Operators and Matrices
Volume 11, Number 4 (2017), 1033–1046 doi:10.7153/oam-2017-11-71

ON A CLASS OF DIFFERENTIAL–DIFFERENCE OPERATORS IN SPACES OF ANALYTIC FUNCTIONS

STEPAN LINCHUK AND YURIY LINCHUK

(*Communicated by Y.-T. Poon*)

Abstract. We define the differential-difference operator which generalizes the Dunkl operator and the Bessel-Struve operator in the space of analytic functions in an arbitrary starlike p-symmetric domain of the complex plane with respect to the origin, where p is some positive integer. We investigate conditions of the equivalence of this operator to the power of the usual differentiation. We describe the commutant of this operator. We establish the hypercyclicity and the chaoticity of this operator. We investigate properties of diagonal operators induced by the Taylor coefficients of the generalized hypergeometric function.

1. Introduction

Let G be an arbitrary domain of the complex plane. By $\mathscr{H}(G)$ denotes the space of all analytic functions in G equipped with the topology of compact convergence. By $\mathscr{L}(\mathscr{H}(G))$ we denote the set of all linear continuous operators in the space $\mathscr{H}(G)$.

Let α be an arbitrary complex number. If G is a symmetric domain of the complex plane with respect to the origin, then the Dunkl operator Λ_α acts in the space $\mathscr{H}(G)$ according to the rule

$$(\Lambda_\alpha f)(z) = f'(z) + \alpha \frac{f(z) - f(-z)}{z}.$$

This operator has been introduced by Dunkl in [9]. The Dunkl operator and some of its modifications in spaces of analytic functions were studied in [2, 3, 4, 5, 14]. Let G be an arbitrary domain of the complex plane containing the origin and $\alpha \in \mathbb{C}$. The Bessel-Struve operator l_α acts in the space $\mathscr{H}(G)$ according to the rule

$$(l_\alpha f)(z) = f''(z) + \alpha \frac{f'(z) - f'(0)}{z}.$$

In [10] the harmonic analysis associated with the Bessel-Struve operator in the space of entire functions $\mathscr{H}(\mathbb{C})$ was studied in the case $\alpha > 0$. Similar investigations can

Mathematics subject classification (2010): 47B38.
Keywords and phrases: Dunkl operator, Bessel-Struve operator, space of analytic functions, transmutation operator, diagonal operator, hypercyclic operator.

Paper OaM-11-71

one find in [11] for the space of analytic functions in circular domains. Further investigations of various properties of the Bessel-Struve operator in various functional spaces can be found in [12, 13, 15].

In this paper we define the differential-difference operator which generalizes the Dunkl operator and the Bessel-Struve operator. We study some properties of this differential-difference operator in the space of analytic functions in domains. Let m and p be arbitrary fixed positive integers, $\omega = \exp\left(\frac{2\pi i}{p}\right)$. Suppose G is an arbitrary starlike domain of $\mathbb{C}$ with respect to $z=0$ that is invariant under the rotation about the origin by the angle $\frac{2\pi}{p}$, i.e., $\omega G = G$. Note that a domain G of the complex plane is called p-symmetric with respect to the origin if $\omega G = G$.

For any complex numbers α_j, $j=\overline{0,p-1}$, by B we denote the operator which acts in the space $\mathscr{H}(G)$ according to the rule

$$(Bf)(z) = f^{(m)}(z) + \sum_{j=0}^{p-1} \alpha_j \frac{f^{(m-1)}(\omega^j z) - f^{(m-1)}(0)}{z}, \tag{1}$$

$z \in G\backslash\{0\}$, $(Bf)(0) = \left(1 + \sum\limits_{j=0}^{p-1} \alpha_j \omega^j\right) f^{(m)}(0)$. If G is a starlike domain of the complex plane with respect to the origin, then the operator B can be represented in the following form:

$$(Bf)(z) = f^{(m)}(z) + \sum_{j=0}^{p-1} \alpha_j \int_0^{\omega^j} f^{(m)}(tz)dt. \tag{2}$$

It follows from (2) that the operator B acts linearly and continuously in the space $\mathscr{H}(G)$.

If $m=1$, $p=2$ and $\alpha_1 = -\alpha_0$, then the operator B coincides with the Dunkl operator Λ_{α_0}. If $m=2$, $p=1$ or $m=p=2$, $\alpha_1 = 0$, then the operator B coincides with the Bessel-Struve operator l_{α_0}. Therefore, the operator B generalizes the Dunkl operator and the Bessel-Struve operator. The Dunkl operator and the Bessel-Struve operator are equivalent to the differentiation and the square of differentiation respectively in different functional spaces. This fact is significantly used in studying properties of the Dunkl operator and the Bessel-Struve operator.

2. Preliminaries

Recall that two linear continuous operators A and B acting in a topological vector space H are called equivalent if there exists an isomorphism S of H such that $SA = BS$. Herewith the operator S is called a transmutation of A into B. Transmutation operators were first introduced by Delsarte in [6] for studying of the equivalence of differential operators.

In order to prove the equivalence of the operators l_α and $\frac{d^2}{dz^2}$ the transmutation operator has been constructed in [10, 11] in the form of the infinite operator series. This method was introduced by J. Delsarte and J. L. Lions in [7] for the proof of the equivalence of differential operators of finite order in the space of entire functions. Therefore, such transmutation operators are called the Delsarte-Lions series. In order to prove the continuity and the isomorphicity of the Delsarte-Lions operator series in $\mathscr{H}(\mathbb{C})$ the possibility of expanding an arbitrary entire function in a power series was used significantly. The above-mentioned method for constructing of the transmutation operator for differential operators does not work for solving analogous problems in the space of analytic functions in non-circular domains G, since functions of the space $\mathscr{H}(G)$ cannot be expanded as a power series.

In this paper we study the conditions of equivalence of the operator B to the operator of m-fold differentiation in the space $\mathscr{H}(G)$. For the pair of these operators we construct the transmutation operator in the form of a diagonal operator. For this purpose, we shall study the general properties of diagonal operators in the space $\mathscr{H}(G)$. Results obtained in the process will be used to describe the commutant of the operator B. In addition, we establish the conditions of the hypercyclicity and the chaoticity of one class of operators.

3. Properties of diagonal operators induced by the Taylor coefficients of the generalized hypergeometric function

Let G be an arbitrary domain of the complex plane containing the origin. Then the formula $(\Delta f)(z)=\frac{f(z)-f(0)}{z}$, $z\neq 0$, $(\Delta f)(0)=f'(0)$ defines the Pommiez operator Δ. The operator Δ acts linearly and continuously in the space $\mathscr{H}(G)$.

If G is a starlike domain of the complex plane with respect to the origin, then by $\mathscr{J}$ denote the integration operator which acts linearly and continuously in the space $\mathscr{H}(G)$ according to the rule $(\mathscr{J}f)(z)=\int\limits_0^z f(t)dt$; the integration is carried out along the straight line segment joining the points 0 and z. By U_z and E denote the operator of multiplication by the independent variable and the identity operator respectively. By D denote the operator of differentiation. For any $a\in\mathbb{C}$, $n\in\mathbb{N}$ denote by $(a)_n$ the Pochhammer symbol, i.e., $(a)_n=a(a+1)\dots(a+n-1)$, $n\geqslant 1$, $(a)_0=1$.

LEMMA 1. *Let G be an arbitrary starlike domain of the complex plane, $a\in\mathbb{C}$. Then the diagonal operator P_a acting on powers of z according to the rule $P_a(z^n)=\frac{(a)_n}{n!}z^n$, $n=0,1,\dots$ can be extended to an operator of the class $\mathscr{L}(\mathscr{H}(G))$.*

Proof. Let m be an arbitrary fixed positive integer. For any $a\in\mathbb{C}$, $0<Re a<m$, by Q_a denote the operator acting according to the rule

$$(Q_af)(z)=\frac{1}{\Gamma(a)\Gamma(m-a)}\left(\prod_{j=1}^{m-1}(U_zD+jE)\right)\int\limits_0^1(1-t)^{m-a-1}t^{a-1}f(zt)dt.$$

Note that the operator Q_a belongs to the class $\mathscr{L}(\mathscr{H}(G))$. For any $n=0,1,\dots$ we have

$$(Q_a)(z^n) = \frac{1}{\Gamma(a)\Gamma(m-a)} \left(\prod_{j=1}^{m-1} (n+j) \right) \frac{\Gamma(m-a)\Gamma(n+a)}{\Gamma(n+m)} z^n = \frac{(a)_n}{n!} z^n.$$

Therefore, for all $0 < Rea < m$ the operator P_a can be extended to an operator from the class $\mathscr{L}(\mathscr{H}(G))$ by the following formula $P_a = Q_a$. By virtue of the arbitrariness of m we get that the operator P_a can be extended to an operator from the class $\mathscr{L}(\mathscr{H}(G))$ for all $a \in \mathbb{C}$, $Rea > 0$.

By δ_k denote the linear continuous functional acting on the space $\mathscr{H}(G)$ by the rule $\delta_k(f) = f^{(k)}(0)$, $k = 0, 1, \ldots$. Let $m \in \mathbb{N}$, $a \in \mathbb{C}$, $Rea > -m$. Then the formula

$$R_a = (a)_m \mathscr{J}^m P_{a+m} \Delta^m + \sum_{k=0}^{m-1} \frac{(a)_k}{(k!)^2} U_z^k \delta_k$$

defines the operator R_a of the class $\mathscr{L}(\mathscr{H}(G))$. If $n < m$, then we have

$$R_a(z^n) = \sum_{k=0}^{m-1} \frac{(a)_k}{(k!)^2} z^k \delta_k(z^n) = \frac{(a)_n}{n!} z^n.$$

If $n \geqslant m$, then

$$\begin{aligned} R_a(z^n) &= ((a)_m \mathscr{J}^m P_{a+m} \Delta^m)\, z^n = (a)_m (\mathscr{J}^m P_{a+m})(z^{n-m}) \\ &= (a)_m \frac{(a+m)_{n-m}}{(n-m)!} \mathscr{J}^m (z^{n-m}) = (a)_m \frac{(a+m)_{n-m}}{(n-m)!} \frac{(n-m)!}{n!} z^n = \frac{(a)_n}{n!} z^n. \end{aligned}$$

Thus, $P_a(z^n) = R_a(z^n)$, $n = 0, 1, \ldots$. Therefore, for all $a \in \mathbb{C}$, $Rea > -m$, the operator P_a can be extended to an operator of the class $\mathscr{L}(\mathscr{H}(G))$ by the rule $P_a = R_a$. Since m is an arbitrary fixed positive integer, the operator P_a can be extended to an operator of the class $\mathscr{L}(\mathscr{H}(G))$ for all $a \in \mathbb{C}$. $\square$

Henceforth throughout the paper by P_a, $a \in \mathbb{C}$, we denote the operator of the class $\mathscr{L}(\mathscr{H}(G))$ such that $P_a(z^n) = \frac{(a)_n}{n!} z^n$, $n = 0, 1, \ldots$.

LEMMA 2. *Suppose G is an arbitrary starlike domain of the complex plane with respect to the origin, $a \in \mathbb{C}$. The operator P_a is an isomorphism of the space $\mathscr{H}(G)$ if and only if*

$$a \neq -n,\ n = 0, 1, \ldots. \tag{3}$$

Proof. Necessity. Assume that the operator P_a is an isomorphism of the space $\mathscr{H}(G)$. Assume that (3) does not hold. Hence P_a has a non-trivial zero in the space $\mathscr{H}(G)$. We obtain a contradiction, since P_a is an isomorphism of the space $\mathscr{H}(G)$.

Sufficiency. Suppose that (3) holds. Let us choose an arbitrary $m \in \mathbb{N}$ such that $Rea > -m$. By $\tilde{P}_a$ denote the operator acting by the rule

$$(\tilde{P}_a f)(z) = \left(\prod_{j=0}^{m-1} (a+j) \right)^{-1} \left(\prod_{j=0}^{m} (U_z D + (a+j)E) \right) \int_0^1 (1-t)^{a+m-1} f(tz)\, dt.$$

The operator $\tilde{P}_a$ belongs to the class $\mathscr{L}(\mathscr{H}(G))$. For all $n=0,1,\dots$ we have

$$\tilde{P}_a(z^n) = \frac{\prod\limits_{j=0}^{m}(n+a+j)}{\prod\limits_{j=0}^{m-1}(a+j)} \frac{\Gamma(a+m)\Gamma(n+1)}{\Gamma(a+m+n+1)} z^n$$

$$= \frac{\prod\limits_{j=n}^{n+m}(a+j)}{\prod\limits_{j=0}^{m-1}(a+j)} \frac{\Gamma(a+m)n!}{\prod\limits_{j=m}^{m+n}(a+j)} \frac{z^n}{\Gamma(a+m)} = \frac{n!}{(a)_n} z^n.$$

Therefore, $(P_a\tilde{P}_a)(z^n) = (\tilde{P}_a P_a)(z^n) = z^n$ for all $n=0,1,\dots$. Since the system $(z^n)_{n=0}^{\infty}$ is complete in $\mathscr{H}(G)$, the operator P_a is an isomorphism of the space $\mathscr{H}(G)$ and $P_a^{-1} = \tilde{P}_a$. $\square$

THEOREM 1. *Suppose G is an arbitrary starlike domain of the complex plane with respect to the origin, $a,b \in \mathbb{C}$ and*

$$b \neq -n,\ n=0,1,\dots. \tag{4}$$

Then the operator $S_{a,b}$ acting on powers of z according to the rule

$$S_{a,b}(z^n) = \frac{(a)_n}{(b)_n} z^n,\ \ n=0,1,\dots,$$

can be extended to an operator of the class $\mathscr{L}(\mathscr{H}(G))$.

Proof. Since (4) holds, by Lemma 2 the operator P_b is an isomorphism of the space $\mathscr{H}(G)$. For all $n=0,1,\dots$ we have $S_{a,b}(z^n) = (P_a P_b^{-1})(z^n)$. Therefore, the operator $S_{a,b}$ can be extended to an operator of the class $\mathscr{L}(\mathscr{H}(G))$ by the rule $S_{a,b} = P_a P_b^{-1}$. $\square$

THEOREM 2. *Let G be an arbitrary starlike domain of the complex plane with respect to the origin, $a,b \in \mathbb{C}$. Suppose that conditions* (3), (4) *hold. Then the operator $S_{a,b}$ is an isomorphism of the space $\mathscr{H}(G)$.*

Proof. The operators P_a, P_b are isomorphisms of the space $\mathscr{H}(G)$. Since $S_{a,b} = P_a P_b^{-1}$, the proof is complete. $\square$

COROLLARY 1. *Suppose G is an arbitrary starlike domain of the complex plane with respect to the origin. Let l be a fixed positive integer, $a=(a_j)_{j=1}^{l}$, $b=(b_j)_{j=1}^{l} \in \mathbb{C}^l$, $a_j, b_j \neq -k$, $k=0,1,\dots$ for all $j=\overline{1,l}$. Then the operator $S_{a,b}$ acting on powers of z according to the rule*

$$S_{a,b} z^n = \frac{(a_1)_n (a_2)_n \dots (a_l)_n}{(b_1)_n (b_2)_n \dots (b_l)_n} z^n,\ \ n=0,1,\dots,$$

can be extended to an isomorphism of the class $\mathscr{L}(\mathscr{H}(G))$.

Proof. If $l=1$, then Corollary 1 coincides with Theorem 2. For all $l\in\mathbb{N}$ Lemma 1 is obtained by induction over l. □

Let G be an arbitrary starlike m-symmetric domain of the complex plane with respect to the origin. By $\mathscr{H}_k(G)$, $k=\overline{0,m-1}$, denote the closed subspaces of the space $\mathscr{H}(G)$ which are determined as follows:

$$\mathscr{H}_k(G)=\{f\in\mathscr{H}(G): f(\omega z)=\omega^k f(z), \forall z\in G\},$$

$\omega=\exp\frac{2\pi i}{m}$. It was shown in [8] that the space $\mathscr{H}(G)$ is a direct sum of subspaces $\mathscr{H}_k(G)$, $k=\overline{0,m-1}$, i.e.

$$\mathscr{H}(G)=\mathscr{H}_0(G)\oplus\mathscr{H}_1(G)\oplus\ldots\oplus\mathscr{H}_{m-1}(G).$$

An arbitrary $f\in\mathscr{H}(G)$ can be represented uniquely in the form $f(z)=\sum\limits_{j=0}^{m-1}f_j(z)$, $z\in G$, where $f_j\in\mathscr{H}_j(G)$, $j=\overline{0,m-1}$. Herewith

$$f_j(z)=(P_jf)(z)=\frac{1}{m}\sum_{l=0}^{m-1}\omega^{-jl}f(\omega^l z).$$

The operators P_j are projectors in the space $\mathscr{H}(G)$ and $\mathscr{H}_j(G)=P_j(\mathscr{H}(G))$, $j=\overline{0,m-1}$. Thus, an arbitrary function $f\in\mathscr{H}(G)$ can be represented in the form

$$f(z)=\sum_{j=0}^{m-1}(P_jf)(z). \tag{5}$$

By G^m denote the set $\{z^m : z\in G\}$. Clearly, the domain G^m is starlike with respect to the origin, since the domain G has the same property. It was shown in [8] that the subspace $\mathscr{H}_j(G)$, $j=\overline{0,m-1}$, is isomorphic to the space $\mathscr{H}(G^m)$. Herewith the operator U_j acting by the rule $(U_jg)(z)=z^jg(z^m)$, $j=\overline{0,m-1}$ is an isomorphism of $\mathscr{H}(G^m)$ on $\mathscr{H}_j(G)$. From the above assertions we deduce the following result.

LEMMA 3. *Let G be an arbitrary starlike m-symmetric domain of the complex plane with respect to the origin. Then an arbitrary function f of $\mathscr{H}(G)$ can be represented uniquely in the form*

$$f(z)=\sum_{j=0}^{m-1}z^jf_j(z^m), \tag{6}$$

$z\in G$, where $f_j\in\mathscr{H}(G^m)$, $j=\overline{0,m-1}$.

THEOREM 3. *Suppose m and l are positive integers. Let G be an arbitrary starlike m-symmetric domain of the complex plane with respect to the origin. Let $c^{(r)}, a_j^{(r)}, b_j^{(r)}$, $r=\overline{0,m-1}$, $j=\overline{1,l}$, be complex numbers such that $c^{(r)}\neq 0$, $a_j^{(r)}\neq -n$,*

$b_j^{(r)} \neq -n,\ n = 0,1,\ldots.$ *Then the operator* T *acting on powers of* z *according to the rule*

$$Tz^{mn+r} = c^{(r)} \frac{\left(a_1^{(r)}\right)_n \left(a_2^{(r)}\right)_n \cdots \left(a_l^{(r)}\right)_n}{\left(b_1^{(r)}\right)_n \left(b_2^{(r)}\right)_n \cdots \left(b_l^{(r)}\right)_n} z^{mn+r}, \tag{7}$$

$r = \overline{0,m-1},\ n = 0,1,\ldots,$ *can be extended to an isomorphism of the space* $\mathscr{H}(G)$.

Proof. Let us denote

$$\gamma_{mn+r} = c^{(r)} \frac{\left(a_1^{(r)}\right)_n \left(a_2^{(r)}\right)_n \cdots \left(a_l^{(r)}\right)_n}{\left(b_1^{(r)}\right)_n \left(b_2^{(r)}\right)_n \cdots \left(b_l^{(r)}\right)_n}, \tag{8}$$

$r = \overline{0,m-1},\ n = 0,1,\ldots.$ First, it will be proved that the operator T can be extended to an operator of the class $\mathscr{L}(\mathscr{H}(G))$.

For any $r = \overline{0,m-1}$ by T_r denote the operator acting on powers of z according to the rule $T_r(z^n) = \gamma_{mn+r} z^n,\ n = 0,1,\ldots.$ It follows from (8) and Corollary 1 that the operator $T_r,\ r = \overline{0,m-1}$, is an isomorphism of the space $\mathscr{H}(G^m)$. Let us choose an arbitrary function f of $\mathscr{H}(G)$. Let us represent the function f in the form (6), where $f_j \in \mathscr{H}(G^m),\ j = \overline{0,m-1}$. By $\tilde{T}$ denote the operator acting by the rule

$$(\tilde{T}f)(z) = \sum_{j=0}^{m-1} z^j \left(T_j f_j\right)(z^m),\ z \in G.$$

Since $T_j \in \mathscr{L}(\mathscr{H}(G^m))$ for all $j = \overline{0,m-1}$, the operator $\tilde{T}$ belongs to the space $\mathscr{L}(\mathscr{H}(G))$. For all $r = \overline{0,m-1},\ n = 0,1,\ldots$ we have $\tilde{T}z^{mn+r} = \gamma_{mn+r} z^{mn+r}$. Therefore, the operator $\tilde{T}$ is an extention of the operator T to the operator of the class $\mathscr{L}(\mathscr{H}(G))$. Note that the sequences $(\gamma_n)_{n=0}^{\infty}$ and $\left(\frac{1}{\gamma_n}\right)_{n=0}^{\infty}$ have the same structure. Hence, the operator T' acting on powers of z by the rule $T'z^n = \frac{1}{\gamma_n} z^n,\ n = 0,1,\ldots,$ can be extended to an operator from the class $\mathscr{L}(\mathscr{H}(G))$. For all $n = 0,1,\ldots$ we have $(TT')(z^n) = (T'T)(z^n) = z^n$. Since the system $\{z^n\}_{n=0}^{\infty}$ is complete in the space $\mathscr{H}(G)$, $TT' = T'T = E$. Therefore, the operator T' is inverse to the operator T. Thus, the operator T is an isomorphism of the space $\mathscr{H}(G)$. $\square$

4. Equivalence of the operator B to the power of the usual differentiation in the space $\mathscr{H}(G)$

If the operator B is defined by (1), then $B(z^n) = 0,\ n = \overline{0,m-1}$, and $B(z^n) = \frac{n!}{(n-m+1)!} \lambda_n z^{n-m}$,

$$\lambda_n = n - m + 1 + \sum_{j=0}^{p-1} \alpha_j \omega^{j(n-m+1)}, \tag{9}$$

$n = m, m+1, \ldots.$

THEOREM 4. *Suppose p and m are positive integers such that p is divisible by m. Let G be an arbitrary starlike p-symmetric domain of the complex plane with respect to the origin. The operator B is equivalent to the operator D^m in the space $\mathscr{H}(G)$ if and only if*

$$\lambda_n \neq 0,\ n = m, m+1, \ldots. \tag{10}$$

Proof. Necessity. Assume that the operator B is equivalent to the operator D^m in the space $\mathscr{H}(G)$. Since dimensions of kernels of equivalent operators are equal among themselves and $dim\,\mathrm{Ker}(D^m) = m$, $dim\,\mathrm{Ker}(B) = m$. It follows from (1) that $B(z^n) = 0$ for all $n = \overline{0, m-1}$. Therefore, $z^n \in \mathrm{Ker}(B)$, $n = \overline{0, m-1}$. Since $dim\,\mathrm{Ker}(B) = m$, $B(z^n) \neq 0$, $n \geqslant m$. Herewith $B(z^n) = \frac{n!}{(n-m+1)!}\lambda_n z^{n-m}$, $n \geqslant m$. Therefore, (10) holds.

Sufficiency. Assume that (10) holds. Let γ_r, $r = \overline{0, m-1}$, be arbitrary non-zero complex numbers. Let

$$\gamma_{mn+r} = \gamma_r \frac{\prod\limits_{\nu=0}^{n-1}(\nu m + r + 1)}{\prod\limits_{\nu=0}^{n-1}\lambda_{(\nu+1)m+r}}, \tag{11}$$

$n = 1, 2, \ldots$, $r = \overline{0, m-1}$. By M denote the operator acting on powers of z according to the rule

$$M z^{mn+r} = \gamma_{mn+r} z^{mn+r}, \tag{12}$$

$n = 0, 1, \ldots$, $r = \overline{0, m-1}$. Let us show that the operator M can be extended to an isomorphism of the space $\mathscr{H}(G)$. Let us transform the numbers γ_{pn+r}, where $r = \overline{0, p-1}$, $n = 0, 1, \ldots$. Since p is divisible by m, $p = ml$, where $l \in \mathbb{N}$. Let us represent r in the form $r = mq + r_1$, where $q = \overline{0, l-1}$, $r_1 = \overline{0, m-1}$. Then

$$\begin{aligned}
\gamma_{pn+r} &= \gamma_{m(ln+q)+r_1} = \gamma_{r_1} \frac{\prod\limits_{\nu=0}^{ln+q-1}(\nu m + r_1 + 1)}{\prod\limits_{\nu=0}^{ln+q-1}\lambda_{(\nu+1)m+r_1}} \\
&= \gamma_{r_1} \frac{\prod\limits_{\nu=0}^{q-1}(\nu m + r_1 + 1)\prod\limits_{\nu=q}^{ln+q-1}(\nu m + r_1 + 1)}{\prod\limits_{\nu=0}^{q-1}\lambda_{(\nu+1)m+r_1}\prod\limits_{\nu=q}^{ln+q-1}\lambda_{(\nu+1)m+r_1}} = c_r \frac{\prod\limits_{\mu=0}^{ln-1}((\mu+q)m + r_1 + 1)}{\prod\limits_{\mu=0}^{ln-1}\lambda_{(\mu+q+1)m+r_1}}
\end{aligned}$$

with $c^{(r)} = \gamma_{r_1}\prod\limits_{\nu=0}^{q-1}\frac{\nu m + r_1 + 1}{\lambda_{(\nu+1)m+r_1}}$, $r = mq + r_1$ and the product on an empty set of indices is equal 1. Thus,

$$\gamma_{pn+r} = c^{(r)} \frac{\prod\limits_{\mu=0}^{ln-1}((\mu+q)m + r_1 + 1)}{\prod\limits_{\mu=0}^{ln-1}\lambda_{(\mu+q+1)m+r_1}}. \tag{13}$$

Let us transform the numerator and denominator of the right-hand side of (13). Since $\prod_{\mu=0}^{ln-1} d_\mu = \prod_{s=0}^{l-1}\prod_{k=0}^{n-1} d_{kl+s}$,

$$\begin{aligned}\prod_{\mu=0}^{ln-1} \lambda_{(\mu+q+1)m+r_1} &= \prod_{s=0}^{l-1}\prod_{k=0}^{n-1} \lambda_{(kl+s+q+1)m+r_1}\\ &= \prod_{s=0}^{l-1}\prod_{k=0}^{n-1}\left(mkl+ms+mq+r_1+1+\sum_{j=0}^{ml-1}\alpha_j\omega^{j(ms+mq+r_1+1)}\right)\\ &= \prod_{s=0}^{l-1}\prod_{k=0}^{n-1}(mkl+\lambda_{m(s+q+1)+r_1}) = \prod_{s=0}^{l-1}(ml)^n\prod_{k=0}^{n-1}\left(k+\frac{\lambda_{m(s+q+1)+r_1}}{ml}\right)\\ &= \prod_{s=0}^{l-1}(ml)^n\left(\frac{\lambda_{m(s+q+1)+r_1}}{ml}\right)_n = (ml)^{nl}\prod_{s=0}^{l-1}\left(\frac{\lambda_{m(s+q+1)+r_1}}{ml}\right)_n.\end{aligned}$$

Similarly,

$$\begin{aligned}\prod_{\nu=0}^{ln-1}((\mu+q)m+r_1+1) &= \prod_{s=0}^{l-1}\prod_{k=0}^{n-1}((kl+s+q)m+r_1+1)\\ &= \prod_{s=0}^{l-1}(ml)^n\prod_{k=0}^{n-1}\left(k+\frac{ms+mq+r_1+1}{ml}\right)\\ &= \prod_{s=0}^{l-1}(ml)^n\left(\frac{ms+mq+r_1+1}{ml}\right)_n\\ &= (ml)^{nl}\prod_{s=0}^{l-1}\left(\frac{ms+mq+r_1+1}{ml}\right)_n.\end{aligned}$$

Denote $a_s^{(r)} = \frac{ms+mq+r_1+1}{ml}$, $b_s^{(r)} = \frac{\lambda_{m(s+q+1)+r_1}}{ml}$. Then (13) can be represented in the following form:

$$\gamma_{pn+r} = c^{(r)}\prod_{s=0}^{l-1}\frac{\left(a_s^{(r)}\right)_n}{\left(b_s^{(r)}\right)_n}.$$

It follows from (10) that $c^{(r)} \neq 0$, $a_s^{(r)} \neq -n$, $b_s^{(r)} \neq -n$, $r = \overline{0,p-1}$, $s = \overline{0,l-1}$, $n = 0,1,\ldots$. Then by virtue of Theorem 3 the operator M can be extended to an isomorphism of the space $\mathscr{H}(G)$.

For all $n = 1,2,\ldots$, $r = \overline{0,m-1}$ we have

$$\begin{aligned}(BM)(z^{mn+r}) &= \frac{(mn+r)!}{(m(n-1)+r+1)!}\lambda_{mn+r}\gamma_{mn+r}z^{m(n-1)+r}\\ &= \frac{(mn+r)!}{(m(n-1)+r)!}\gamma_{m(n-1)+r}z^{m(n-1)+r} = (MD^m)(z^{mn+r}).\end{aligned}$$

Hence, $(BM)(z^n) = (MD^m)(z^n)$, $n = 0,1,\ldots$. Since the system $(z^n)_{n=0}^\infty$ is complete in the space $\mathscr{H}(G)$ and $\{B,M,D^m\} \subset \mathscr{L}(\mathscr{H}(G))$, the above equalities imply that

$BM = MD^m$. Therefore, the operator B is equivalent to the operator D^m in the space $\mathscr{H}(G)$. $\square$

The proof of sufficient conditions of Theorem 4 implies the following result.

COROLLARY 2. *Let p and m be arbitrary integers such that p is divisible by m. Let G be an arbitrary starlike p-symmetric domain of the complex plane with respect to the origin. Suppose α_j, $j = \overline{0, p-1}$ are complex numbers, the numbers λ_n, $n = m, m+1, \dots$ are defined by (9) and (10) holds. Then the operator M acting on powers of z according to the rule (12) can be extended to an isomorphism of the space $\mathscr{H}(G)$, where γ_{mn+r}, $n = 1, 2, \dots$, $r = \overline{0, m-1}$ are defined by (11), γ_r, $r = \overline{0, m-1}$ are complex numbers.*

For the Dunkl operator Λ_α, $\alpha \in \mathbb{C}$, we have $\lambda_n = n + \alpha\left(1 + (-1)^{n+1}\right)$, $n = 1, 2, \dots$. Therefore, Theorem 4 implies that if G is a starlike symmetric domain of the complex plane with respect to the origin, then the Dunkl operator Λ_α is equivalent to the operator D in the space $\mathscr{H}(G)$ if and only if

$$\alpha \neq \frac{1-2k}{2}, \; k = 1, 2, \dots.$$

In particular, this allows to find the result obtained in [10] by Gasmi and Sifi in the space $\mathscr{H}(\mathbb{C})$ since they deal with $\alpha > 0$ (proposition 2.1 and remark 2.2).

For the Bessel-Struve operator l_α, $\alpha \in \mathbb{C}$, we have $\lambda_n = n - 1 + \alpha$, $n = 2, 3, \dots$. Therefore, Theorem 4 implies that if G is a starlike symmetric domain of the complex plane with respect to the origin, then the Bessel-Struve operator l_α is equivalent to the operator D^2 in the space $\mathscr{H}(G)$ if and only if

$$\alpha \neq -n, \; n = 1, 2, \dots.$$

In particular, this allows to find the result obtained in Theorem 2.2 [16] by Ben Salem and Kallel in the space $\mathscr{H}(\mathbb{C})$ since they deal with $\alpha > 0$ (proposition 2.1 and remark 2.2).

5. Commutant of the operator B

Let $(\beta_n)_{n=0}^\infty$ be a sequence of non-zero complex numbers. The operators of generalized differentiation D_β and generalized integration $\mathscr{J}_\beta$ induced by the sequence $(\beta_n)_{n=0}^\infty$ act on powers of z according to the rules $D_\beta(z^n) = \frac{\beta_{n-1}}{\beta_n} z^{n-1}$, $n = 1, 2, \dots$, $D_\beta(1) = 0$, $\mathscr{J}_\beta(z^n) = \frac{\beta_{n+1}}{\beta_n} z^{n+1}$, $n = 0, 1, \dots$ respectively.

LEMMA 4. *Let p and m be arbitrary integers such that p is divisible by m. Let G be an arbitrary starlike p-symmetric domain of the complex plane with respect to the origin. Let α_j, $j = \overline{0, p-1}$ be complex numbers. Suppose that the numbers λ_n, $n = m, m+1, \dots$, are defined by (9) and (10) holds. Suppose that the numbers γ_{mn+r}, $n = 1, 2, \dots$, $r = \overline{0, m-1}$, are defined by (11), where γ_r, $r = \overline{0, m-1}$ are non-zero* *complex numbers. Then the operators* D_β *and* $\mathscr{J}_\beta$ *induced by the sequence* $\beta_n = \frac{\gamma_n}{n!}$, $n = 0,1,\ldots$ *can be extended to operators of the class* $\mathscr{L}(\mathscr{H}(G))$ *by the following formulas*

$$D_\beta = MDM^{-1}, \ \mathscr{J}_\beta = M\mathscr{J}M^{-1}, \tag{14}$$

where the operator M *is defined by Corollary 2.*

Proof. For all $n = 0,1,\ldots$ we have

$$D_\beta(z^n) = (MDM^{-1})(z^n), \quad \mathscr{J}_\beta(z^n) = (M\mathscr{J}M^{-1})(z^n).$$

Therefore, the operators D_β and $\mathscr{J}_\beta$ can be extended to operators of the class $\mathscr{L}(\mathscr{H}(G))$ by formulas (14). $\square$

COROLLARY 3. *Under the conditions of Lemma 4 the equality* $B = D_\beta^m$ *holds.*

Proof. It follows from Theorem 4 and Lemma 4 that $B = MD^mM^{-1}$, $D_\beta = MDM^{-1}$. Therefore, $B = (MDM^{-1})^m = D_\beta^m$. $\square$

COROLLARY 4. *Under the conditions of Lemma 4 the operator* D_β *is equivalent to the operator* D *and the operator* $\mathscr{J}_\beta$ *is equivalent to the operator* $\mathscr{J}$ *in the space* $\mathscr{H}(G)$.

Let us describe the commutant of the operator B in the space $\mathscr{H}(G)$, i.e., we find all operators $T \in \mathscr{L}(\mathscr{H}(G))$ such that $TB = BT$. For this purpose, we need the description of the commutant of the operator D^m in the space $\mathscr{H}(G)$. It was shown in [17] that if G is a bounded starlike m-symmetric domain of the complex plane with respect to the origin, then the general form of operators T_1 of the class $\mathscr{L}(\mathscr{H}(G))$ such that $T_1D^m = D^mT_1$ can be represented by the formula

$$T_1 = \sum_{r=0}^{m-1} P_r \mathscr{J}^r \sum_{j=0}^{\infty} c_j^{(r)} D^j, \tag{15}$$

where $(P_rf)(z) = \frac{1}{m}\sum\limits_{l=0}^{m-1} \omega^{-rl} f(\omega^l z)$, $\omega = \exp\frac{2\pi i}{m}$. Herewith the following condition

$$\lim_{j\to\infty} \sqrt[j]{j!|c_j^{(r)}|} = 0, \ r = \overline{0,m-1} \tag{16}$$

holds.

THEOREM 5. *Let* p *and* m *be arbitrary positive integers such that* p *is divisible by* m. *Let* G *be an arbitrary bounded starlike* p*-symmetric domain of the complex plane with respect to the origin. Let* α_j, $j = \overline{0,p-1}$, *be complex numbers. Suppose that the numbers* λ_n, $n = m, m+1, \ldots$, *are defined by* (9) *and* (10) *holds. The operator*

T acts linearly and continuously in the space $\mathscr{H}(G)$ and commutes with the operator B if and only if

$$T=\sum_{r=0}^{m-1} P_r \mathscr{J}_\beta^r \sum_{j=0}^{\infty} c_j^{(r)} D_\beta^j, \tag{17}$$

where $c_j^{(r)}$ are complex numbers such that (16) holds.

Proof. By virtue of Theorem 4 the operator B in the space $\mathscr{H}(G)$ is equivalent to the operator D^m and $BM=MD^m$, where M is an isomorphism of the space $\mathscr{H}(G)$ constructed in Theorem 4 (see also Corollary 2). Therefore, the operator T of the class $\mathscr{L}(\mathscr{H}(G))$ commutes with B if and only if $T=MT_1M^{-1}$, where T_1 is an operator of the class $\mathscr{L}(\mathscr{H}(G))$ such that $T_1D^m=D^mT_1$. The general form of linear continuous operators $T_1\in\mathscr{L}(\mathscr{H}(G))$ such that $T_1D^m=D^mT_1$ is described by formula (15).

Since equalities (14) hold and the operators P_r commute with M, an operator $T\in\mathscr{L}(\mathscr{H}(G))$ commutes with B if and only if

$$\begin{aligned} T &= MT_1M^{-1}=M\left(\sum_{r=0}^{m-1} P_r \mathscr{J}^r \sum_{j=0}^{\infty} c_j^{(r)} D^j\right)M^{-1}\\ &=\sum_{r=0}^{m-1} P_r (M\mathscr{J}M^{-1})^r \sum_{j=0}^{\infty} c_j^{(r)} (MDM^{-1})^j=\sum_{r=0}^{m-1} P_r \mathscr{J}_\beta^r \sum_{j=0}^{\infty} c_j^{(r)} D_\beta^j, \end{aligned}$$

where $c_j^{(r)}$, $j=0,1,\dots$, $r=\overline{0,m-1}$, are complex numbers such that (16) holds. □

6. Hypercyclicity and chaoticity of one class of operators

We begin with some definitions and concepts of the theory of dynamical systems [1]. Let A be an operator of the class $\mathscr{L}(\mathscr{H}(G))$. A function $f\in\mathscr{H}(G)$ is called a hypercyclic element of the operator A if the system of functions $(A^n f)_{n=0}^{\infty}$ is dense in the space $\mathscr{H}(G)$. An operator A is called hypercyclic if A has a hypercyclic element. A function f of $\mathscr{H}(G)$ is called a periodic element of A if there exists a positive integer n such that the equality $A^n f=f$ holds. An operator A is called chaotic if A has the set of periodic elements which is dense in $\mathscr{H}(G)$.

THEOREM 6. *Let p and m be arbitrary positive integers such that p is divisible by m. Let G be an arbitrary bounded starlike p-symmetric domain of the complex plane with respect to the origin. Suppose α_j, $j=\overline{0,p-1}$ are complex numbers. Suppose that the numbers λ_n, $n=m,m+1,\dots$ are defined by (9) and (10) holds. If $(c_n)_{n=0}^{\infty}$ is a sequence of complex numbers such that $\lim\limits_{n\to\infty}\sqrt[n]{n!|c_n|}=0$ and there exists at least one $n\geqslant 1$ such that $c_n\neq 0$, then the operator $T=\sum\limits_{n=0}^{\infty} c_nD_\beta^n$ is hypercyclic and chaotic in the space $\mathscr{H}(G)$.*

Proof. By virtue of Lemma 4 the operator of generalized differentiation D_β is equivalent to D in the space $\mathscr{H}(G)$, $D_\beta = MDM^{-1}$, where M is the isomorphism of $\mathscr{H}(G)$ defined in Corollary 2. Therefore,

$$T = \sum_{n=0}^{\infty} c_n D_\beta^n = M\left(\sum_{n=0}^{\infty} c_n D^n\right) M^{-1} = MT_1M^{-1}.$$

But, the operator $T_1 = \sum_{n=0}^{\infty} c_n D^n$ is hypercyclic and chaotic in the space $\mathscr{H}(G)$, since the operator T_1 commutes with D and T_1 differs from the scalar operator [5]. It follows from the equality $T = MT_1M^{-1}$ that the operator T is hypercyclic and chaotic in the space $\mathscr{H}(G)$. $\square$

From Theorem 6 and Corollary 3 we deduce the following result.

COROLLARY 5. *Under the conditions of Theorem 6 the differential-difference operator B is hypercyclic and chaotic in the space $\mathscr{H}(G)$.*

Acknowledgement. The authors are grateful to the referee's for his valuable comments.

REFERENCES

[1] F. BAYART, E. MATHERON, *Dynamics of linear operator*, Cambridge Tracts in Mathematics, **179**, Cambridge University Press, Cambridge, 2009.

[2] N. BEN SALEM, S. KALLEL, *Integro-differential-difference equations associated with the Dunkl operator and entire functions*, Comment. Math. Univ. Carolinae. **45**, 4 (2004), 699–725.

[3] L. BERNAL-GONZALEZ, A. BONILLA, *Rate of growth of hypercyclic and frequently hypercyclic functions for the Dunkl Operator*, Mediterr. J. Math. **13**, 5 (2016), 3359–3372.

[4] J. J. BETANCOR, M. SIFI, K. TRIMECHE, *Hypercyclic and chaotic convolution operators associated with the Dunkl operator on $\mathbb{C}$*, Acta Math. Hung. **106**, (1–2) (2005), 101–116.

[5] A. V. BRATISHCHEV, *Chaotic commutating with the Dunkl differentiation linear operators on the space of analytic functions*, Vestnik of DSTU. **9**, 2 (2009), 196–207, (in Russian).

[6] J. DELSARTE, *Sur certaines transformations fonctionelles relative aux equations linearines aux derives partielles du second ordre*, C. R. Math. Acad. Sci. Paris. **206**, (1938), 178–182.

[7] J. DELSARTE, J. L. LIONS, *Transmutations d'operateurs differentieles dans le domaine complexe*, Comm. Math. Helv. **32**, 2 (1957), 113–128.

[8] I. H. DIMOVSKI, *Convolutional Calculus*, Kluwer, Dordrecht, 1990.

[9] C. F. DUNKL, *Differential difference operators associated to reflection groups*, Trans. Amer. Math. Soc. **311**, (1989), 167–183.

[10] A. GASMI, M. SIFI, *The Bessel-Struve intertwining operator on $\mathbb{C}$ and mean-periodic functions*, Int. J. Math. Sci. **59**, (2004), 3171–3185.

[11] A. GASMI, M. SIFI, *Analytic mean-periodic functions associated with the Bessel-Struve operator on a disk*, Global J. Pure and Appl. Math. **1**, 1 (2005), 55–68.

[12] S. HAMEM, L. KAMOUN, S. NEGZAOUI, *Cowling-Price type theorem related to Bessel-Struve transform*, Arab J. Math. Sci. **19**, 2 (2013), 187–198.

[13] L. KAMOUN, M. SIFI, *Bessel-Struve intertwining operator and generalized Taylor series on the real line*, Integral Transforms and Spec. Funct. **16**, 1 (2005), 39–55.

[14] YU. S. LINCHUK, *Generalized Dunkl-Opdam operator and its properties in the spaces of functions analytic in domains*, J. Math. Sci. **220**, 1 (2017), 1–14.

[15] S. NEGZAOUI, *Beurling-Hormander's theorem related to Bessel-Struve transform*, Integral Transforms Spec. Funct. **27**, 9 (2016), 685–697.

[16] N. B. SALEM, S. KALLEL, *Mean-periodic functions associated with the Dunkl operators*, Integral Transforms Spec. Funct. **15**, 2 (2004), 155–179.

[17] M. YU. TSAR'KOV, *Isomorphismen einiger analytischer Raume, welche mit dem Grad des Differentiations operators kommutieren*, Teor. Funkts., Funkts. Anal. Prilozh. **11**, (1970), 86–92.

(Received February 10, 2017)

Stepan Linchuk
Department of Mathematical Analysis
Chernivtsi National University
Kotsjubyns'koho 2, Chernivtsi 58012, Ukraine
e-mail: nesslin.new@gmail.com

Yuriy Linchuk
Department of Mathematical Analysis
Chernivtsi National University
Kotsjubyns'koho 2, Chernivtsi 58012, Ukraine
e-mail: yustlin@gmail.com