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SOME PROPERTIES OF G–FRAMES FOR HILBERT SPACE OPERATORS

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Abstract. In this paper, we present some new characterizations of g-frames for a bounded operator. We discuss the g-frame for an operator K which has closed range and give a necessary and sufficient conditions for a family of bounded operators to be a K-g-frame. We also give a characterization of the dual for a K-g-frame. Moreover, We use quotient operators to characterize K-g-frames and find that the results on K-g-frames can be proved by theory of quotient operators.

1. Introduction

Frames were first introduced by Duffin and Schaeffer [7] in the study of nonharmonic Fourier series, and reintroduced in 1986 by Daubechies, et al. [5] and popularized from then on. Frames have established themselves by now as a standard notion in applied mathematics and engineering. Nice properties of frames have made them useful in filter bank theory [14], compressed sensing [3], coding theory [15, 16], probability statistics [8, 18], and signal and image processing [11].

Sun in [21] introduced the concept of g-frame in Hilbert space. G-frames are natural generalizations of frames which cover many other recent generalizations of frames, such as bounded quasi-projections [9], fusion frames [2] and pseudo-frames [17]. The author in [1] gave the notion of K-g-frame. It shows that K-g-frames possess higher generality than g-frames in the sense that the lower frame bound condition holds only for the elements in the range of K. Hence K-g-frames provide more flexibility and thus make the study of them interesting.

There are some important results on K-frames which have established by [10]. It shows that may properties for ordinary frames may not hold for K-frames, such as the corresponding synthesis operator for K-frames is not surjective, the frame operator for K-frames is not invertible and so on. We refer reader to [19, 20, 22] for more details about the results of K-frames. Since the structure of g-frame is more complicated than that of ordinary frame, it is necessary to generalize some of the known results in K-frames to K-g-frames. Moreover, we give some new characterizations of K-g-frames by the range of K.

Throughout the paper, $\mathscr{H}$ and $\mathscr{K}$ are two Hilbert spaces and $\{\mathscr{H}_i\}_{i\in I}$ is a sequence of closed subspaces of $\mathscr{K}$ where I is a subset of $\mathbb{Z}$ and $\mathscr{L}(\mathscr{H},\mathscr{H}_i)$ is the

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collection of all bounded linear operators from $\mathscr{H}$ into $\mathscr{H}_i$. For $T \in \mathscr{L}(\mathscr{H})$, we denote $\mathscr{D}(T)$, $\mathscr{R}(T)$ and $\mathscr{N}(T)$ for domain, range and ker of T, respectively. And we denote by $I_{\mathscr{H}}$ the identity operator on $\mathscr{H}$.

We need recall some basic definitions and results of g-frames and K-g-frames in Hilbert space.

We call a sequence $\{\Lambda_i\}_{i\in I}$ a generalized frame, or simply a g-frame, for $\mathscr{H}$ with respect to $\{\mathscr{H}_i\}_{i\in I}$ if there are two positive constants A and B such that

$$A\|f\|^2 \leqslant \sum_{i\in I}\|\Lambda_i f\|^2 \leqslant B\|f\|^2, \quad \forall f \in \mathscr{H}.$$

We call A and B the lower and upper frame bounds, respectively. If $\{\Lambda_i\}_{i\in I}$ possesses an upper frame bound, but not necessarily a lower bound, we call it a Bessel g-sequence with Bessel bound B.

We say $\{\Lambda_i\}_{i\in I}$ a g-frame sequence, if it is a g-frame for $\overline{\operatorname{span}}\{\Lambda_i^*(\mathscr{H}_i)\}_{i\in I}$.

Now define

$$\left(\sum_{i\in I}\oplus\mathscr{H}_i\right)_{\ell^2} := \left\{\{f_i\}_{i\in I} | f_i \in \mathscr{H}_i,\ \|\{f_i\}_{i\in I}\|_2^2 = \sum_{i\in I}\|f_i\|^2 < \infty\right\},$$

with pointwise operators and inner product as

$$\langle\{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle = \sum_{i\in I}\langle f_i, g_i\rangle.$$

In [21], Sun showed that every g-frame can be considered as a frame. More precisely, let $\{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathscr{H}$ and $\{e_{i,j}\}_{j\in J_i}$ be an orthonormal basis for $\mathscr{H}_i$, then there exists a frame $\{u_{i,j}\}_{i\in I, j\in J_i}$ of $\mathscr{H}$ such that

$$u_{i,j} = \Lambda_i^* e_{i,j}, \tag{1}$$

and

$$\Lambda_i f = \sum_{j\in J_i}\langle f, u_{i,j}\rangle e_{i,j}, \quad \forall f \in \mathscr{H}.$$

We call $\{u_{i,j}\}_{i\in I, j\in J_i}$ the frame induced by $\{\Lambda_i\}_{i\in I}$ with respect to $\{e_{i,j}\}_{i\in I, j\in J_i}$. The next lemma is a characterization of g-frame by a frame.

LEMMA 1. [21] *Let* $\{\Lambda_i\}_{i\in I}$ *be a family of linear operators and* $u_{i,j}$ *be defined as in* (1). *Then* $\{\Lambda_i\}_{i\in I}$ *is a g-frame for* $\mathscr{H}$ *if and only if* $\{u_{i,j}\}_{i\in I, j\in J_i}$ *is a frame for* $\mathscr{H}$.

DEFINITION 1. Let $K \in \mathscr{L}(\mathscr{H})$. We call a sequence $\{\Lambda_i\}_{i\in I}$ a K-g-frame for $\mathscr{H}$ with respect to $\{\mathscr{H}_i\}_{i\in I}$ if there are two positive constants A and B such that

$$A\|K^* f\|^2 \leqslant \sum_{i\in I}\|\Lambda_i f\|^2 \leqslant B\|f\|^2, \quad \forall f \in \mathscr{H}.$$

We call A and B the lower and upper frame bounds, respectively.

We call $\{\Lambda_i\}_{i\in I}$ a tight K-g-frame if $A\|K^* f\|^2 = \sum_{i\in I}\|\Lambda_i f\|^2$, for every $f \in \mathscr{H}$. If we have only the second inequality, we call it a K-Bessel g-sequence. In [1], the synthesis operator of $\{\Lambda_i\}_{i\in I}$ is defined by

$$T_\Lambda : \left(\sum_{i\in I}\oplus\mathscr{H}_i\right)_{\ell^2} \longrightarrow \mathscr{H}, \quad T_\Lambda(\{f_i\}_{i\in I}) = \sum_{i\in I}\Lambda_i^* f_i,$$

and the K-g-frame operator S_Λ is defined as follows

$$S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{i\in I} \Lambda_i^* \Lambda_i f, \quad \forall f \in \mathscr{H}.$$

The following theorem is a characterization of K-g-frames in [1].

THEOREM 1. *Let $K \in \mathscr{L}(\mathscr{H})$. Then the following statements are equivalent:*

1. *$\{\Lambda_i\}_{i\in I}$ is a K-g-frame;*
2. *$\{\Lambda_i\}_{i\in I}$ is a Bessel g-sequence for $\mathscr{H}$ and there exists a Bessel g-sequence $\{\Gamma_i\}_{i\in I}$ for $\mathscr{H}$ such that*

$$Kf = \sum_{i\in I} \Lambda_i^* \Gamma_i f, \quad \forall f \in \mathscr{H}.$$

The following lemmas are fundamental results in the study of the K-g-frames.

LEMMA 2. [6] *Let $U, V \in \mathscr{L}(\mathscr{H})$. The following statements are equivalent:*

(i) *$\mathscr{R}(U) \subset \mathscr{R}(V)$.*

(ii) *$UU^* \leqslant \lambda VV^*$ for some $\lambda \geqslant 0$.*

(iii) *There exists $Q \in \mathscr{L}(\mathscr{H})$ such that $U = VQ$.*

Moreover, if (i), (ii) and (iii) are valid, then there exists a unique operator Q such that

1. *$\|Q\|^2 = \inf\{\mu : UU^* \leqslant \mu VV^*\}$,*
2. *$\mathscr{N}(U) = \mathscr{N}(Q)$, and*
3. *$\mathscr{R}(Q) \subset \overline{\mathscr{R}(V^*)}$.*

LEMMA 3. [4] *Let $\mathscr{H}$ be a Hilbert space, and suppose that $T \in \mathscr{L}(\mathscr{H})$ has a closed range. Then there exists an operator $T^\dagger \in \mathscr{L}(\mathscr{H})$ for which*

$$\mathscr{N}(T^\dagger) = \mathscr{R}(T)^\perp, \ \ \mathscr{R}(T^\dagger) = \mathscr{N}(T)^\perp, \ \ TT^\dagger f = f, \ \ f \in \mathscr{R}(T).$$

We call the operator $T^\dagger$ the pseudo-inverse of T.

2. Characterizations of K-g-frames

In this section, we give some equivalent characterizations of K-g-frames. First we need the following lemma which is a generalization of Theorem 3.5 in [22].

LEMMA 4. *Let $\{\Lambda_i\}_{i\in I}$ be a Bessel g-sequence for $\mathscr{H}$ with frame operator S_Λ. Then $\{\Lambda_i\}_{i\in I}$ is a K-g-frame if and only if there exists $\lambda > 0$ such that $S_\Lambda \geqslant \lambda KK^*$.*

Proof. $\{\Lambda_i\}_{i\in I}$ is K-g-frame with frame bounds A,B and frame operator S_Λ if and only if

$$A\|K^*f\|^2 \leqslant \sum_{i\in I}\|\Lambda_i f\|^2 = \langle S_\Lambda f, f\rangle \leqslant B\|f\|^2, \quad \forall f\in \mathscr{H},$$

that is,

$$\langle AKK^*f,f\rangle \leqslant \langle S_\Lambda f,f\rangle \leqslant \langle Bf,f\rangle, \quad \forall f\in \mathscr{H}.$$

So the conclusion holds. □

The following theorem shows that every Bessel g-sequence can be a K-g-frame.

THEOREM 2. *Let $\{\Lambda_i\}_{i\in I}$ be a Bessel g-sequence for $\mathscr{H}$ with frame operator S_Λ. Then $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathscr{H}$ if and only if $K = S_\Lambda^{1/2}T$, for some $T\in \mathscr{L}(\mathscr{H})$.*

Proof. By Lemma 4, $\{\Lambda_i\}_{i\in I}$ is a K-g-frame if and only if there exists $\lambda>0$ such that

$$\lambda KK^* \leqslant S_\Lambda = S_\Lambda^{1/2}(S_\Lambda^{1/2})^*.$$

Therefore by Lemma 2 the conclusion hold. □

COROLLARY 1. *Let $\{\Lambda_i\}_{i\in I}$ be a sequence of bounded operators for $\mathscr{H}$ and u_{ij} be defined as in (1). Let $K\in \mathscr{L}(\mathscr{H})$, then $\{\Lambda_i\}_{i\in I}$ is a K-g-frame if and only if $\{u_{i,j}\}_{i\in I, j\in J_i}$ is a K-frame for $\mathscr{H}$.*

The following proposition gives a condition for a Bessel g-sequence to be a K-g-frame as well as other operator T.

PROPOSITION 1. *Let $K\in \mathscr{L}(\mathscr{H})$ and $\{\Lambda_i\}_{i\in I}$ be a K-g-frame for $\mathscr{H}$. Let $T\in \mathscr{L}(\mathscr{H})$ with $\mathscr{R}(T)\subset \mathscr{R}(K)$, then $\{\Lambda_i\}_{i\in I}$ is a T-g-frame for $\mathscr{H}$.*

Proof. Suppose that $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathscr{H}$. Then there exist $0<A\leqslant B<\infty$ such that

$$A\|K^*f\|^2 \leqslant \sum_{i\in I}\|\Lambda_i f\|^2 \leqslant B\|f\|^2, \quad \forall f\in \mathscr{H}. \tag{2}$$

Since $\mathscr{R}(T)\subset \mathscr{R}(K)$, by Lemma 4, there exists $\lambda>0$ such that $TT^*\leqslant \lambda KK^*$. Then $\frac{1}{\lambda}\|T^*f\|^2 \leqslant \|K^*f\|^2$. From (2), we have

$$\frac{A}{\lambda}\|T^*f\|^2 \leqslant A\|K^*f\|^2 \leqslant \sum_{i\in I}\|\Lambda_i f\|^2 \leqslant B\|f\|^2, \quad \forall f\in \mathscr{H}.$$

Hence $\{\Lambda_i\}_{i\in I}$ is a T-g-frame for $\mathscr{H}$. □

Let $K=I$, we have the following corollary.

COROLLARY 2. *Let $\{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathscr{H}$. Let $K\in \mathscr{L}(\mathscr{H})$, then $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathscr{H}$.*

Proof. In fact, $\{\Lambda_i\}_{i\in I}$ can be viewed as an $I_{\mathscr{H}}$-g-frame for $\mathscr{H}$. Since $\mathscr{R}(K) \subset \mathscr{R}(I_{\mathscr{H}})$, by proposition 1, the conclusion is hold. □

The Corollary 2 is equivalent to Theorem 2.3 in [1].

Now we characterize K-g-frame in terms of range of K.

THEOREM 3. *Let $K \in \mathscr{L}(\mathscr{H})$ with closed range and $\overline{\text{span}}\{\Lambda_i^*(\mathscr{H}_i)\}_{i\in I} \subset \mathscr{R}(K)$. Then $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathscr{H}$ if and only if $\{\Lambda_i\}_{i\in I}$ is a g-frame on $\mathscr{R}(K)$.*

Proof. Suppose that $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathscr{H}$, then there exist $0 < A \leqslant B < \infty$ such that

$$A\|K^*f\|^2 \leqslant \sum_{i\in I}\|\Lambda_i f\|^2 \leqslant B\|f\|^2, \quad \forall f \in \mathscr{H}.$$

Thus for all $f \in \mathscr{H}$, we have

$$\|K^*f\|^2 \leqslant \frac{1}{A}\sum_{i\in I}\|\Lambda_i f\|^2.$$

Since $\mathscr{R}(K)$ is closed, by Lemma 3, there exists $K^\dagger$ of K such that $f = KK^\dagger f$, $\forall f \in \mathscr{R}(K)$. And then for all $f \in \mathscr{R}(K)$,

$$\|f\|^4 = |\langle KK^\dagger f, f\rangle|^2 = |\langle K^\dagger f, K^*f\rangle|^2 \leqslant \|K^\dagger f\|^2\|K^*f\|^2 \leqslant \|K^\dagger\|^2\|f\|^2\frac{1}{A}\sum_{i\in I}\|\Lambda_i f\|^2.$$

Hence

$$\frac{A}{\|K^\dagger\|^2}\|f\|^2 \leqslant \sum_{i\in I}\|\Lambda_i f\|^2 \leqslant B\|f\|^2, \quad \forall f \in \mathscr{R}(K).$$

Therefore, $\{\Lambda_i\}_{i\in I}$ is a g-frame on $\mathscr{R}(K)$.

Conversely, suppose that $\{\Lambda_i\}_{i\in I}$ is a g-frame on $\mathscr{R}(K)$, then there exist $0 < C \leqslant D < \infty$ such that

$$C\|f\|^2 \leqslant \sum_{i\in I}\|\Lambda_i f\|^2 \leqslant D\|f\|^2, \quad \forall f \in \mathscr{R}(K).$$

Clearly, $\{\Lambda_i\}_{i\in I}$ is a Bessel g-sequence for $\mathscr{H}$. Now we prove that $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathscr{H}$. We show that $\{\Lambda_i\}_{i\in I}$ has a lower bound for all $f \in \mathscr{H}$. Suppose that $\|K^*\| = \|K\| \neq 0$, for $f \in \mathscr{H}$, we have

$$\|K^*f\| \leqslant \|K^*\|\|f\| = \|K\|\|f\|,$$

and then $\|f\| \geqslant \frac{1}{\|K\|}\|K^*f\|$, for all $f \in \mathscr{H}$. Since $\mathscr{R}(K)$ is closed, we have $\mathscr{H} = \mathscr{R}(K)\bigoplus\mathscr{R}(K)^\perp$. For any $f \in \mathscr{H}$, let $f = f_1 + f_2$, where $f_1 \in \mathscr{R}(K)$ and $f_2 \in \mathscr{R}(K)^\perp$. So

$$\sum_{i\in I}\|\Lambda_i f_1\|^2 \geqslant C\|f_1\|^2 \geqslant \frac{C}{\|K\|^2}\|K^*f_1\|^2.$$

Since $f_2 \in \mathscr{R}(K)^\perp$, $K^*f_2 = 0$ and

$$\frac{C}{\|K\|^2}\|K^*f_2\|^2 = 0 \leqslant C\|f_2\|^2 \leqslant \sum_{i\in I}\|\Lambda_i f_2\|^2.$$

And since $\overline{\text{span}}\{\Lambda_i^*(\mathscr{H}_i)\}_{i\in I} \subset \mathscr{R}(K)$, for any $f \in \mathscr{H}$, we have

$$\frac{C}{\|K\|^2}\|K^*f\|^2 \leqslant C\|f\|^2 \leqslant \sum_{i\in I}\|\Lambda_i f\|^2. \quad \square$$

REMARK 1. If $\mathscr{R}(K) = \mathscr{H}$, then $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathscr{H}$ as well as a g-frame for $\mathscr{H}$.

Let $\{\Lambda_i\}_{i\in I}$ be a K-g-frame, we know that the frame operator of $\{\Lambda_i\}_{i\in I}$ may be not invertible, so there is no classical canonical dual for $\{\Lambda_i\}_{i\in I}$. Next, we will give a characterization of duals for a K-g-frame.

DEFINITION 2. Suppose $K \in \mathscr{L}(\mathscr{H})$ and $\{\Lambda_i\}_{i\in I}$ is a K-frame for $\mathscr{H}$. A g-Bessel sequence $\{\Gamma_i\}_{i\in I}$ for $\mathscr{H}$ is called a K-dual g-frame of $\{\Lambda_i\}_{i\in I}$ if

$$Kf = \sum_{i\in I} \Lambda_i^* \Gamma_i f, \quad \forall f \in \mathscr{H}.$$

The following theorem provides a necessary and sufficient conditions for a Bessel g-sequence to be a K-dual g-frame. Note that $\{\delta_i\}_{i\in I}$ denotes the canonical basis of $\ell^2(I)$. Let $\{e_{ij}\}_{i\in I, j\in J_i}$ be an orthonormal basis for $\mathscr{H}_i$, then roughly speaking $\{e_{ij\delta_i}\}_{i\in I, j\in J_i}$ is an orthonormal basis of $(\sum_{i\in I} \oplus \mathscr{H}_i)_{\ell^2}$, because for any $\{f_i\}_{i\in I} \in (\sum_{i\in I} \oplus \mathscr{H}_i)_{\ell^2}$,

$$f_i = \sum_{i\in I} f_i \delta_i = \sum_{i\in I} \sum_{j\in J_i} \langle f_i, e_{ij} \rangle e_{ij} \delta_i.$$

THEOREM 4. *Suppose that $K \in \mathscr{L}(\mathscr{H})$ and $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathscr{H}$ with the synthesis operator T_Λ. Then the dual $\{\Gamma_i\}_{i\in I}$ is a K-dual g-frame of $\{\Lambda_i\}_{i\in I}$ if and only if there exists a bounded operator $\Phi : (\sum_{i\in I} \oplus \mathscr{H}_i) \longrightarrow \mathscr{H}$ such that $K^* = \Phi T_\Lambda^*$ and $\Gamma^* e_{i,j} = \Phi(e_{ij}\delta_i)$, $j \in J_i$, $i \in I$. Moreover, K-dual g-frame $\{\Gamma_i\}_{i\in I}$ is a K^*-g-frame.*

Proof. Suppose that $\{\Gamma_i\}_{i\in I}$ is a K-dual g-frame of $\{\Lambda_i\}_{i\in I}$. Then the synthesis operator for $\{\Gamma_i\}_{i\in I}$ satisfies the conditions. In fact, $\{\Gamma_i\}_{i\in I}$ is a Bessel g-sequence and for all $f \in \mathscr{H}$, we have

$$K^* f = \sum_{i\in I} \Gamma_i^* \Lambda_i f.$$

Let Φ be the synthesis operator of $\{\Gamma_i\}_{i\in I}$, then

$$\Phi(e_{ij}\delta_i) = \sum_{i\in I} \Gamma_i^* e_{ij} \delta_i = \sum_{i\in I} u_{ij} \delta_i = \sum_{j\in J_i} u_{ij} = \sum_{i\in J_i} \langle e_{ij}, e_{ij} \rangle u_{ij} = \Gamma_i^* e_{ij}.$$

So a calculation as above shows that

$$K^* f = \sum_{i\in I} \Gamma_i^* \Lambda_i f = \sum_{i\in I} \Gamma_i^* (\sum_{j\in J_i} \langle f, u_{ij} \rangle e_{ij}) = \Phi(\sum_{i,j} \langle \Lambda_i f, e_{ij} \rangle e_{ij} \delta_i) = \Phi T_\Lambda^* f.$$

So $K^* = \Phi T_\Lambda^*$.

Conversely, if $\{f_i\}_{i\in I} \in (\sum_{i\in I} \oplus \mathscr{H}_i)_{\ell^2}$, then we have

$$\{f_i\}_{i\in I} = \sum_{i\in I} f_i \delta_i = \sum_{i\in I} \sum_{j\in J_i} \langle f_i, e_{ij} \rangle e_{ij} \delta_i.$$

Roughly speaking $\{e_{i,j}\delta_i\}_{i\in I, j\in J_i}$ is an orthonormal basis of $(\sum_{i\in I} \oplus \mathscr{H}_i)_{\ell^2}$. Let $u_{i,j}$ be defined as in (1). If $K^* = \Phi T_\Lambda^*$ and $\Gamma^* e_{i,j} = \Phi(e_{ij}\delta_i)$, $j \in J_i$, $i \in I$ then for all $f \in \mathscr{H}$

we have

$$K^*f = \Phi T_\Lambda^* f = \Phi(\sum_{i,j} \langle \Lambda_i f, e_{ij} \rangle e_{ij}\delta_i) = \sum_{i\in I}\sum_{j\in J} \langle f, \Lambda_i^* e_{ij} \rangle \Phi(e_{ij}\delta_i)$$
$$= \sum_{i\in I}\sum_{j\in J} \langle f, u_{ij} \rangle \Gamma_i^* e_{ij} = \sum_{i\in I} \Gamma_i^* (\sum_{j\in J_i} \langle f, u_{ij} \rangle e_{ij}) = \sum_{i\in I} \Gamma_i^* \Lambda_i f.$$

Consequently, $Kf = \sum_{i\in I} \Lambda_i^* \Gamma_i f$, meaning that $\{\Gamma_i\}_{i\in I}$ is a K-dual g-frame of $\{\Lambda_i\}_{i\in I}$.

Moreover, let B and D be the Bessel bounds for $\{\Lambda_i\}_{i\in I}$ and $\{\Gamma_i\}_{i\in I}$, respectively. For any $f \in \mathscr{H}$, we have

$$\|(K^*)^* f\|^2 = \|Kf\|^2 = \|T_\Lambda \Phi^* f\|^2 \leqslant \|T_\Lambda\|^2 \|\Phi^* f\|^2 \leqslant B \sum_{i\in I} \|\Gamma_i f\|^2 \leqslant D\|f\|^2,$$

thus,

$$\frac{1}{B}\|(K^*)^* f\|^2 \leqslant \sum_{i\in I} \|\Gamma_i f\|^2 \leqslant \frac{D}{B}\|f\|^2, \quad \forall f \in \mathscr{H}.$$

Hence, K-dual g-frame $\{\Gamma_i\}_{i\in I}$ is a K^*-g-frame. □

REMARK 2. When $K = I_{\mathscr{H}}$, the K-g-frame is exactly g-frame, in this case, the K-dual is exactly the canonical dual g-frame.

We end this section by giving the following results concerning the constructions of new K-frames.

THEOREM 5. *Let $K \in \mathscr{L}(\mathscr{H})$ and let $\{\Lambda_i\}_{i\in I}$ be a K-g-frame. For $T \in \mathscr{L}(\mathscr{H})$ with $TK^* = K^*T$, then $\Lambda_i T$ is a K-g-frame for $\mathscr{H}$.*

Proof. Suppose that $\{\Lambda_i\}_{i\in I}$ is a K-g-frame with bounds A and B. Now for any $f \in \mathscr{H}$, we have

$$\sum_{i\in I} \|\Lambda_i T f\|^2 \geqslant A\|K^* T f\|^2 = A\|TK^* f\|^2 \geqslant A\|T\|\|K^* f\|^2,$$

and

$$\sum_{i\in I} \|\Lambda_i T f\|^2 \leqslant B\|Tf\|^2 \leqslant B\|T\|^2\|f\|^2.$$

Hence, $\Lambda_i T$ is a K-g-frame for $\mathscr{H}$. □

COROLLARY 3. *Let $K \in \mathscr{L}(\mathscr{H})$ and let $\{\Lambda_i\}_{i\in I}$ be a K-g-frame. For $T \in \mathscr{L}(\mathscr{H})$, then $\Lambda_i T^*$ is a TK-g-frame for $\mathscr{H}$.*

COROLLARY 4. *Let $\{\Lambda_i\}_{i\in I}$ be a g-frame. For $K \in \mathscr{L}(\mathscr{H})$, then $\Lambda_i K^*$ is a K-g-frame for $\mathscr{H}$.*

3. K-g-frames with operator quotient

In this section, we characterize K-g-frame by operator quotient.

DEFINITION 3. [12] Let U and V be bounded (linear) operators on a Hilbert space $\mathcal{H}$ with the kernel condition

$$\mathcal{N}(V) \subset \mathcal{N}(U).$$

Then the quotient $[U/V]$ is a map from $\mathcal{R}(V)$ to $\mathcal{R}(U)$ defined by $Vf \longmapsto Uf$ for all $f \in \mathcal{H}$.

We note that $W = [U/V]$ is a linear operator on $\mathcal{H}$ if and only if $\mathcal{N}(V) \subset \mathcal{N}(U)$. In this case $\mathcal{D}(W) = \mathcal{R}(V)$, $\mathcal{R}(W) \subset \mathcal{R}(U)$ and $WV = U$. The quotient $[U/V]$ is called a semiclosed operator and its collection is closed under sum and product [13]. The authors of [20] fined that there is a relationship between K-frames and operator quotient operator. So we present few results on K-g-frame techniques on quotients of bounded operators. These results are inspired by the results in [20]. But there are some different properties in our results because g-frames are more complicated than ordinary frames.

THEOREM 6. *Let $K \in \mathcal{H}$ and $\{\Lambda_i\}_{i\in I}$ be a Bessel g-sequence in $\mathcal{H}$ with the frame operator S_Λ.Then $\{\Lambda_i\}_{i\in I}$ is a K-g-frame if and only if the quotient operator $[K^*/S_\Lambda^{1/2}]$ is bounded.*

Proof. $\Longrightarrow$: Since $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathcal{H}$, there exists a constant $A > 0$ such that

$$A\|K^*f\|^2 \leqslant \sum_{i\in I}\|\Lambda_i f\|^2 = \langle S_\Lambda f, f\rangle, \quad \forall f \in \mathcal{H}.$$

That is, $A\|K^*f\|^2 \leqslant \|S_\Lambda^{1/2}f\|^2$ for all $f \in \mathcal{H}$. Define $W : \mathcal{R}(S_\Lambda^{1/2}) \to \mathcal{R}(K^*)$ by

$$W(S_\Lambda^{1/2}f) = K^*f, \ \forall f \in \mathcal{H}.$$

Then W is well-defined because $\mathcal{N}(S_\Lambda^{1/2}) \subset \mathcal{N}(K^*)$. For all $f \in \mathcal{H}$, we have

$$\|WS_\Lambda^{1/2}f\| = \|K^*f\| \leqslant \frac{1}{\sqrt{A}}\|S_\Lambda^{1/2}f\|.$$

So W is bounded. From the notion of quotient of bounded operators, W can be expressed as $[K^*/S_\Lambda^{1/2}]$.

$\Longleftarrow$: Suppose that the quotient operator $[K^*/S_\Lambda^{1/2}]$ is bounded. Then there exists $\lambda > 0$ such that

$$\|K^*f\|^2 \leqslant \lambda\|S_\Lambda^{1/2}f\|^2, \quad \forall f \in \mathcal{H}.$$

Thus

$$\frac{1}{\lambda}\|K^*f\|^2 \leqslant \|S_\Lambda^{1/2}f\|^2 = \langle S_\Lambda f, f\rangle = \sum_{i\in I}\|\Lambda_i f\|^2,$$

for all $f \in \mathcal{H}$. Hence $\{\Lambda_i\}_{i\in I}$ is a K-g-frame for $\mathcal{H}$. $\square$

Let $K = I_{\mathcal{H}}$, we get the following corollary.

COROLLARY 5. *Let $\{\Lambda_i\}_{i\in I}$ be a Bessel g-sequence for $\mathscr{H}$ with the frame operator S_Λ.Then $\{\Lambda_i\}_{i\in I}$ is a g-frame if and only if the frame operator S_Λ is bounded.*

THEOREM 7. *Let $\{\Lambda_i\}_{i\in I}$ be a K-g-frame with the frame operator S_Λ and $T \in \mathscr{L}(\mathscr{H})$. Then the following are equivalent:*

(1) $\{\Lambda_i T^*\}_{i\in I}$ *is a TK-g-frame;*

(2) $[(TK)^*/S_\Lambda^{1/2}T^*]$ *is bounded;*

(3) $[(TK)^*/(TS_\Lambda T^*)^{1/2}]$ *is bounded.*

Proof. (1)$\Rightarrow$(2); Suppose that $\{\Lambda_i T\}_{i\in I}$ is a TK-g-frame. Then there exist $\lambda > 0$ such that

$$\lambda\|(TK)^*f\|^2 \leqslant \sum_{i\in I}\|\Lambda_i T^* f\|^2 = \langle S_\Lambda T^* f, T^* f\rangle = \|S_\Lambda^{1/2}T^* f\|^2, \ \forall f \in \mathscr{H}.$$

Hence $[(TK)^*/S_\Lambda^{1/2}T^*]$ is bounded.

(2)$\Rightarrow$(3); Suppose $[(TK)^*/S_\Lambda^{1/2}T^*]$ is bounded. Then there exists $\mu > 0$ such that

$$\|(TK)^*f\|^2 \leqslant \mu\|S_\Lambda^{1/2}T^* f\|^2, \ \forall f \in \mathscr{H}.$$

Since

$$\begin{aligned}\|(TS_\Lambda T^*)^{1/2}f\|^2 &= \left\langle (TS_\Lambda T^*)^{1/2}f, (TS_\Lambda T^*)^{1/2}f\right\rangle = \langle (TS_\Lambda T^*)f, f\rangle \\ &= \langle S_\Lambda T^* f, T^* f\rangle = \|S_\Lambda^{1/2}T^* f\|^2,\end{aligned}$$

for all $f \in \mathscr{H}$, we have

$$\frac{1}{\mu}\|(TK)^*f\|^2 \leqslant \|(TS_\Lambda T^*)^{1/2}f\|^2.$$

Therefore $[(TK)^*/(TS_\Lambda T^*)^{1/2}]$ is bounded.

(3)$\Rightarrow$(1); Suppose $[(TK)^*/(TS_\Lambda T^*)^{1/2}]$ is bounded. Then there exists $\mu > 0$ such that

$$\|(TK)^*f\|^2 \leqslant \mu\|(TS_\Lambda T^*)^{1/2}f\|^2, \quad \forall f \in \mathscr{H}.$$

Consider

$$\sum_{i\in I}\|\Lambda_i Tf\|^2 = \langle S_\Lambda T^* f, ^* Tf\rangle = \langle TS_\Lambda T^* f, f\rangle,$$

So $TS_\Lambda T^*$ is positive and self-adjoint, its square root exists, and it is denoted by $(TS_\Lambda T^*)^{1/2}$. Hence

$$\sum_{i\in I}\|(\Lambda_i T)f\|^2 = \|(TS_\Lambda T^*)^{1/2}f\|^2 \geqslant \frac{1}{\mu}\|(TK)^*f\|^2, \ \forall f \in \mathscr{H}.$$

Hence $\{\Lambda_i T\}_{i\in I}$ is a TK-g-frame. $\square$

COROLLARY 6. *Let $K \in \mathscr{L}(\mathscr{H})$ and $\{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathscr{H}$. Then the following are equivalent:*

1. $\{\Lambda_i K^*\}_{i\in I}$ *is a* K*-g-frame for* $\mathscr{H}$ *;*

2. $[K^*/S^{1/2}]$ *is bounded.*

COROLLARY 7. *Let* $K \in \mathscr{L}(\mathscr{H})$ *and* $\{\Theta_i\}_{i\in I}$ *be a g-orthonormal basis for* $\mathscr{H}$*. Then the following are equivalent:*

1. $\{\Theta_i K^*\}$ *is a* K*-g-frame for* $\mathscr{H}$ *;*

2. $[K^*/I_{\mathscr{H}}]$ *is bounded.*

REMARK 3. The Theorem 7 proofs the conclusions in Corollary 3 and Corollary 4.

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