For every $(r,x) \in [0,+\infty[\times\mathbb{R}$, the translation operator $\tau_{(r,x)}$ with the Riemann-Liouville operator is defined on $L^p(d\nu_\alpha)$; $p \in [1,+\infty]$ by

$$\tau_{(r,x)}(f)(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^{\pi} f\left(\sqrt{r^2+s^2+2rs\cos\theta}, x+y\right)\sin^{2\alpha}(\theta)d\theta. \tag{2.7}$$

(ii) The convolution product of $f,g \in L^1(d\nu_\alpha)$ is defined for every $(r,x) \in [0,+\infty[\times\mathbb{R}$, by

$$f*g(r,x) = \int_o^{+\infty}\int_{\mathbb{R}}\tau_{(r,-x)}(\check{f})(s,y)g(s,y)d\nu_\alpha(s,y), \tag{2.8}$$

whenever the integral exists, where $\check{f}(s,y) = f(s,-y)$.

Moreover, we have the following properties:

• The translation operator satisfies the following product formula

$$\tau_{(r,x)}(\varphi_{\lambda_0,\lambda})(s,y) = \varphi_{\lambda_0,\lambda}(r,x)\varphi_{\lambda_0,\lambda}(s,y). \tag{2.9}$$

• For all $(r,x) \in [0,+\infty[$; $f \in L^1(d\nu_\alpha)$,

$$\int_0^{+\infty}\int_{\mathbb{R}} \tau_{(r,x)}(f)(s,y)d\nu_\alpha(s,y) = \int_0^{+\infty}\int_{\mathbb{R}} f(s,y)d\nu_\alpha(s,y). \tag{2.10}$$

• For every $(r,x) \in [0,+\infty[\times\mathbb{R}$, $\tau_{(r,x)}$ is a positive bounded operator on $L^p(d\nu_\alpha)$; $p \in [1,+\infty]$, and for ever $f \in L^p(d\nu_\alpha)$,

$$\left|\left|\tau_{(r,x)}(f)\right|\right|_{p,\nu_\alpha} \leqslant ||f||_{p,\nu_\alpha}. \tag{2.11}$$

• If $p,q,r \in [1,+\infty]$ such that $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$, then for every $f \in L^p(d\nu_\alpha)$ and $g \in L^q(d\nu_\alpha)$, the function $f*g$ belongs to the space $L^r(d\nu_\alpha)$ and we have

$$||f*g||_{r,\nu_\alpha} \leqslant ||f||_{p,\nu_\alpha}||g||_{q,\nu_\alpha}. \tag{2.12}$$

Now, using the eigenfunction $\varphi_{\lambda_0,\lambda}$ given by the relation (2.1), we can define the Fourier transform.

DEFINITION 2.2. For every $f \in L^1(d\nu_\alpha)$, the Fourier transform of f is defined by

$$\forall(\lambda_0,\lambda) \in \Upsilon,\ \mathscr{F}_\alpha(f)(\lambda_0,\lambda) = \int_0^{+\infty}\int_{\mathbb{R}} f(r,x)\varphi_{\lambda_0,\lambda}(r,x)d\nu_\alpha(r,x). \tag{2.13}$$

In the following, we give some properties of this transform (see [22, 23]):

• For every $f \in L^1(d\nu_\alpha)$, the function $\mathscr{F}_\alpha(f)$ is bounded on the set Υ and for every $(\lambda_0,\lambda) \in \Upsilon$, $\left|\mathscr{F}_\alpha(f)(\lambda_0,\lambda)\right| \leqslant ||f||_{1,\nu_\alpha}$.

• For every $f \in L^1(d\nu_\alpha)$ and $(r,x) \in [0,+\infty[\times\mathbb{R}$, the function $\tau_{(r,x)}(f)$ belongs to $L^1(d\nu_\alpha)$ and we have

$$\forall(\lambda_0,\lambda) \in \Upsilon,\ \mathscr{F}_\alpha\big(\tau_{(r,x)}(f)\big)(\lambda_0,\lambda) = \overline{\varphi_{\lambda_0,\lambda}(r,x)}\mathscr{F}_\alpha(f)(\lambda_0,\lambda). \tag{2.14}$$

• For all $f,g \in L^1(d\nu_\alpha)$, the function $f*g$ belongs to $L^1(d\nu_\alpha)$ and

$$\forall(\lambda_0,\lambda) \in \Upsilon,\ \mathscr{F}_\alpha(f*g)(\lambda_0,\lambda) = \mathscr{F}_\alpha(f)(\lambda_0,\lambda)\mathscr{F}_\alpha(g)(\lambda_0,\lambda). \tag{2.15}$$

• For every $f \in L^1(d\nu_\alpha)$; $\mathscr{F}_\alpha(f)(\lambda_0,\lambda) = \widetilde{\mathscr{F}_\alpha}(f)\big(\sqrt{\lambda_0^2+\lambda^2},\lambda\big)$, where $\widetilde{\mathscr{F}_\alpha}$ is the mapping defined on $L^1(d\nu_\alpha)$ by the relation (1.6).

• (Inversion formula) For every $f \in L^1(d\nu_\alpha)$, such that $\widetilde{\mathscr{F}_\alpha}(f)$ belongs to $L^1(d\nu_\alpha)$ and for almost every $(r,x) \in [0,+\infty[\times\mathbb{R}$, we have

$$\begin{aligned} f(r,x) &= \int_0^{\infty}\int_{\mathbb{R}} \widetilde{\mathscr{F}_\alpha}(f)(\mu,\lambda)\ j_\alpha(r\mu)\ e^{i\lambda x}\ d\nu_\alpha(\mu,\lambda) \\ &= \widetilde{\mathscr{F}_\alpha}\big(\widetilde{\mathscr{F}_\alpha}(f)\big)(r,-x). \end{aligned} \tag{2.16}$$

• (Plancherel theorem) The transform $\widetilde{\mathscr{F}}_\alpha$ can be extended to an isometric isomorphism from $L^2(d\nu_\alpha)$ onto itself and for every $f \in L^2(d\nu_\alpha)$,

$$\widetilde{\mathscr{F}}_\alpha^{-1}(f) = \widetilde{\mathscr{F}}_\alpha(\check{f}) = \check{\widetilde{\mathscr{F}}_\alpha}(f). \tag{2.17}$$

• For every $f \in L^1(d\nu_\alpha)$; $g \in L^p(d\nu_\alpha)$; $p \in \{1,2\}$, the function $f * g$ belongs to $L^p(d\nu_\alpha)$ and we have

$$\widetilde{\mathscr{F}}_\alpha(f * g) = \widetilde{\mathscr{F}}_\alpha(f)\,\widetilde{\mathscr{F}}_\alpha(g). \tag{2.18}$$

• For all $f,g \in L^2(d\nu_\alpha)$, the function $f * g$ belongs to $\mathscr{C}_{0,e}(\mathbb{R}^2)$ (the space of continuous functions f on $\mathbb{R}^2$, even with respect to the first variable and such that $\lim\limits_{r^2+x^2\to+\infty} f(r,x) = 0$) and we have

$$f * g = \widetilde{\mathscr{F}}_\alpha^{-1}\big(\widetilde{\mathscr{F}}_\alpha(f)\,\widetilde{\mathscr{F}}_\alpha(g)\big) = \widetilde{\mathscr{F}}_\alpha\big(\check{\widetilde{\mathscr{F}}_\alpha(f)}\,\widetilde{\mathscr{F}}_\alpha(g)\big). \tag{2.19}$$

• For all $f,g \in L^2(d\nu_\alpha)$, the function $f * g$ belongs to $L^2(d\nu_\alpha)$ if and only if $\widetilde{\mathscr{F}}_\alpha(f)\,\widetilde{\mathscr{F}}_\alpha(g) \in L^2(d\nu_\alpha)$ and in this case;

$$\widetilde{\mathscr{F}}_\alpha(f * g) = \widetilde{\mathscr{F}}_\alpha(f)\,\widetilde{\mathscr{F}}_\alpha(g). \tag{2.20}$$

3. The windowed Fourier transform associated with the Riemann-Liouville operator

In the sequel, we introduce the windowed Fourier transform associated with the Riemann-Liouville operator and we give some properties, the main references are given in [4, 5, 7, 8].

For every $g \in L^2(d\nu_\alpha)$ and $(\xi_1,\xi_2) \in [0,+\infty[\times\mathbb{R}$; the modulation of g by (ξ_1,ξ_2) is defined by

$$M_{(\xi_1,\xi_2)}(g) = \widetilde{\mathscr{F}}_\alpha\Big(\sqrt{\tau_{(\xi_1,\xi_2)}\big(|\widetilde{\mathscr{F}}_\alpha(g)|^2\big)}\Big) = g_{(\xi_1,\xi_2)}. \tag{3.1}$$

From the Plancherel theorem for $\widetilde{\mathscr{F}}_\alpha$ and the relations (2.10) and (2.16), we have

$$||M_{(\xi_1,\xi_2)}(g)||_{2,\nu_\alpha} = ||g||_{2,\nu_\alpha}. \tag{3.2}$$

For a non-zero window function g in $L^2(d\nu_\alpha)$; and all (r,x), $(\xi_1,\xi_2) \in [0,+\infty[\times\mathbb{R}$; we define the function $g_{(r,x),(\xi_1,\xi_2)}$ by

$$g_{(r,x),(\xi_1,\xi_2)} = \tau_{(r,x)}\big(M_{(\xi_1,\xi_2)}(g)\big). \tag{3.3}$$

For every $f \in L^2(d\nu_\alpha)$, we define the windowed Fourier transform associated with the Riemann-Liouville operator (WFTRL) by

$$\mathscr{V}_g(f)\big((r,x),(\xi_1,\xi_2)\big) = \int_0^\infty \int_{\mathbb{R}} f(s,y)\,\overline{g_{(r,x),(\xi_1,\xi_2)}(s,y)}\,d\nu_\alpha(s,y), \tag{3.4}$$

which can be also written in the form

$$\mathscr{V}_g(f)\big((r,x),(\xi_1,\xi_2)\big) = \langle f|g_{(r,x),(\xi_1,\xi_2)}\rangle_{\nu_\alpha} = f * g_{(\xi_1,\xi_2)}(r,-x) \tag{3.5}$$

Moreover, from Cauchy-Schwarz's inequality and relation (3.2), we get

$$||\mathscr{V}_g(f)||_{\infty,\nu_\alpha\otimes\nu_\alpha} \leqslant ||f||_{2,\nu_\alpha}\ ||g||_{2,\nu_\alpha}. \tag{3.6}$$

The WFTRL possesses the following properties

PROPOSITION 3.1. *Let g be a non-zero window function g in $L^2(d\nu_\alpha)$.*
i. For every $f \in L^2(d\nu_\alpha)$, we have the Plancherel-type theorem for $\mathscr{V}_g$

$$||\mathscr{V}_g(f)||_{2,\nu_\alpha\otimes\nu_\alpha} = ||f||_{2,\nu_\alpha}\ ||g||_{2,\nu_\alpha}. \tag{3.7}$$

ii. For all $f,h \in L^2(d\nu_\alpha)$, we have the orthogonality-type relation for $\mathscr{V}_g$

$$\langle \mathscr{V}_g(f)|\mathscr{V}_g(h)\rangle_{\nu_\alpha\otimes\nu_\alpha} = ||g||^2_{2,\nu_\alpha}\ \langle f|h\rangle_{\nu_\alpha}. \tag{3.8}$$

Proof. i) From the relations (2.16), (2.20), (3.1), (3.5) and Fubini's theorem, we have

$$\begin{aligned}
||\mathscr{V}_g(f)||^2_{2,\nu_\alpha\otimes\nu_\alpha} &= \int\int_{\big([0,+\infty[\times\mathbb{R}\big)^2} \big|f * g_{(\xi_1,\xi_2)}(r,x)\big|^2\ d\nu_\alpha(r,x)\ d\nu_\alpha(\xi_1,\xi_2) \\
&= \int_0^\infty \int_{\mathbb{R}} \Big(\int_0^\infty \int_{\mathbb{R}} \big|\widetilde{\mathscr{F}}_\alpha(f)(u,v)\big|^2\ \tau_{(\xi_1,\xi_2)}\big(\big|\widetilde{\mathscr{F}}_\alpha(g)\big|^2\big)(u,-v)\ d\nu_\alpha(u,v)\Big) d\nu_\alpha(\xi_1,\xi_2) \\
&= \int_0^\infty \int_{\mathbb{R}} \big|\widetilde{\mathscr{F}}_\alpha(f)(u,v)\big|^2 \Big(\int_0^\infty \int_{\mathbb{R}} \tau_{(u,-v)}\big(\big|\widetilde{\mathscr{F}}_\alpha(g)\big|^2\big)(\xi_1,\xi_2)\ d\nu_\alpha(\xi_1,\xi_2)\Big) d\nu_\alpha(u,v) \\
&= \int_0^\infty \int_{\mathbb{R}} \big|\widetilde{\mathscr{F}}_\alpha(f)(u,v)\big|^2 d\nu_\alpha(u,v) \int_0^\infty \int_{\mathbb{R}} \big|\widetilde{\mathscr{F}}_\alpha(g)(\xi_1,\xi_2)\big|^2 d\nu_\alpha(\xi_1,\xi_2) \\
&= ||f||^2_{2,\nu_\alpha}\ ||g||^2_{2,\nu_\alpha}.
\end{aligned}$$

ii) Follows from i) by polarization. □

From the relations (3.6), (3.7) and the Riesz-Thorin interpolation theorem [21], we have

THEOREM 3.2. *For every non-zero window function g in $L^2(d\nu_\alpha)$ and $f \in L^2(d\nu_\alpha)$, the function $\mathscr{V}_g(f)$ belongs to $L^q(d\nu_\alpha \otimes d\nu_\alpha)$; $2 \leqslant q \leqslant \infty$, with*

$$||\mathscr{V}_g(f)||_{q,\nu_\alpha\otimes\nu_\alpha} \leqslant ||f||_{2,\nu_\alpha}\ ||g||_{2,\nu_\alpha}. \tag{3.9}$$

In the following, we establish a reconstruction formula for $\mathscr{V}_g$.

THEOREM 3.3. (Reconstruction formula) *Let $g \in L^2(d\nu_\alpha)$ be a non-zero window function. Then, for every f in $L^2(d\nu_\alpha)$, such that $\mathscr{V}_g(f) \in L^1(d\nu_\alpha \otimes d\nu_\alpha)$, we have*

$$f(s,y) = \frac{1}{||g||^2_{2,\nu_\alpha}} \iint_{([0,+\infty[\times\mathbb{R})^2} \mathscr{V}_g(f)((r,x),(\xi_1,\xi_2))g_{(r,x),(\xi_1,\xi_2)}(s,y)d\nu_\alpha(r,x)d\nu_\alpha(\xi_1,\xi_2), \tag{3.10}$$

weakly in $L^2(d\nu_\alpha)$.

Proof. From relations (3.4), (3.8) and by Fubini's theorem, we have for all h in $L^2(d\nu_\alpha)$

$$\begin{aligned}
\langle f|h\rangle_{\nu_\alpha} &= \frac{1}{||g||^2_{2,\nu_\alpha}}\langle \mathscr{V}_g(f)|\mathscr{V}_g(h)\rangle_{\nu_\alpha\otimes\nu_\alpha}\\
&= \frac{1}{||g||^2_{2,\nu_\alpha}}\iint_{([0,+\infty[\times\mathbb{R})^2}\mathscr{V}_g(f)((r,x),(\xi_1,\xi_2))\overline{\mathscr{V}_g(h)((r,x),(\xi_1,\xi_2))}d\nu_\alpha(r,x)d\nu_\alpha(\xi_1,\xi_2)\\
&= \frac{1}{||g||^2_{2,\nu_\alpha}}\iint_{[0,+\infty[\times\mathbb{R}}\Big(\iint_{\left([0,+\infty[\times\mathbb{R}\right)^2}\mathscr{V}_g(f)((r,x),(\xi_1,\xi_2))\; g_{(r,x),(\xi_1,\xi_2)}(s,y)\\
&\quad \times d\nu_\alpha(r,x)d\nu_\alpha(\xi_1,\xi_2)\Big)\overline{h(s,y)}d\nu_\alpha(s,y)\\
&= \Big\langle \frac{1}{||g||^2_{2,\nu_\alpha}}\iint_{([0,+\infty[\times\mathbb{R})^2}\mathscr{V}_g(f)((r,x),(\xi_1,\xi_2))g_{(r,x),(\xi_1,\xi_2)}(.,.)d\nu_\alpha(r,x)d\nu_\alpha(\xi_1,\xi_2)|h\Big\rangle_{\nu_\alpha}
\end{aligned}$$

which gives the result. □

4. Mean dispersion theorem for the WFTRL

In this section, g will be a fixed nonzero window function in $L^2(d\nu_\alpha)$ with $||g||_{2,\nu_\alpha} = 1$ and Σ be a subset of the time-frequency set $\left([0,+\infty[\times\mathbb{R}\right)^2$ of finite measure $0 < \nu_\alpha\otimes\nu_\alpha(\Sigma) < \infty$. We introduce a pair of orthogonal projections on $L^2(d\nu_\alpha\otimes d\nu_\alpha)$. The first, denoted P_g, is the orthogonal projection from $L^2(d\nu_\alpha\otimes d\nu_\alpha)$ onto $\mathscr{V}_g(L^2(d\nu_\alpha))$ and the second is the time-frequency limiting operator defined by:

$$P_\Sigma(F) = F\,\mathbf{1}_\Sigma;\; F\in L^2(d\nu_\alpha\otimes d\nu_\alpha). \tag{4.1}$$

We begin this section by the following useful lemma which shows that, for a given window function g, the Hilbert space $\mathscr{V}_g(L^2(d\nu_\alpha))$ has a reproducing kernel. This allows to express the orthogonal projection operator over $\mathscr{V}_g(L^2(d\nu_\alpha))$.

LEMMA 4.1. (Reproducing kernel Hilbert space) *The space $\mathscr{V}_g(L^2(d\nu_\alpha))$ possesses a reproducing kernel* [19] *given by*

$$\mathscr{K}_g\big((r,x),(\xi_1,\xi_2),(s,y),(\mu,\lambda)\big) = \frac{1}{||g||^2_{2,\nu_\alpha}}\;\mathscr{V}_g\big(\tau_{(s,y)}(g_{(\mu,\lambda)})\big)\big((r,x),(\xi_1,\xi_2)\big). \tag{4.2}$$

Furthermore, the kernel $\mathscr{K}_g$ is pointwise bounded, that is

$$|\mathscr{K}_g\big((r,x),(\xi_1,\xi_2),(s,y),(\mu,\lambda)\big)| \leqslant 1,\;\; \forall (r,x),(\xi_1,\xi_2),(s,y),(\mu,\lambda)\in[0,+\infty[\times\mathbb{R}. \tag{4.3}$$

Proof. For every $\big((s,y),(\mu,\lambda)\big)\in\left([0,+\infty[\times\mathbb{R}\right)^2$; the function $\tau_{(s,y)}(g_{(\mu,\lambda)})$ belongs to $L^2(d\nu_\alpha)$, consequently, for every $\big((s,y),(\mu,\lambda)\big)\in\left([0,+\infty[\times\mathbb{R}\right)^2$; the function $\mathscr{K}_g\big((.,.),(.,.),(s,y),(\mu,\lambda)\big)$ belongs to $\mathscr{V}_g(L^2(d\nu_\alpha))$. Moreover, from Plancherel theorem, the relations (2.11) and (3.2),

$$\begin{aligned}||\mathscr{K}_g\big((.,.),(.,.),(s,y),(\mu,\lambda)\big)||^2_{2,\nu_\alpha\otimes\nu_\alpha} &= \frac{1}{||g||^4_{2,\nu_\alpha}}\,||\mathscr{V}_g\big(\tau_{(s,y)}(g_{(\mu,\lambda)})\big)||^2_{2,\nu_\alpha\otimes\nu_\alpha}\\ &= \frac{1}{||g||^2_{2,\nu_\alpha}}\,||\tau_{(s,y)}(g_{(\mu,\lambda)})||^2_{2,\nu_\alpha} \leqslant 1. \quad (4.4)\end{aligned}$$

(Reproducing property)Let $F \in \mathscr{V}_g(L^2(d\nu_\alpha))$; $F = \mathscr{V}_g(f)$; $f \in L^2(d\nu_\alpha)$. For every $\big((s,y),(\mu,\lambda)\big) \in \big([0,+\infty[\times\mathbb{R}\big)^2$ and applying orthogonality property, we get

$$\begin{aligned}\langle F|\mathscr{K}_g\big((.,.),(.,.),(s,y),(\mu,\lambda)\big)\rangle_{\nu_\alpha\otimes\nu_\alpha} &= \frac{1}{||g||^2_{2,\nu_\alpha}}\,\langle \mathscr{V}_g(f)|\mathscr{V}_g\big(\tau_{(s,y)}(g_{(\mu,\lambda)})\big)\rangle_{\nu_\alpha\otimes\nu_\alpha}\\ &= \langle f|\tau_{(s,y)}(g_{(\mu,\lambda)})\rangle_{\nu_\alpha} = \mathscr{V}_g(f)\big((s,y),(\mu,\lambda)\big)\\ &= F\big((s,y),(\mu,\lambda)\big). \quad (4.5)\end{aligned}$$

Now, by the relations (2.11), (3.2), (3.3) and (3.5), we deduce that

$$\begin{aligned}|\mathscr{K}_g\big((r,x),(\xi_1,\xi_2),(s,y),(\mu,\lambda)\big)| &= \frac{1}{||g||^2_{2,\nu_\alpha}}\,|\langle\tau_{(s,y)}(g_{(\mu,\lambda)})|g_{(r,x),(\xi_1,\xi_2)}\rangle_{\nu_\alpha}|\\ &\leqslant \frac{1}{||g||^2_{2,\nu_\alpha}}\,||\tau_{(s,y)}(g_{(\mu,\lambda)})||_{2,\nu_\alpha}\,||g_{(r,x),(\xi_1,\xi_2)}||_{2,\nu_\alpha}\\ &\leqslant 1.\end{aligned}$$

This achieves the proof. □

The following proposition shows that the WFTRL cannot be concentrated inside a set with finite measure arbitrary small.

PROPOSITION 4.2. *Let* $\Sigma \subset \big([0,+\infty[\times\mathbb{R}\big)^2$ *such that* $0 < \nu_\alpha\otimes\nu_\alpha(\Sigma) < 1$. *Then, for every* $f \in L^2(d\nu_\alpha)$,

$$||\mathbf{1}_{\Sigma^c}\mathscr{V}_g(f)||_{2,\nu_\alpha\otimes\nu_\alpha} \geqslant \sqrt{1-\nu_\alpha\otimes\nu_\alpha(\Sigma)}\,||f||_{2,\nu_\alpha}\,||g||_{2,\nu_\alpha}. \quad (4.6)$$

Proof. From (3.6), we get

$$||\mathbf{1}_\Sigma\mathscr{V}_g(f)||^2_{2,\nu_\alpha\otimes\nu_\alpha} \leqslant ||f||^2_{2,\nu_\alpha}\,||g||^2_{2,\nu_\alpha}\,\nu_\alpha\otimes\nu_\alpha(\Sigma).$$

Now since

$$||\mathbf{1}_{\Sigma^c}\mathscr{V}_g(f)||^2_{2,\nu_\alpha\otimes\nu_\alpha} = ||\mathscr{V}_g(f)||^2_{2,\nu_\alpha\otimes\nu_\alpha} - ||\mathbf{1}_\Sigma\mathscr{V}_g(f)||^2_{2,\nu_\alpha\otimes\nu_\alpha},$$

from relation (3.7), we obtain

$$||\mathbf{1}_{\Sigma^c}\mathscr{V}_g(f)||^2_{2,\nu_\alpha\otimes\nu_\alpha} \geqslant ||f||^2_{2,\nu_\alpha}\,||g||^2_{2,\nu_\alpha}(1-\nu_\alpha\otimes\nu_\alpha(\Sigma)).$$

Hence, the proof is complete. □

The following proposition shows that the operator norm $||P_\Sigma P_g||$ can be estimated by Hilbert-Schmidt norm $||P_\Sigma P_g||_{HS}$.

PROPOSITION 4.3. *The operator $P_\Sigma P_g$ is an Hilbert-Shmidt operator such that*

$$||P_\Sigma P_g|| \leqslant ||P_\Sigma P_g||_{HS} \leqslant \sqrt{\nu_\alpha \otimes \nu_\alpha(\Sigma)}. \tag{4.7}$$

where $||.||$ denotes here the classical operator norm and $||.||_{HS}$ denotes the Hilbert-Schmidt norm. Moreover, if $||P_\Sigma P_g|| < 1$, then for every $f \in L^2(d\nu_\alpha)$,

$$||\boldsymbol{1}_{\Sigma^c}\mathscr{V}_g(f)||_{2,\nu_\alpha\otimes\nu_\alpha} \geqslant \sqrt{1-||P_S\, P_g||^2}\; ||f||_{2,\nu_\alpha}\; ||g||_{2,\nu_\alpha}. \tag{4.8}$$

Proof. Let $F \in L^2(d\nu_\alpha \otimes d\nu_\alpha)$; $F = P_g(F)+G$; $G \perp \mathscr{V}_g(L^2(d\nu_\alpha))$. For all (s,y), $(\mu,\lambda) \in [0,+\infty[\times\mathbb{R}$; the function $\mathscr{K}_g\big((.,.),(.,.),(s,y),(\mu,\lambda)\big)$ belongs to $\mathscr{V}_g(L^2(d\nu_\alpha))$; so

$$\begin{aligned}\langle F|\mathscr{K}_g\big((.,.),(.,.),(s,y),(\mu,\lambda)\big)\rangle_{\nu_\alpha\otimes\nu_\alpha} &= \langle P_g(F)|\mathscr{K}_g\big((.,.),(.,.),(s,y),(\mu,\lambda)\big)\rangle_{\nu_\alpha\otimes\nu_\alpha}\\ &= P_g(F)\big((s,y),(\mu,\lambda)\big),\end{aligned} \tag{4.9}$$

because $\mathscr{K}_g$ is a reproducing Kernel of $\mathscr{V}_g(L^2(d\nu_\alpha))$.

Hence, for every $F \in L^2(d\nu_\alpha \otimes d\nu_\alpha)$,

$$\begin{aligned}&P_\Sigma\, P_g(F)\big((s,y),(\mu,\lambda)\big)\\ &= \int_{([0,+\infty[\times\mathbb{R})^2} \mathscr{H}_g\big((s,y),(\mu,\lambda),(r,x),(\xi_1,\xi_2)\big)\; F\big((r,x),(\xi_1,\xi_2)\big) d\nu_\alpha(r,x) d\nu_\alpha(\xi_1,\xi_2),\end{aligned} \tag{4.10}$$

where the Kernel $\mathscr{H}_g$ is given by

$$\mathscr{H}_g\big((s,y),(\mu,\lambda),(r,x),(\xi_1,\xi_2)\big) = \mathbf{1}_\Sigma\big((s,y),(\mu,\lambda)\big)\overline{\mathscr{K}_g\big((r,x),(\xi_1,\xi_2),(s,y),(\mu,\lambda)\big)}.$$

Now, by Fubini's theorem and the relation (4.4),

$$\begin{aligned}&\iint_{([0,+\infty[\times\mathbb{R})^4} |\mathscr{H}_g((s,y),(\mu,\lambda),(r,x),(\xi_1,\xi_2))|^2 d\nu_\alpha(s,y) d\nu_\alpha(\mu,\lambda) d\nu_\alpha(r,x) d\nu_\alpha(\xi_1,\xi_2)\\ &= \iint_{([0,+\infty[\times\mathbb{R})^2} \mathbf{1}_\Sigma((s,y),(\mu,\lambda)) ||\mathscr{K}_g((.,.),(.,.),(s,y),(\mu,\lambda))||^2_{2,\nu_\alpha\otimes\nu_\alpha} d\nu_\alpha(s,y) d\nu_\alpha(\mu,\lambda)\\ &\leqslant \nu_\alpha \otimes \nu_\alpha(\Sigma).\end{aligned} \tag{4.11}$$

The relations (4.10) and (4.11) show that $P_\Sigma P_g$ is an Hilbert-Shmidt operator and that

$$\begin{aligned}&||P_\Sigma P_g||^2_{HS}\\ &= \int_{([0,+\infty[\times\mathbb{R})^4} |\mathscr{H}_g((s,y),(\mu,\lambda),(r,x),(\xi_1,\xi_2))|^2 d\nu_\alpha(s,y) d\nu_\alpha(\mu,\lambda) d\nu_\alpha(r,x) d\nu_\alpha(\xi_1,\xi_2)\\ &\leqslant \nu_\alpha \otimes \nu_\alpha(\Sigma).\end{aligned}$$

This involves that

$$||P_\Sigma P_g|| \leqslant ||P_\Sigma P_g||_{HS} \leqslant \sqrt{\nu_\alpha \otimes \nu_\alpha(\Sigma)}. \tag{4.12}$$

On other hand, for every $f \in L^2(d\nu_\alpha)$

$$||\mathbf{1}_{\Sigma^c}\mathscr{V}_g(f)||^2_{2,\nu_\alpha\otimes\nu_\alpha} = ||\mathscr{V}_g(f)||^2_{2,\nu_\alpha\otimes\nu_\alpha} - ||\mathbf{1}_{\Sigma}\mathscr{V}_g(f)||^2_{2,\nu_\alpha\otimes\nu_\alpha}.$$

Now since

$$\mathbf{1}_{\Sigma}\mathscr{V}_g(f) = P_\Sigma P_g(\mathscr{V}_g(f)),$$

and then from the relation (3.6), we get

$$\begin{aligned} ||\mathbf{1}_{\Sigma^c}\mathscr{V}_g(f)||^2_{2,\nu_\alpha\otimes\nu_\alpha} &\geqslant (1-||P_\Sigma P_g||^2)\ ||\mathscr{V}_g(f)||^2_{2,\nu_\alpha\otimes\nu_\alpha} \\ &= (1-||P_\Sigma\, P_g||^2)\ ||f||^2_{2,\nu_\alpha}\ ||g||^2_{2,\nu_\alpha}. \end{aligned} \tag{4.13}$$

Hence, the proof is complete. □

THEOREM 4.4. *Let $\mathscr{K}$ be a finite subset of $\mathbb{N}^2$ and let $(\varphi_{m,n})_{(m,n)\in\,\mathscr{K}}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$. Then, for every nonempty finite subset $\mathscr{K}\subset\mathbb{N}^2$, we have*

$$\sum_{(m,n)\in\,\mathscr{K}} \left(1-||\mathbf{1}_{\Sigma^c}\mathscr{V}_g(\varphi_{m,n})||_{2,\nu_\alpha\otimes\nu_\alpha}\right) \leqslant \nu_\alpha\otimes\nu_\alpha(\Sigma). \tag{4.14}$$

Proof. Let $(h_{n,m})_{(n,m)\in\mathbb{N}^2}$ be an orthonormal basis of $L^2(d\nu_\alpha\otimes d\nu_\alpha)$, since $P_\Sigma P_g$ is a Hilbert Schmidt operator satisfying relation (4.7), we deduce that the positive operator $P_gP_\Sigma P_g$ satisfies

$$\sum_{(m,n)\in\mathbb{N}^2} \langle P_gP_\Sigma P_g h_{m,n} \mid h_{m,n}\rangle_{\nu_\alpha\otimes\nu_\alpha} = ||P_\Sigma P_g||^2_{HS} \leqslant \nu_\alpha\otimes\nu_\alpha(\Sigma) < \infty, \tag{4.15}$$

which means according to [24, Theorems 2.6 and 2.7], that $P_\Sigma P_g P_\Sigma$ is a trace class operator, with

$$tr(P_gP_\Sigma P_g) = ||P_\Sigma P_g||^2_{HS} \leqslant \nu_\alpha\otimes\nu_\alpha(\Sigma). \tag{4.16}$$

Actually, since $(\varphi_{m,n})_{(m,n)\in\,\mathscr{K}}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$, then by relation (3.8) we deduce that $(\mathscr{V}_g(\varphi_{m,n}))_{(m,n)\in\,\mathscr{K}}$ be an orthonormal sequence in $L^2(d\nu_\alpha\otimes d\nu_\alpha)$, hence

$$\begin{aligned} \sum_{(m,n)\in\,\mathscr{K}} \langle P_\Sigma\mathscr{V}_g(\varphi_{m,n}) \mid \mathscr{V}_g(\varphi_{m,n})\rangle_{\nu_\alpha\otimes\nu_\alpha} &= \sum_{(m,n)\in\,\mathscr{K}} \langle P_gP_\Sigma P_g\mathscr{V}_g(\varphi_{m,n}) \mid \mathscr{V}_g(\varphi_{m,n})\rangle_{\nu_\alpha\otimes\nu_\alpha} \\ &\leqslant tr(P_gP_\Sigma P_g). \end{aligned} \tag{4.17}$$

Then by (4.16) we obtain

$$\sum_{(m,n)\in\,\mathscr{K}} \langle P_\Sigma\mathscr{V}_g(\varphi_{m,n}) \mid \mathscr{V}_g(\varphi_{m,n})\rangle_{\nu_\alpha\otimes\nu_\alpha} \leqslant \nu_\alpha\otimes\nu_\alpha(\Sigma). \tag{4.18}$$

Then by Cauchy-Schwartz's inequality,

$$\begin{aligned} \langle P_\Sigma\mathscr{V}_g(\varphi_{m,n}) \mid \mathscr{V}_g(\varphi_{m,n})\rangle_{\nu_\alpha\otimes\nu_\alpha} &= 1-\langle P_{\Sigma^c}\mathscr{V}_g(\varphi_{m,n}) \mid \mathscr{V}_g(\varphi_{m,n})\rangle_{\nu_\alpha\otimes\nu_\alpha} \\ &\geqslant 1-||1_{\Sigma^c}\mathscr{V}_g(\varphi_{m,n})||_{2,\nu_\alpha\otimes\nu_\alpha}. \end{aligned} \tag{4.19}$$

Therefore by (4.18), we deduce the desired result. □

DEFINITION 4.5. Let ε be a positive real number. We say that $\mathscr{V}_g(f)$ is ε-time-concentrated on Σ, if

$$||\mathbf{1}_{\Sigma^c}\mathscr{V}_g(f)||_{2,\nu_\alpha\otimes\nu_\alpha} \leqslant \varepsilon||f||_{2,\nu_\alpha}\ ||g||_{2,\nu_\alpha}. \tag{4.20}$$

Then by Theorem 4.4, we shall deduce the following proposition which show that, if the WFTRL of an othornormal sequence are ε- time-frequency concentrated in a given centred ball of $\big([0,+\infty[\times\mathbb{R}\big)^2$, then such sequence is necessary finite.

PROPOSITION 4.6. *Let* $0<\varepsilon<1$ *and let* $\mathscr{K}\subset\mathbb{N}^2$ *be a nonempty subset and let* $(\varphi_{m,n})_{(m,n)\in\mathscr{K}}$ *be an orthonormal sequence in* $L^2(d\nu_\alpha)$. *If* $\mathscr{V}_g(\varphi_{m,n})$ *is* ε*-time-concentrated on* $B_\rho^+=\Big\{\big((r,x),(s,y)\big)\in\big([0,+\infty[\times\mathbb{R}\big)^2;\ r^2+x^2+s^2+y^2\leqslant\rho^2\Big\}$, *then, the subset* $\mathscr{K}$ *is finite and*

$$card(\mathscr{K})\leqslant\frac{\rho^{4\alpha+6}}{2^{2\alpha+3}\,\Gamma(2\alpha+4)(1-\varepsilon)}. \tag{4.21}$$

Proof. Let $\mathscr{K}_1\subset\mathscr{K}$ be a nonempty finite subset, then by Theorem 4.4, it follows that

$$\sum_{(m,n)\in\mathscr{K}_1}\big(1-||\mathbf{1}_{(B_\rho^+)^c}\mathscr{V}_g(\varphi_{m,n})||_{2,\nu_\alpha\otimes\nu_\alpha}\big)\leqslant\nu_\alpha\otimes\nu_\alpha(B_\rho^+). \tag{4.22}$$

However for every $(m,n)\in\mathscr{K}_1$, $||\mathbf{1}_{(B_\rho^+)^c}\mathscr{V}_g(\varphi_{m,n})||_{2,\nu_\alpha\otimes\nu_\alpha}\leqslant\varepsilon$, and we have

$$\nu_\alpha\otimes\nu_\alpha\big(B_\rho^+\big)=\frac{\rho^{4\alpha+6}}{2^{2\alpha+3}\,\Gamma(2\alpha+4)}. \tag{4.23}$$

This involves that for every finite subset $\mathscr{K}_1$ of $\mathscr{K}$, we have

$$\mathrm{card}(\mathscr{K}_1)\leqslant\frac{\rho^{4\alpha+6}}{(1-\varepsilon)\ 2^{2\alpha+3}\,\Gamma(2\alpha+4)}.$$

which means that $\mathscr{K}$ is a finite subset and satisfies relation (4.21). □

Therefore if the generalized dispersion of the elements of an orthonormal sequence is uniformly bounded then this sequence is finite and we can give a bound on the number of elements in that sequence. More precisely:

COROLLARY 4.7. *Fix* $A>0$. *Let* $\mathscr{K}$ *be a nonempty subset of* $\mathbb{N}^2$ *and let* $(\varphi_{m,n})_{(m,n)\in\mathscr{K}}$ *be an orthonormal sequence in* $L^2(d\nu_\alpha)$ *that satisfies*

$$||\ |((r,x),(\xi_1,\xi_2))|^s\mathscr{V}_g(\varphi_{m,n})||_{2,\nu_\alpha\otimes\nu_\alpha}^{1/s}\leqslant A. \tag{4.24}$$

Then $\mathscr{K}$ *is a finite subset and*

$$card(\mathscr{K})\leqslant\frac{2^{(2\alpha+3)(2/s-1)+1}}{\Gamma(2\alpha+4)}\,A^{4\alpha+6}. \tag{4.25}$$

Proof. Since

$$||\mathbf{1}_{(B_\rho^+)^c}\mathscr{V}_g(\varphi_{m,n})||_{2,\nu_\alpha\otimes\nu_\alpha} \leqslant \rho^{-s}||\,|((r,x),(\xi_1,\xi_2))|^s\mathscr{V}_g(\varphi_{m,n})||_{2,\nu_\alpha\otimes\nu_\alpha},$$

then if we choose $\rho = A\,2^{1/s}$, we deduce that for every $(m,n)\in\mathscr{K}$; $\mathscr{V}_g(\varphi_{m,n})$ is $\frac{1}{2}$-time-concentrated in the ball B_ρ^+. Therefore from Proposition 4.6 we obtain the desired result. □

LEMMA 4.8. *Let* $s>0$ *and let* $(\varphi_{m,n})_{(m,n)\in\mathbb{N}^2}$ *be an orthonormal sequence in* $L^2(d\nu_\alpha)$. *Then, there exists* $j_0\in\mathbb{Z}$ *such that*

$$\forall (m,n)\in\mathbb{N}^2,\quad ||\,|((r,x),(\xi_1,\xi_2))|^s\mathscr{V}_g(\varphi_{m,n})||_{2,\nu_\alpha\otimes\nu_\alpha}^{1/s} \geqslant 2^{j_0}. \tag{4.26}$$

Proof. For every $j\in\mathbb{Z}$, let

$$P_j = \left\{(m,n)\in\mathbb{N}^2;\ 2^{j-1}\leqslant ||\,|((r,x),(\xi_1,\xi_2))|^s\mathscr{V}_g(\varphi_{m,n})||_{2,\nu_\alpha\otimes\nu_\alpha}^{1/s} < 2^j\right\}.$$

Then, $\mathbb{N}^2 = \bigcup_{j\in\mathbb{Z}} P_j$, $P_{j_1}\cap P_{j_2}=\emptyset$ if $j_1\neq j_2$ and for every $(m,n)\in P_j$,

$$||\,|((r,x),(\xi_1,\xi_2))|^s\mathscr{V}_g(\varphi_{m,n})||_{2,\nu_\alpha\otimes\nu_\alpha}^{1/s} \leqslant 2^j. \tag{4.27}$$

Applying Corollary 4.7, we deduce that P_j is finite and

$$\mathrm{card}(P_j) \leqslant \frac{2^{(2\alpha+3)(\frac{2}{s}-1)+1}(2^j)^{4\alpha+6}}{\Gamma(2\alpha+4)}. \tag{4.28}$$

Thus, for j negative and $|j|$ sufficiently large, we get $\mathrm{card}(P_j)=0$ or $P_j=\emptyset$. This means that there exists $j_0\in\mathbb{Z}$ such that $\forall j<j_0$, $P_j=\emptyset$. So,

$$\mathbb{N}^2 = \bigcup_{j\in\mathbb{Z}} P_j = \bigcup_{j=j_0}^{+\infty} P_j. \quad \square \tag{4.29}$$

Then we have an analogue of Shapiro's uncertainty principle for the WFTRL and the proof of this Theorem is inspired from the paper of Malinnikova [15], who proved a similar result for the usual Fourier transform (1.3).

THEOREM 4.9. (Shapiro's uncertainty principle for the WFTRL)
Let $(\varphi_{m,n})_{(m,n)\in\mathbb{N}^2}$ *be an orthonormal sequence in* $L^2(d\nu_\alpha)$, *then for every positive real number* s *and for every nonempty finite subset* $\mathscr{K}\subset\mathbb{N}^2$, *we have*

$$\sum_{(m,n)\in\mathscr{K}} \left(||\,|(r,x)|^s\mathscr{V}_g(\varphi_{m,n})||^2_{2,\nu_\alpha\otimes\nu_\alpha} + ||\,|(\xi_1,\xi_2)|^s\mathscr{V}_g(\varphi_{m,n})||^2_{2,\nu_\alpha\otimes\nu_\alpha}\right) \tag{4.30}$$

$$\geqslant c_{s,\alpha}\; card(\mathscr{K})^{1+\frac{s}{4\alpha+6}}.$$

where $c_{s,\alpha} = \left(\dfrac{(2^{4\alpha+6}-1)\Gamma(2\alpha+4)}{2^{(2\alpha+3)(4+\frac{3}{s})+2}}\right)^{\frac{s}{2\alpha+3}}$.

Proof. Let j_0 be defined in Lemma 4.8. For every $k \geqslant j_0$, we put $Q_k = \bigcup_{j=j_0}^{k} P_j$.
From the relation (4.28),

$$\mathrm{card}(Q_k) = \sum_{j=j_0}^{k} card(P_j) \leqslant a_{s,\alpha}\, 2^{(4\alpha+6)(k+1)}, \tag{4.31}$$

where $a_{s,\alpha} = \dfrac{2^{(2\alpha+3)(2/s-1)+1}}{(2^{4\alpha+6}-1)\Gamma(2\alpha+4)}$.

i) If $\mathrm{card}(\mathscr{K}) > 2\, a_{s,\alpha}\, 2^{(4\alpha+6)(j_0+1)}$; let $k > j_0$ such that

$$2\, a_{s,\alpha}\, 2^{(4\alpha+6)k} \leqslant \mathrm{card}(\mathscr{K}) < 2\, a_{s,\alpha}\, 2^{(4\alpha+6)(k+1)}. \tag{4.32}$$

From the relations (4.31) and (4.32), we have

$$\mathrm{card}(Q_{k-1}) \leqslant \frac{\mathrm{card}(\mathscr{K})}{2}\,. \tag{4.33}$$

On the other hand,

$$\begin{aligned}
&\sum_{(m,n)\in \mathscr{K}} \| |((r,x),(\xi_1,\xi_2))|^s \mathscr{V}_g(\varphi_{m,n})\|^2_{2,\nu_\alpha\otimes\nu_\alpha} \\
&= \sum_{(m,n)\in \mathscr{K}\cap Q_{k-1}} \| |((r,x),(\xi_1,\xi_2))|^s \mathscr{V}_g(\varphi_{m,n})\|^2_{2,\nu_\alpha\otimes\nu_\alpha} \\
&\quad + \sum_{(m,n)\in \mathscr{K}\setminus Q_{k-1}} \| |((r,x),(\xi_1,\xi_2))|^s \mathscr{V}_g(\varphi_{m,n})\|^2_{2,\nu_\alpha\otimes\nu_\alpha} \\
&\geqslant \sum_{(m,n)\in \mathscr{K}\setminus Q_{k-1}} \| |((r,x),(\xi_1,\xi_2))|^s \mathscr{V}_g(\varphi_{m,n})\|^2_{2,\nu_\alpha\otimes\nu_\alpha}.
\end{aligned}$$

But, for every $(m,n) \in \mathscr{K} \setminus Q_{k-1}$,

$$\| |((r,x),(\xi_1,\xi_2))|^s \mathscr{V}_g(\varphi_{m,n})\|^2_{2,\nu_\alpha\otimes\nu_\alpha} \geqslant 4^{(k-1)s}$$

Then, from the relations (4.32) and (4.33), we deduce that

$$\begin{aligned}
&\sum_{(m,n)\in \mathscr{K}} \| |((r,x),(\xi_1,\xi_2))|^s \mathscr{V}_g(\varphi_{m,n})\|^2_{2,\nu_\alpha\otimes\nu_\alpha} \\
&\geqslant 4^{(k-1)s}\, \mathrm{card}(\mathscr{K} \setminus Q_{k-1}) \geqslant 4^{(k-1)s}\, \frac{\mathrm{card}(\mathscr{K})}{2} \\
&\geqslant \left(\frac{(2^{4\alpha+6}-1)\Gamma(2\alpha+4)}{2^{(2\alpha+3)(3+\frac{3}{s})+2}} \right)^{\frac{s}{2\alpha+3}} \mathrm{card}(\mathscr{K})^{1+\frac{s}{2\alpha+3}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{(m,n)\in \mathscr{K}} \left(\| |(r,x)|^s \mathscr{V}_g(\varphi_{m,n})\|^2_{2,\nu_\alpha\otimes\nu_\alpha} + \| |(\xi_1,\xi_2)|^s \mathscr{V}_g(\varphi_{m,n})\|^2_{2,\nu_\alpha\otimes\nu_\alpha} \right) \\
&\qquad \geqslant c_{s,\alpha} \mathrm{card}(\mathscr{K})^{1+\frac{s}{2\alpha+3}}.
\end{aligned}$$

ii) If $\text{card}(\mathcal{K}) \leqslant 2\, a_{s,\alpha}\, 2^{(4\alpha+6)(j_0+1)}$. By Lemma 4.8, we have

$$\sum_{(m,n)\in\,\mathcal{K}} ||\, |((r,x),(\xi_1,\xi_2))|^s \mathscr{V}_g(\varphi_{m,n})||^2_{2,\nu_\alpha\otimes\nu_\alpha}$$

$$\geqslant \text{card}(\mathcal{K})\, 4^{(j_0-1)s} \geqslant \left(\frac{(2^{4\alpha+6}-1)\Gamma(2\alpha+4)}{2^{(2\alpha+3)(3+\frac{3}{s})+2}}\right)^{\frac{s}{2\alpha+3}} \text{card}(\mathcal{K})^{1+\frac{s}{2\alpha+3}}. \tag{4.34}$$

Then,

$$\sum_{(m,n)\in\,\mathcal{K}} (||\, |(r,x)|^s \mathscr{V}_g(\varphi_{m,n})||^2_{2,\nu_\alpha\otimes\nu_\alpha} + ||\, |(\xi_1,\xi_2)|^s \mathscr{V}_g(\varphi_{m,n})||^2_{2,\nu_\alpha\otimes\nu_\alpha})$$

$$\geqslant c_{s,\alpha}\text{card}(\mathcal{K})^{1+\frac{s}{2\alpha+3}}.$$

This complete the proof. $\square$

The last dispersion inequality implies in particular that, there does not exist an infinite sequence $(\varphi_{m,n})_{(m,n)\in\mathbb{N}^2}$ of $L^2(d\nu_\alpha)$ such that the two sequences

$$(||\, |(r,x)|^s \mathscr{V}_g(\varphi_{m,n})||^2_{2,\nu_\alpha\otimes\nu_\alpha})_{(m,n)\in\mathbb{N}^2}$$

and

$$(||\, |(\xi_1,\xi_2)|^s \mathscr{V}_g(\varphi_{m,n})||^2_{2,\nu_\alpha\otimes\nu_\alpha})_{(m,n)\in\mathbb{N}^2}$$

are simultaneously bounded. More precisely:

COROLLARY 4.10. *Let $(\varphi_{m,n})_{(m,n)\in\,\mathbb{N}^2}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$, then for every positive real number s and for every nonempty finite subset $\mathcal{K}\subset\mathbb{N}^2$, we have*

$$\sup_{(m,n)\in\,\mathcal{K}} \{||\, |(r,x)|^s \mathscr{V}_g(\varphi_{m,n})||^2_{2,\nu_\alpha\otimes\nu_\alpha} + ||\, |(\xi_1,\xi_2)|^s \mathscr{V}_g(\varphi_{m,n})||^2_{2,\nu_\alpha\otimes\nu_\alpha}\} \tag{4.35}$$

$$\geqslant\; c_{s,\alpha}\; card(\mathcal{K})^{\frac{s}{2\alpha+3}}.$$

In particular, we obtain the following strong uncertainty principle for the WFTRL who proved a similar result for the usual Fourier transform (1.4):

$$\sup_{(m,n)\in\,\mathbb{N}^2} \left(||\, |(r,x)|^s \mathscr{V}_g(\varphi_{m,n})||^2_{2,\nu_\alpha\otimes\nu_\alpha} + ||\, |(\xi_1,\xi_2)|^s \mathscr{V}_g(\varphi_{m,n})||^2_{2,\nu_\alpha\otimes\nu_\alpha}\right) = +\infty. \tag{4.36}$$

Next, we shall prove an analogue of Shapiro's umbrella theorem for the WFTRL. More precisely:

THEOREM 4.11. (Shapiro's umbrella theorem for the WFTRL) *Let $\mathcal{K}\subset\mathbb{N}^2$ be a nonempty subset and $(\varphi_{m,n})_{(m,n)\in\,\mathcal{K}}$ be an orthonormal sequence in $L^2(d\nu_\alpha)$, if there is a function $\psi\in L^2(d\nu_\alpha\otimes d\nu_\alpha)$ such that*

$$|\, \mathscr{V}_g(\varphi_{m,n})\big((r,x),(\xi_1,\xi_2)\big)\,| \leqslant \psi\big((r,x),(\xi_1,\xi_2)\big),$$

for every $(m,n)\in\mathcal{K}$ and for almost every $\big((r,x),(\xi_1,\xi_2)\big)\in\big([0,+\infty[\times\mathbb{R}\big)^2$, then $\mathcal{K}$ is finite.

Proof. Following the idea of Malinnikova [15, Corollary 2], for every positive real number $0<\varepsilon<1$, there is a subset $\Delta_{\psi,\varepsilon}\subset\big([0,+\infty[\times\mathbb{R}\big)^2$ such that

$$\nu_\alpha\otimes\nu_\alpha(\Delta_{\psi,\varepsilon})=\inf\big\{\nu_\alpha\otimes\nu_\alpha(\Sigma);||\,\mathbf{1}_{\Sigma^c}\psi\,||_{2,\nu_\alpha\otimes\nu_\alpha}\leqslant\varepsilon\big\},$$

and

$$\int\int_{([0,+\infty[\times\mathbb{R})^2\setminus\Delta_{\psi,\varepsilon}}|\,\psi\big((r,x),(\xi_1,\xi_2)\big)\,|^2\,d\nu_\alpha(r,x)d\nu_\alpha(\xi_1,\xi_2)=\varepsilon^2.$$

Hence, according to the hypothesis, for every $(m,n)\in\mathscr{K}$, we have

$$\int\int_{([0,+\infty[\times\mathbb{R})^2\setminus\Delta_{\psi,\varepsilon}}|\,\mathscr{V}_g(\varphi_{m,n})\big((r,x),(\xi_1,\xi_2)\big)\,|^2\,d\nu_\alpha(r,x)d\nu_\alpha(\xi_1,\xi_2)\leqslant\varepsilon^2. \tag{4.37}$$

Let $\mathscr{K}_1\subset\mathscr{K}$ be a nonempty finite subset, then by Theorem 4.4, it follows that

$$\sum_{(m,n)\in\mathscr{K}_1}\big(1-||\mathbf{1}_{(\Delta_{\psi,\varepsilon})^c}\mathscr{V}_g(\varphi_{m,n})||_{2,\nu_\alpha\otimes\nu_\alpha}\big)\leqslant\nu_\alpha\otimes\nu_\alpha(\Delta_{\psi,\varepsilon}).$$

Therefore, by (4.37), we can conclude that for every finite subset $\mathscr{K}_1$ of $\mathscr{K}$, we have

$$\mathrm{card}(\mathscr{K}_1)\leqslant\frac{1}{1-\varepsilon}\,\nu_\alpha\otimes\nu_\alpha(\Delta_{\psi,\varepsilon}).$$

which means that $\mathscr{K}$ is a finite subset and

$$\mathrm{card}(\mathscr{K})\leqslant\frac{1}{1-\varepsilon}\,\nu_\alpha\otimes\nu_\alpha(\Delta_{\psi,\varepsilon}).\quad\square \tag{4.38}$$

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