

## RESONANCE FREE REGIONS AND NON-HERMITIAN SPECTRAL OPTIMIZATION FOR SCHRÖDINGER POINT INTERACTIONS

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*Abstract.* Resonances of Schrödinger Hamiltonians with point interactions are considered. The main object under the study is the resonance free region under the assumption that the centers, where the point interactions are located, are known and the associated “strength” parameters are unknown and allowed to bear additional dissipative effects. To this end we consider the boundary of the resonance free region as a Pareto optimal frontier and study the corresponding optimization problem for resonances. It is shown that upper logarithmic bound on resonances can be made uniform with respect to the strength parameters. The necessary conditions on optimality are obtained in terms of first principal minors of the characteristic determinant. We demonstrate the applicability of these optimality conditions on the case of 4 equidistant centers by computing explicitly the resonances of minimal decay for all frequencies. This example shows that a resonance of minimal decay is not necessarily simple, and in some cases it is generated by an infinite family of feasible resonators.

### 1. Introduction

#### 1.1. Statement of problem, motivation, and related studies

In the present paper, we study resonance free regions and extremal resonances of ‘one particle, finitely many centers Hamiltonian’  $H_\alpha = -\Delta_{\alpha,Y}$  associated with the formal expression  $-\Delta u(x) + \sum_{j=1}^N \mu(\alpha_j) \delta(x - y_j) u(x)$ ,  $x \in \mathbb{R}^3$ ,  $N \in \mathbb{N}$ , where  $\Delta$  is the self-adjoint Laplacian acting in the complex Lebesgue space  $\mathcal{L}^2(\mathbb{R}^3)$ ,  $\delta(\cdot - y_j)$  is the Dirac measure at  $y_j \in \mathbb{R}^3$ ,  $\mu(\alpha_j)$  is a complex-valued function of the strength parameter  $\alpha_j$ ,  $j = 1, \dots, N$  (see [1, 2, 3, 6] and Section 2 for basic definitions). The question of optimization of the principal eigenvalue of self-adjoint Schrödinger Hamiltonians with  $\delta$ -type or point interactions attracted recently considerable attention especially in a quantum mechanics context [14, 17, 16, 18, 36]. This line of research was motivated by the isoperimetric problem posed in [14].

In comparison with variational problems involving eigenvalues of self-adjoint operators, the resonance spectral problem describes the dissipation of energy to the outer medium and so it is of a non-Hermitian type. The facts that resonances move under perturbations in two-dimensions of the complex plane and that degenerate (multiple) resonances can split in non-differentiable branches lead to essentially new difficulties

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and effects for the application of variational techniques [23, 45, 10, 9, 25, 26, 27, 28]. In particular, the problem of optimization of an individual resonance takes the flavor of Pareto optimization if one considers it as an  $\mathbb{R}^2$ -valued objective function and the boundary of the resonance free region as a Pareto frontier [27, 28]. Numerical optimization of 1-D resonances produced by point interactions were initiated recently in [40].

Estimates on poles of scattering matrices and resonances have being studied in Mathematical Physics at least since the Lax-Phillips upper logarithmic bound on resonances' imaginary parts [33] and constitute an active area of research [13, 19, 47]. Optimization of resonances may be seen as an attempt to obtain sharp estimates on resonance free regions. This point of view and the study of resonances associated with random Schrödinger operators were initial sources of the interest in this problem [22, 23, 45].

The present growth of interest in numerical [21, 24, 25, 37, 41] and analytical [26, 27, 28] aspects of resonance optimization is stimulated by a number of optical engineering studies of resonators with high quality factor (high-Q cavities), see [12, 34, 37, 39] and references therein.

In this paper, we assume that the tuple of centers  $Y = (y_j)_1^N \in (\mathbb{R}^3)^N$  (locations of the  $\delta$ -interactions) is fixed and known, but the N-tuple  $\alpha = (\alpha_j)_1^N$  of scalar free 'strength' parameters  $\alpha_j$  of point interactions is unknown. The associated point interactions Hamiltonians  $H_\alpha = -\Delta_{\alpha,Y}$  can be defined in several ways as densely defined closed operators in the Hilbert space  $\mathcal{L}^2(\mathbb{R}^3)$  [3, 5, 20], in particular, via a Krein-type formula for the difference of the perturbed and unperturbed resolvents of operators  $H_\alpha$  and  $-\Delta$ , respectively. Eigenvalues and (continuation) resonances  $k$  of the corresponding operator  $H_\alpha$  are connected with the special  $N \times N$ -matrix function  $\Gamma_{\alpha,Y}(z)$  which appears naturally as a part of the expression for  $(-\Delta_{\alpha,Y} - z^2)^{-1} - (-\Delta - z^2)^{-1}$ , see Section 2. If one denotes by  $\Sigma(\alpha, Y)$  the set of zeroes of  $\det \Gamma_{\alpha,Y}(\cdot)$ , then the set  $\Sigma_{\text{res}}(\alpha, Y)$  of resonances  $k$  associated with  $H_\alpha$  can be defined by

$$\Sigma_{\text{res}}(\alpha, Y) := \Sigma(\alpha, Y) \cap (\mathbb{C}_- \cup \mathbb{R}), \quad \text{see [3, 5],} \tag{1.1}$$

where  $\mathbb{C}_-$  is the lower half of the complex plane.

The functions  $\det \Gamma_{\alpha,Y}(\cdot)$  take the form of exponential polynomials, for those there exists a well-developed theory with a number of applications in Analysis and connections to the studies of the Riemann zeta function [7, 35, 38]. Pólya's results on positions and distribution of zeros of exponential polynomials were refined and generalized in many works leading, in particular, to the Pólya-Dickson theorem [7]. This theorem implies, for example, that the imaginary parts of resonances of  $H_\alpha$  satisfy upper and lower logarithmic bounds (see Lemma 2.1 and (5.1) below), in this way establishing and strengthening for point interactions the Lax-Phillips result [33]. From this point of view, the present work can be seen as an attempt to obtain more refined bounds on zeros of special exponential polynomials employing Pareto optimization techniques of [26, 27, 28].

While our main goal is to consider the resonance free regions in the case where the  $\alpha_j$  run through the compactification  $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$  of the real line, our technique

also leads us to the study of ‘dissipative point interactions’ corresponding to the case  $\alpha_j \in \overline{\mathbb{C}}_- := \mathbb{C}_- \cup \mathbb{R} \cup \{\infty\}$ . It is not difficult to see (see Section 2) that the corresponding operators  $H_\alpha$  are well-defined, closed, and maximal dissipative in the sense that the  $iH_\alpha$  are maximal accretive (i.e.,  $\text{Re}(iH_\alpha u, u) \geq 0$  for all  $u$  in the domain  $\text{dom} H_\alpha$  of  $H_\alpha$  and  $(iH_\alpha + \lambda)\text{dom} H_\alpha = \mathcal{L}^2(\mathbb{R}^3)$  for  $\lambda > 0$ ). So  $H_\alpha$  can be considered as pseudo-Hamiltonians in the terminology of [15]. Following the logic of the resolvent continuation it is natural to extend the definition of resonances given by formula (1.1) to the case  $\alpha \in (\overline{\mathbb{C}}_-)^N$ .

Assuming that each of the parameters  $\alpha_j, j = 1, \dots, N$ , is allowed to run through some set  $\mathbb{A} \subset \overline{\mathbb{C}}_-$  we consider the associated operators  $H_\alpha$  as feasible points (see [8] for basic notions of the optimization of vector-valued objective functions) and denote the associated feasible set of operators by  $\mathbb{F}_\mathbb{A}$ . The resonance free region for the family  $\mathbb{F}_\mathbb{A}$  is defined as  $\mathbb{C} \setminus \Sigma_{\text{res}}[\mathbb{F}_\mathbb{A}]$  where  $\Sigma_{\text{res}}[\mathbb{F}_\mathbb{A}] := \bigcup_{H_\alpha \in \mathbb{F}_\mathbb{A}} \Sigma_{\text{res}}(\alpha, Y)$  is the set of achievable resonances.

### 1.2. Main results and some examples

The main results of the present paper are:

- It is shown in Theorem 5.1 that upper logarithmic bounds on imaginary parts of resonances can be modified to become uniform estimates over  $\mathbb{F}_{\overline{\mathbb{C}}_-}$  and  $\mathbb{F}_\mathbb{R}$ .
- To achieve more detailed results on the resonance free region, we employ the Pareto optimization approach and consider Hamiltonians  $H_\alpha \in \mathbb{F}_\mathbb{A}$  that produce resonances on the boundary  $\text{bd} \Sigma_{\text{res}}[\mathbb{F}_\mathbb{A}]$  of the set of achievable resonances. When the set  $\mathbb{A}$  of feasible strength parameters  $\alpha_j, j = 1, \dots, N$ , is closed in the topology of the compactification  $\overline{\mathbb{C}}_-$ , such extremal feasible operators  $H_\alpha$  do exist since the set  $\Sigma_{\text{res}}[\mathbb{F}_\mathbb{A}]$  is closed (see Theorem 4.1). The function of minimal decay rate  $\tau(\cdot)$  [26] provides a convenient way to describe the part of  $\text{bd} \Sigma_{\text{res}}[\mathbb{F}_\mathbb{A}]$  closest to  $\mathbb{R}$  (see Definition 3.1 and the discussions in Section 8). The associated extremal resonances  $k$  and operators  $H_\alpha$  are said to be of minimal decay for their particular frequencies  $f = \text{Re} k$ .
- In Section 6 we obtain various necessary conditions on  $H_\alpha$  to be extremal over  $\mathbb{F}_\mathbb{R}$  and  $\mathbb{F}_{\overline{\mathbb{C}}_-}$  in terms of first minors of a regularized version of  $\det \Gamma_{\alpha, Y}$ . This is done with the use of the multi-parameter perturbations technique of [27].
- The effectiveness of the conditions of Section 6 can be seen in the equidistant cases when  $|y_j - y_{j'}| = L$  for all  $j \neq j'$ . Namely, we provide an explicit calculation of resonances of minimal decay and associated tuples  $\alpha$  for the case where  $\{y_j\}_1^4$  constitute the vertices of a regular tetrahedron (see Section 7).

In the process of deriving the above results, we obtained several examples that are of independent interest since they address the questions arising often in the study of resonances and their optimization.

Namely, it occurs in the case of vertices of a regular tetrahedron that the optimal  $\alpha$  does not always consist of equal  $\alpha_j$  and that, for some of resonances of minimal decay,

there exists an infinite family of optimizers  $H_\alpha$  preserving only one of the symmetries (see the discussion in Section 8). This gives a negative answer to the multidimensional part of the question of uniqueness of optimizers for a given  $\text{Re}k$ , which was posed in [27, Section 8] (see also [23, 29]).

The assumption that a resonance  $k$  is of multiplicity 1 essentially simplifies its perturbation theory (see (4.4)), and therefore this assumption is often explicitly or implicitly used in intuitive arguments. While it is known that generic resonances are simple [13] (i.e., of multiplicity 1), there are no reasons to assume that resonances of minimal decay are generic. Example 8.4 describes  $H_\alpha \in \mathbb{F}_{\mathbb{R}}$  that produce resonances of minimal decay with multiplicity  $\geq 2$ .

Nonzero resonances on the real line are often assumed to be connected with eigenvalues embedded into the essential spectrum. Remark 3.1 provides a very simple example of a dissipative Schrödinger Hamiltonian that generates a resonance  $k$  in  $\mathbb{R}_-$ , but has no embedded eigenvalue at  $k^2$ .

NOTATION. The following standard sets are used: the lower ( $-$ ) and upper ( $+$ ) complex half-planes  $\mathbb{C}_\pm = \{z : \pm \text{Im}z > 0\}$ ,  $\mathbb{C}_I, \mathbb{C}_{II}, \mathbb{C}_{III}$ , and  $\mathbb{C}_{IV}$  are the open quadrants in  $\mathbb{C}$  corresponding to the combinations of signs  $(+, +), (-, +), (-, -)$ , and  $(+, -)$  for  $(\text{Re}z, \text{Im}z)$ , open half-lines  $\mathbb{R}_\pm = \{x \in \mathbb{R} : \pm x > 0\}$ , open discs  $\mathbb{D}_\varepsilon(\zeta) := \{z \in \mathbb{C} : |z - \zeta| < \varepsilon\}$ , and the boundary  $\text{bd}S$  of a subset  $S$  of a normed space  $U$ . For  $u_0 \in U$  and  $z \in \mathbb{C}$ , we write  $zS + u_0 := \{zu + u_0 : u \in S\}$ . The convex cone generated by  $S$  (all nonnegative linear combinations of elements of  $S$ ) is denoted by  $\text{Cone}S$ . If a certain map  $g$  is defined on  $S$ ,  $g[S]$  is its image (when it is convenient, we write without brackets, e.g.  $\text{Re}S$  for  $S \subset \mathbb{C}$ .) The diameter of  $S$  is  $\text{diam}(S) := \sup_{u_0, u_1 \in S} \|u_0 - u_1\|_U$ . By  $\partial_z f, \partial_{\alpha_j} f$ , etc., we denote (ordinary or partial) derivatives with respect to (w.r.t.)  $z, \alpha_j$ , etc.;  $\text{deg} p$  stands for the degree of a polynomial  $p$  of one or several variables.  $\delta_{ij}$  is the Kronecker delta.

### 2. Nonconservative point interactions

Let us fix a set  $Y = \{y_j\}_{j=1}^N$  consisting of  $N$  distinct points  $y_1, \dots, y_N$  in  $\mathbb{R}^3$ . For every tuple  $\alpha = (\alpha_j)_{j=1}^N \in \mathbb{R}^N$ , there exists the self-adjoint Hamiltonian  $H_\alpha = -\Delta_{\alpha, Y}$  in  $\mathcal{L}^2(\mathbb{R}^3)$  with point interactions at the centers  $y_j$  that has for all  $z \in \mathbb{C}_I$  the resolvent  $(-\Delta_{\alpha, Y} - z^2)^{-1}$  with the integral kernel

$$(-\Delta_{\alpha, Y} - z^2)^{-1}(x, x') = G_z(x - x') + \sum_{j, j'=1}^N G_z(x - y_j) [\Gamma_{\alpha, Y}]_{j, j'}^{-1} G_z(x' - y_{j'}), \tag{2.1}$$

where  $x, x' \in \mathbb{R}^3 \setminus Y$  and  $x \neq x'$ , see [3, Section II.1.1]. Here  $G_z(x - x') := \frac{e^{iz|x-x'|}}{4\pi|x-x'|}$  is the integral kernel associated the resolvent  $(-\Delta - z^2)^{-1}$  of the kinetic energy Hamiltonian  $-\Delta$ , and  $[\Gamma_{\alpha, Y}]_{j, j'}^{-1}$  denotes the  $j, j'$ -element of the inverse to the matrix

$$\Gamma_{\alpha, Y}(z) = \left[ \left( \alpha_j - \frac{iz}{4\pi} \right) \delta_{jj'} - \tilde{G}_z(y_j - y_{j'}) \right]_{j, j'=1}^N \text{ with } \tilde{G}_z(x) := \begin{cases} G_z(x), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

In the case of one center ( $N = 1$ ) and  $\alpha_1 \in \mathbb{C}$ , the above definition leads to the  $m$ -accretive operator  $iH_{\alpha_1}$  when  $\alpha_1 \in \mathbb{C}_-$ , and the  $m$ -accretive operator  $(-i)H_{\alpha_1}$  when  $\alpha_1 \in \mathbb{C}_+$  (see [4] and [3, Sections I.1.1 and I.2.1]).

The aim of this section is to extend the above definition to all tuples  $\alpha \in \mathbb{C}^N$ . Later we will use the case  $\alpha \in (\mathbb{C}_- \cup \mathbb{R})^N$  as a technical tool for optimization of resonances over  $\alpha \in \mathbb{R}^N$ .

Here and below  $\deg p$  is the degree of the polynomial  $p$  of one or several variables and  $\text{diam}(Y) := \max_{1 \leq j, j' \leq N} |y_j - y_{j'}|$  is the diameter of  $Y$ .

As it was pointed out to us by the referee, the following lemma could be obtained from the theory of zeroes of exponential polynomials [35, 7] which goes back to Pólya. We provide here a short self-contained proof that while not using the general theory, shows how one of Pólya’s arguments works.

**LEMMA 2.1.** *For every  $\alpha \in \mathbb{C}^N$ , there exist  $c_{i,j} = c_{i,j}(\alpha, Y) > 0$ ,  $i, j = 1, 2$ , such that all zeros  $k$  of  $\det \Gamma_{\alpha, Y}(\cdot)$  satisfy*

$$-c_{2,1} \ln(|\text{Re } k| + 1) - c_{2,2} \leq \text{Im } k \leq -c_{1,1} \ln(|\text{Re } k| + 1) + c_{1,2}. \tag{2.2}$$

*Proof.* Consider  $\det \Gamma_{\alpha, Y}(z)$  as a function in  $z$  only. Then there exists a unique representation

$$\det \Gamma_{\alpha, Y}(z) = (-4\pi)^{-N} D(z), \quad D(z) = \sum_{l=0}^v p_l(z) e^{izq_l}, \tag{2.3}$$

where the numbers  $v = v(\alpha, Y) \in \mathbb{N} \cup \{0\}$ ,  $q_l = q_l(\alpha, Y) \geq 0$ , and the nontrivial polynomials  $p_l(z)$  (i.e.,  $p_l \not\equiv 0$ ) with coefficient depending on  $\alpha$  and  $Y$  are such that

$$0 = q_0 < q_1 < \dots < q_v \leq N \text{diam}(Y).$$

Clearly,  $p_0(z) = \prod_{j=1}^N (iz - 4\pi\alpha_j)$  and  $\deg p_l \leq N - 2$  for all  $1 \leq l \leq v$ .

If  $v = 0$ ,  $\det \Gamma_{\alpha, Y}(z) = p_0(z)$  and so the statement of the lemma is obvious.

Note that  $v = 0$  if and only if  $N = 1$ . Indeed, for  $N \geq 2$  it is easy to see that  $q_1 = 2 \min_{j \neq j'} |y_j - y_{j'}|$  and the terms containing  $e^{izq_1}$  do not cancel.

Let  $N \geq 2$  and  $v \geq 1$ . We prove (2.2) in several regions of  $\mathbb{C}$  and then take the largest of the corresponding constants  $c_{i,j}$ . First, note that (2.2) is obvious in any disc  $\mathbb{D}_r(0)$  and also for  $z \in \mathbb{C}_+ \cup \mathbb{R}$  (due to asymptotics of exponential terms in (2.3)).

Let  $z \in \mathbb{C}_-$ . Then there exists  $r_1(\alpha) > 0$  and  $C_1(\alpha, Y) > 0$  so that

$$|D(z)| \geq |p_0(z)| - C_1 |z|^{N-2} |e^{izq_v}| \geq 2^{-1} |z|^{N-2} [(|z| + 1)^2 - 2C_1 e^{-q_v \text{Im } z}] \quad \text{for } |z| \geq r_1.$$

Assuming additionally  $z \in \Omega_1 = \{\text{Im } z > -c_{1,1} \ln(|\text{Re } z| + 1) + c_{1,2}\}$ , we see that

$$(|z| + 1)^2 - 2C_1 e^{-q_v \text{Im } z} \geq (|z| + 1)^2 - (|\text{Re } z| + 1)^{q_v c_{1,1}} 2C_1 e^{-q_v c_{1,2}} > 0$$

whenever  $C_1 e^{-q_v c_{1,2}} \leq 1/4$  and  $c_{1,1} \leq 2/q_v$ . Hence, such a choice of  $c_{1,1}, c_{1,2}$  ensures the absence of zeros of  $D$  in  $(\Omega_1 \cap \mathbb{C}_-) \setminus \mathbb{D}_{r_1}(0)$ .

On the other hand, for certain  $r_2(\alpha) > 0$ ,  $C_3(\alpha, Y) > 0$ , and  $C_4(\alpha, Y) > 0$ , it follows from  $|z| \geq r_2$  that

$$|D(z)| \geq C_3 e^{-q_v \operatorname{Im} z} - C_4 |z|^N e^{-q_{v-1} \operatorname{Im} z} = C_3 e^{-q_{v-1} \operatorname{Im} z} (e^{(q_v-1-q_v) \operatorname{Im} z} - |z|^N C_4/C_3).$$

Thus, taking  $c_{2,1} \geq N/(q_v - q_{v-1})$  it is easy to show the existence of  $c_{2,2}$  and  $r_3(\alpha, Y) > r_2$  such that  $z \in \Omega_2 = \{\operatorname{Im} z < -c_{2,1} \ln(|\operatorname{Re} z| + 1) - c_{2,2}\}$  and  $z \in \mathbb{C}_- \setminus \mathbb{D}_{r_3}(0)$  imply  $|D(z)| > 0$ .  $\square$

**PROPOSITION 2.2.** *Let  $\alpha \in \mathbb{C}^N$ . Then there exists a closed operator  $H_\alpha = -\Delta_{\alpha, Y}$  in  $\mathcal{L}^2(\mathbb{R}^3)$  with the spectrum  $\sigma(H_\alpha) = [0, +\infty) \cup \{z^2 : z \in \mathbb{C}_+, \det \Gamma_{\alpha, Y}(z) = 0\}$  and the resolvent  $(H_\alpha - z^2)^{-1}$  defined for  $\{z \in \mathbb{C}_+ : z^2 \notin \sigma(H_\alpha)\}$  by the integral kernel (2.1). If  $\alpha \in (\mathbb{C}_- \cup \mathbb{R})^N$ , the operator  $iH_\alpha$  is  $m$ -accretive in the sense of [30].*

*Proof.* The proof of the first statement can be obtained by modification of the arguments of [3, Section II.1.1] with the use of Lemma 2.1 and the formula

$$(\Gamma_{\alpha, Y}(z))^* = \Gamma_{\bar{\alpha}, Y}(-\bar{z}), \text{ where } \bar{\alpha} := (\bar{\alpha}_j)_{j=1}^\infty \tag{2.4}$$

(here  $\bar{z}$  is the complex conjugate of  $z \in \mathbb{C}$ ).

Let now  $\alpha \in (\mathbb{C}_- \cup \mathbb{R})^N$ . Then, it is easy to see that, for  $z \in i\mathbb{R}_+$  the operator  $i\Gamma_{\alpha, Y}(z)$  is accretive in the  $N$ -dimensional  $\ell^2$ -space. So, if additionally  $\det \Gamma_{\alpha, Y}(z) \neq 0$ , the operator  $(iH_\alpha - iz^2)^{-1}$  and, in turn,  $iH_\alpha$  are accretive. Since the resolvent set of  $H_\alpha$  is nonempty,  $iH_\alpha$  is  $m$ -accretive.  $\square$

### 3. Resonances and related optimization problems

We will use the compactifications  $\bar{\mathbb{C}} = \{\infty\} \cup \mathbb{C}$ ,  $\bar{\mathbb{R}} = \{\infty\} \cup \mathbb{R}$ , and  $\bar{\mathbb{C}}_- := \{\infty\} \cup \mathbb{R} \cup \mathbb{C}_-$ .

To carry over the above definitions of point interactions to the extended  $N$ -tuples  $\alpha \in \bar{\mathbb{C}}^N$ , we put, following [3],  $\Delta_{\alpha, Y} = \Delta_{\tilde{\alpha}, \tilde{Y}}$ , where

$\tilde{\alpha}$  and  $\tilde{Y}$  are produced from  $\alpha$  and  $Y$ , resp.,

$$\text{by removing of the components with numbers } j \text{ satisfying } \tilde{\alpha}_j = \infty. \tag{3.1}$$

(It is assumed in the sequel that if all  $\alpha_j \in \mathbb{C}$ , then  $\tilde{\alpha} := \alpha$ ,  $\tilde{Y} := Y$ ). Using this rule we can formally define the function  $\det \Gamma_{\alpha, Y}(\cdot) := \det \Gamma_{\tilde{\alpha}, \tilde{Y}}(\cdot)$  for arbitrary  $\alpha \in \bar{\mathbb{C}}^N$ .

Points  $k$  belonging to the set  $\Sigma(\alpha, Y)$  of zeroes of the determinant  $\det \Gamma_{\alpha, Y}(z)$  will be called  $\Gamma^{-1}$ -poles (or  $\Gamma_{\alpha, Y}^{-1}$ -poles). The set of (continuation) resonances  $\Sigma_{\text{res}}(\alpha, Y)$  associated with  $H_\alpha$  is defined by (1.1). (This definition is in agreement with the case of real  $\alpha_j$  considered in [3, 5, 20], where also the connection of  $\Gamma_{\alpha, Y}^{-1}$ -poles in  $\mathbb{C}_+$  with eigenvalues of  $H_\alpha$  is addressed. For the origin of this and related approaches to the understanding of resonances, we refer to [5, 13, 20, 43, 46] and the literature therein).

The multiplicity of a resonance or a  $\Gamma^{-1}$ -pole will be understood as the multiplicity of a corresponding zero of the analytic function  $\det \Gamma_{\alpha, Y}(\cdot)$  (see [3]).

For fixed  $Y$ , consider the set

$$\mathbb{F} = \{-\Delta_{\alpha,Y} : \alpha \in S\} \tag{3.2}$$

of operators  $H_\alpha$  with  $N$ -tuples  $\alpha$  belonging to a certain set  $S \subset \overline{\mathbb{C}}^N$ . Let us introduce the sets of all possible resonances  $\Sigma_{\text{res}}[\mathbb{F}]$  and  $\Gamma^{-1}$ -poles  $\Sigma[\mathbb{F}]$  generated by  $H_\alpha \in \mathbb{F}$ ,

$$\Sigma[\mathbb{F}] := \bigcup_{-\Delta_{\alpha,Y} \in \mathbb{F}} \Sigma(\alpha, Y), \quad \Sigma_{\text{res}}[\mathbb{F}] := \bigcup_{-\Delta_{\alpha,Y} \in \mathbb{F}} \Sigma_{\text{res}}(\alpha, Y).$$

We consider  $\mathbb{F}$  as a *feasible set* [8] of operators. The main attention will be paid to the direct products  $S = \mathbb{A}^N$  of the sets  $\mathbb{A} \subset \overline{\mathbb{C}}_-$  of *feasible dissipative  $\alpha_j$ -parameters*. For these direct products, we employ the notation  $\mathbb{F}_{\mathbb{A}} := \{H_\alpha : \alpha \in \mathbb{A}^N\}$ .

Our main goal is to find resonances  $k$  which are extremal over  $\mathbb{F}_{\mathbb{R}}$  or  $\mathbb{F}_{\overline{\mathbb{C}}_-}$  in the framework of the Pareto optimization approach of [26, 27, 28]. In a wide sense, resonances globally Pareto extremal over  $\mathbb{F}$  can be understood as boundary points of the *set of achievable resonances*  $\Sigma_{\text{res}}[\mathbb{F}]$ . Depending on the applied background of more narrow optimization problems, various parts of the boundary  $\text{bd}\Sigma_{\text{res}}[\mathbb{F}]$  can be perceived as optimal resonances (see the discussion in Section 8 and in [28, Section A.2]). Note that our definitions are slightly different from those in [8]. In particular, from our point of view, the use of positive cones for the definition of Pareto optimizers is sometimes too restrictive for the needs of resonance optimization.

One of particular optimization problems can be stated in the following way. If  $k \in \Sigma_{\text{res}}(\alpha, Y)$  is interpreted as a resonance of the wave-type equation  $\partial_t^2 u - \Delta_{\alpha,Y} u = 0$  (cf. [33]) with ‘singular potential term  $V = \sum_{j=1}^N \mu(\alpha_j) \delta(x - y_j) u(x)$ ’, then  $f = \text{Re} k$  can be understood as a (real) frequency of the associated resonant mode and  $\tau = -\text{Im} k \geq 0$  is the corresponding exponential rate of decay (cf. [13, 26, 27]).

We say that  $f \in \mathbb{R}$  is an achievable frequency if  $\alpha \in \text{Re} \Sigma_{\text{res}}[\mathbb{F}]$ . The properties of the set  $\text{Re} \Sigma_{\text{res}}[\mathbb{F}_{\mathbb{R}}]$  are discussed in Section 8.

DEFINITION 3.1. (see [26] for 1-D resonances) Let  $f \in \text{Re} \Sigma_{\text{res}}[\mathbb{F}]$ . The *minimal decay rate*  $\tau_{\min}(f) = \tau_{\min}(f; \mathbb{F})$  for the frequency  $f$  is defined by

$$\tau_{\min}(f; \mathbb{F}) := \inf\{\tau \in [0, +\infty) : f - i\tau \in \Sigma_{\text{res}}[\mathbb{F}]\}.$$

If  $k = f - i\tau_{\min}(f)$  is a resonance of a certain feasible operator  $H_\alpha \in \mathbb{F}$  (i.e., the minimum is achieved), we say that  $k$ ,  $H_\alpha$ , and  $\alpha$  are of *minimal decay* for  $f$ .

EXAMPLE 3.1. Let  $N = 1$  and  $Y = \{y_1\}$ . Then  $\Sigma(\alpha_1, Y)$  consists of one  $\Gamma^{-1}$ -pole  $k = -i4\pi\alpha_1$  of multiplicity 1 [4, 3].

- (i) In the case  $\mathbb{A} = \mathbb{R}$ , one has  $\Sigma[\mathbb{F}_{\mathbb{R}}] = i\mathbb{R} := \{it : t \in \mathbb{R}\}$  and  $\Sigma_{\text{res}}[\mathbb{F}_{\mathbb{R}}] = i(-\infty, 0]$ . The function  $\tau_{\min}(\cdot; \mathbb{R})$  is defined on the set of achievable frequencies consisting of one point  $\text{Re} \Sigma_{\text{res}}[\mathbb{F}_{\mathbb{R}}] = \{0\}$  and one has  $\tau_{\min}(0; \mathbb{F}_{\mathbb{R}}) = 0$ . The resonance  $k = 0$  and the operator  $H_0$  are of minimal decay for the frequency 0.

- (ii) Let  $\mathbb{A} = \overline{\mathbb{C}}_-$ . Then  $\Sigma[\mathbb{F}_{\overline{\mathbb{C}}_-}] = i\mathbb{C}_+ \cup i\mathbb{R}$ . For each  $f \in (-\infty, 0]$ , we have  $\tau_{\min}(f; \mathbb{F}_{\overline{\mathbb{C}}_-}) = 0$ , and see that  $k = f$  and  $H_{1f(4\pi)^{-1}}$  are the resonance and an operator of minimal decay for  $f$ .

REMARK 3.1. It follows from Example 3.1 (ii) that, in the dissipative case, nonzero real resonances are not necessarily associated with embedded eigenvalues of  $H_\alpha$ . Indeed, taking  $N = 1$  and  $\alpha_1 \in i\mathbb{R}_-$ , we see that there exists a real resonance  $k_0 < 0$ . The fact that  $k_0^2$  is not an eigenvalue of  $H_\alpha$  follows easily from the proof of [3, Theorem I.1.1.4].

### 4. Existence of optimizers and perturbation theory

For every  $a = (a_1, \dots, a_N) \in \overline{\mathbb{C}}^N$ , let us denote

$$\text{by } n(a) \text{ the number of parameters } a_j, j = 1, \dots, N, \text{ that are not equal to } \infty, \tag{4.1}$$

$$\text{and, for } k \in \Sigma[\mathbb{F}_{\mathbb{A}}], \text{ by } n_{\min}(k; \mathbb{F}_{\mathbb{A}}) := \min\{n(a) : k \in \Sigma(a, Y) \text{ and } a \in \mathbb{A}^N\} \tag{4.2}$$

the minimal number of centers needed to generate  $k$  over  $\mathbb{F}_{\mathbb{A}}$ .

Let us introduce on the compactification  $\overline{\mathbb{C}}$  of  $\mathbb{C}$  a metric  $\rho_{\overline{\mathbb{C}}}(z_1, z_2)$  generated by the stereographic projection and, e.g., the  $\ell^2$ -distance on the unit sphere  $\mathbb{S}_2 \subset \mathbb{R}^3$ . The direct product  $\overline{\mathbb{C}}^N$  will be considered as a compact metric space with the distance  $\rho_{\overline{\mathbb{C}}^N}(\alpha, \alpha')$  generated by the  $\ell^2$ -distance on  $\mathbb{S}_2^N \subset \mathbb{R}^{3N}$ .

Recall that, for  $S \subset \overline{\mathbb{C}}^N$ , the feasible set  $\mathbb{F}$  of operators is defined by (3.2), and that  $\Sigma[\mathbb{F}] = \bigcup_{\alpha \in S} \Sigma(\alpha, Y)$  is the corresponding set of achievable  $\Gamma^{-1}$ -poles.

THEOREM 4.1. *Let the set  $S$  be closed in the metric space  $(\overline{\mathbb{C}}^N, \rho_{\overline{\mathbb{C}}^N})$ . Then  $\Sigma[\mathbb{F}]$  is a closed set and, for every achievable frequency, there exists an operator  $H_\alpha \in \mathbb{F}$  of minimal decay (in the sense of Definition 3.1).*

This theorem easily follows from the following lemma and the compactness argument. Recall that  $\tilde{\alpha}$  and  $\tilde{Y}$  are defined by (3.1).

LEMMA 4.2. *For every  $a \in \overline{\mathbb{C}}^N$  there exists an open neighborhood  $\mathcal{W} \subset \overline{\mathbb{C}}^N$  of  $a$  (in the topology of  $(\overline{\mathbb{C}}^N, \rho_{\overline{\mathbb{C}}^N})$ ), an open set  $\mathcal{B} \subset \mathbb{C}^N$ , a homeomorphism  $\beta : \mathcal{W} \rightarrow \mathcal{B}$ , and an analytic function  $D_a : \mathcal{B} \times \mathbb{C} \rightarrow \mathbb{C}$  such that, for every  $\alpha \in \mathcal{W}$ , the sets of zeroes of the function  $\det \Gamma_{\tilde{\alpha}, \tilde{Y}}(\cdot) : \mathbb{C} \rightarrow \mathbb{C}$  coincide with the sets of zeroes of the function  $D_a(\beta(\alpha); \cdot) : \mathbb{C} \rightarrow \mathbb{C}$  taking multiplicities into account.*

*Proof.* When  $n(a) = N$  (see (4.1)), the lemma is obvious with  $\beta(\alpha) \equiv \alpha$  and  $D_a(\beta(\alpha); z) \equiv \det \Gamma_{\alpha, Y}(z)$ . Now, let us prove the lemma for the case  $n := n(a) < N$ . Without loss of generality, we can assume that  $a_j \in \mathbb{C}$  for  $1 \leq j \leq n(a)$  and  $a_j = \infty$



for  $j > n(a)$ . Put  $\beta_j = \alpha_j$  for  $1 \leq j \leq n(a)$ . For  $n(a) + 1 \leq j \leq N$  and  $\alpha_j \neq 0$ , let us define  $\beta_j = -1/\alpha_j$  (assuming  $1/\infty = 0$ ). Then the following regularized determinant

$$D_a = (-1)^n \begin{vmatrix} \frac{iz}{4\pi} - \beta_1 & \dots & G_z(y_1 - y_n) & G_z(y_1 - y_{n+1}) & \dots & G_z(y_1 - y_N) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ G_z(y_n - y_1) & \dots & \frac{iz}{4\pi} - \beta_n & G_z(y_n - y_{n+1}) & \dots & G_z(y_n - y_N) \\ \beta_{n+1} G_z(y_{n+1} - y_1) & \dots & \beta_{n+1} G_z(y_{n+1} - y_N) & \frac{iz\beta_{n+1}}{4\pi} + 1 & \dots & \beta_{n+1} G_z(y_{n+1} - y_N) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \beta_N G_z(y_N - y_1) & \dots & \beta_N G_z(y_N - y_n) & \beta_N G_z(y_N - y_{n+1}) & \dots & \frac{iz\beta_N}{4\pi} + 1 \end{vmatrix} \quad (4.3)$$

satisfies the conditions of the lemma.  $\square$

Lemma 4.2 allows one to consider the corrections of resonances and eigenvalues of  $H_\alpha$  under small perturbations of  $a \in \overline{\mathbb{C}}^N$ . The first correction terms under one-parameter perturbations can be described in the following way.

Let  $k$  be an  $m$ -fold zero of the determinant  $D_a(b; \cdot)$  defined by (4.3) at  $b = (a_1, \dots, a_n, 0, \dots, 0)$  and considered as an analytic function of the variables  $z \in \mathbb{C}$  and  $\beta \in \mathbb{C}^N$ . Then, for every analytic function  $\gamma(\zeta)$  that maps  $\mathbb{D}_r(0) \subset \mathbb{C}$  to  $\mathbb{C}^N$  and satisfy  $\gamma(0) = b$ , there exist  $\varepsilon > 0$ ,  $\delta > 0$ , and continuous on  $[0, \varepsilon)$  functions  $\kappa_j(\zeta)$ ,  $j = 1, \dots, m$ , with the asymptotics

$$\kappa_j(\zeta) = k + (C_{\gamma,1} \zeta)^{1/m} + o(\zeta^{1/m}) \text{ as } \zeta \rightarrow 0, \quad C_{\gamma,1} := - \frac{m! \partial_\zeta D_a(\gamma(\zeta), k) |_{\zeta=0}}{\partial_\zeta^m D_a(b, k)}, \quad (4.4)$$

such that all the zeros of  $D_a(\gamma(\zeta), \cdot)$ ,  $\zeta \in [0, \varepsilon)$ , lying in  $\mathbb{D}_\delta(k)$  are given by  $\{\kappa_j(\zeta)\}_1^m$  taking multiplicities into account. In the case  $C_{\gamma,1} \neq 0$ , each branch of  $[\cdot]^{1/m}$  corresponds to exactly one of functions  $\kappa_j$ , and so, all  $m$  values of functions  $\kappa_j(\zeta)$  for small enough  $\zeta > 0$  are distinct zeros of  $D_a(\gamma(\zeta), \cdot)$  of multiplicity 1.

Perturbations of  $b$  in the directions of modified parameters  $\beta_j$  play a special role. Note that  $D_a(\beta_1, \dots, \beta_n, b_{n+1}, \dots, b_N; z) = D_a(\alpha_1, \dots, \alpha_n, 0, \dots, 0; z) = \det \Gamma_{\tilde{\alpha}, \tilde{Y}}(z)$ . Let  $k \in \Sigma(a, Y)$ . If  $a_i \in \mathbb{C}$  (and so  $i \leq n(a)$  and  $\beta_i = \alpha_i$ , under the convention of Lemma 4.2), then the term  $\partial_\zeta D_a(\gamma(0), k)$  corresponding to the perturbation of one of the  $\beta_j$  takes the form of the first principal minor

$$\partial_{\beta_i} D_a(b; k) = \partial_{\alpha_i} \det \Gamma_{\tilde{\alpha}, \tilde{Y}}(k) = \det \Gamma_{\tilde{\alpha}, \tilde{Y}}^{[i]}(k), \quad (4.5)$$

$$\text{where } \Gamma_{\tilde{\alpha}, \tilde{Y}}^{[i]}(k) := \left[ \left( \alpha_j - \frac{ik}{4\pi} \right) \delta_{jj'} - \tilde{G}_z(y_j - y_{j'}) \right]_{\substack{j, j'=1, \dots, n \\ j, j' \neq i}}.$$

If  $a_i = \infty$  (and so  $i > n$ ,  $b_i = 0$ , and  $\beta_i = -1/\alpha_i$ ), one has

$$\partial_{\beta_i} D_a(b, k) = (-1)^n \begin{vmatrix} \frac{ik}{4\pi} - \beta_1 & \dots & G_k(y_1 - y_n) & G_k(y_1 - y_i) \\ \dots & \dots & \dots & \dots \\ G_k(y_n - y_1) & \dots & \frac{ik}{4\pi} - \beta_n & G_k(y_n - y_i) \\ G_k(y_i - y_1) & \dots & G_k(y_i - y_n) & c \end{vmatrix} \quad (4.6)$$

with arbitrary  $c \in \mathbb{C}$ . With  $c = \frac{ik}{4\pi}$  the latter equality is obvious from (4.3). To prove it for arbitrary  $c \in \mathbb{C}$ , note that the first minor in the left upper corner of the determinant in (4.6) is equal to  $\det \Gamma_{\tilde{\alpha}, \tilde{Y}}(k) = 0$ .

Note that when  $z \in \mathbb{C}$  is fixed,  $D_a(\cdot; z)$  is a polynomial in the variables  $\beta_j$  and that  $\partial_{\beta_j}^l D_a(\beta, z) = 0$  for all  $l \geq 2$ . This implies the following lemma.

LEMMA 4.3. *If  $k \in \Sigma(a; Y)$  and  $\partial_{\beta_j} D_a(b; k) = 0$  for certain  $1 \leq j \leq N$ , then  $k \in \Sigma(\alpha; Y)$  for all  $\alpha$  obtained from  $a$  by the change of the  $j$ -th coordinate  $a_j$  to an arbitrary number in  $\overline{\mathbb{C}}$ .*

### 5. Uniform logarithmic bound on resonances

Let  $N \geq 2$  and  $\alpha \in \mathbb{C}^N$ . Then Lemma 2.1 and its proof imply the following 2-side bound on all resonances  $k \in \Sigma_{\text{res}}(\alpha, Y)$ :

$$-\frac{N}{q_v - q_{v-1}} \ln(|\operatorname{Re} k| + 1) - c_{2,2} \leq \operatorname{Im} k \leq -\frac{2}{q_v} \ln(|\operatorname{Re} k| + 1) + c_{1,2}, \tag{5.1}$$

where  $q_v, q_{v-1}, c_{2,2}, c_{2,1}$  are positive constants the depending on  $\alpha$  and  $Y$  defined in the proof of Lemma 2.1. The following theorem shows that the upper bound can be modified in such a way that it becomes uniform with respect to  $\alpha \in \overline{\mathbb{R}}^N$  or  $\alpha \in \overline{\mathbb{C}}_-^N$ .

THEOREM 5.1. *Let  $N \geq 2$  and  $\mathbb{A} \subset \overline{\mathbb{C}}_-$ . Then there exist  $c_1 = c_1(Y) > 0$  such that*

$$\tau_{\min}(f; \mathbb{F}_{\mathbb{A}}) \geq \frac{2}{N \operatorname{diam}(Y)} \ln(|f| + 1) - c_1 \tag{5.2}$$

for all frequencies  $f > 0$  achievable over  $\mathbb{F}_{\mathbb{A}}$ .

*Proof. Step 1.* As a function in  $z$  and  $\alpha \in \mathbb{C}^N$ ,  $\det \Gamma_{\alpha, Y}(z)$  has the following representation

$$\det \Gamma_{\alpha, Y}(z) = (-4\pi)^{-N} \tilde{D}(\alpha, z), \quad \tilde{D}(\alpha, z) = \sum_{l=0}^{\eta} P_l(\alpha, z) e^{izQ_l}, \tag{5.3}$$

which is unique if we assume that the numbers  $\eta = \eta(Y) \in \mathbb{N} \cup \{0\}$ ,  $Q_l = Q_l(Y)$ , and the nontrivial polynomials  $P_l$  in  $z$  and  $\alpha_j$  (with coefficient depending on  $Y$ ) are such that  $0 = Q_0 < Q_1 < \dots < Q_{\eta}$ . In this case, one sees that  $P_0(\alpha, z) = \prod_{j=1}^N (iz - 4\pi\alpha_j)$  and

$$\deg P_l \leq N - 2 \text{ for all } 1 \leq l \leq \eta. \tag{5.4}$$

In the same way as in the proof of Lemma 2.1, the assumption  $N \geq 2$  implies  $\eta \geq 1$ .

*Step 2.* Consider the case  $\mathbb{A} = \mathbb{A}_0 := \mathbb{C}_- \cup \mathbb{R}$  and  $\alpha \in \mathbb{A}_0^N$ . Then all  $\alpha_j$  are finite and we can use (5.3). Denote  $P_{\min}(\alpha, z) := \min_{j \neq j'} |(iz - 4\pi\alpha_j)(iz - 4\pi\alpha_{j'})|$ . It is easy to see that there exists  $C_5 = C_5(Q_{\eta}) > 0$  such that for any  $c_1 \geq C_5$  in the region

$$\Omega_3 := \{ \operatorname{Im} z \leq 0, \operatorname{Re} z > 0, \operatorname{Im} z > -\frac{2}{Q_{\eta}} \ln(\operatorname{Re} z + 1) + c_1 \}$$

the inequalities

$$|iz - 4\pi\alpha_j| \geq \frac{|z|+1}{4} + |\alpha_j| \geq \frac{1}{4} \tag{5.5}$$

hold for all  $\alpha_j \in \mathbb{A}_0$ . Hence, for all  $(z, \alpha) \in \Omega_3 \times \mathbb{A}_0$ , we have

$$P_0(\alpha, z) \neq 0, \quad P_{\min}(z, \alpha) > (|z| + 1)^2/16, \quad \text{and} \tag{5.6}$$

$$|\tilde{D}| \frac{P_{\min}}{|P_0|} \geq P_{\min} - \sum_{l=1}^{\eta} \frac{P_{\min}|P_l|}{|P_0|} e^{-Q_{\eta} \text{Im}z} \geq \frac{(|z| + 1)^2}{16} - 4^{N-2}(N! - 1)e^{-Q_{\eta} \text{Im}z}.$$

The last inequality follows from (5.5), (5.4), and the Leibniz formula for the determinant  $\det \Gamma_{\alpha, Y}$ . Choosing  $c_1$  large enough, one can ensure that  $|\tilde{D}| \frac{P_{\min}}{|P_0|}$  (and so  $\tilde{D}$ ) have no zeros in  $\Omega_3 \times \mathbb{A}_0^N$ .

*Step 3.* From Step 2, the perturbation formula (4.4), and Lemma 4.2, one sees that  $\det \Gamma_{\alpha, Y}(z)$  has no zeros in the larger set  $\Omega_3 \times \mathbb{A}^N$  with  $\mathbb{A} = \overline{\mathbb{C}}_-$ . Now, the statement of theorem follows from the obvious estimate  $Q_{\eta} \leq N \text{diam}(Y)$ .  $\square$

REMARK 5.1. When  $\mathbb{A} = \overline{\mathbb{R}}$ , the estimate (5.2) is valid for all frequencies  $f$  that are achievable over  $\mathbb{F}_{\overline{\mathbb{R}}}$ . Indeed, (2.4) and Example 3.1 imply that

$$\Sigma(\alpha, Y) \text{ is symmetric w.r.t. } i\mathbb{R} \text{ for } \alpha \in \overline{\mathbb{R}}^N \text{ (including multiplicities),} \tag{5.7}$$

$$\tau_{\min}(f; \mathbb{F}_{\overline{\mathbb{R}}}) \text{ is an even function, and } \tau_{\min}(0; \mathbb{F}_{\overline{\mathbb{R}}}) = 0. \tag{5.8}$$

Note also that it follows from (5.7) that  $\Sigma[\mathbb{F}_{\overline{\mathbb{R}}}]$  and  $\Sigma_{\text{res}}[\mathbb{F}_{\overline{\mathbb{R}}}]$  are symmetric w.r.t.  $i\mathbb{R}$ , and that  $\Sigma[\mathbb{F}_{\overline{\mathbb{R}}}] = \Sigma_{\text{res}}[\mathbb{F}_{\overline{\mathbb{R}}}] \cup i[0, +\infty)$ . To see the last equality, it suffices to notice that the m-accretivity statement of Proposition 2.2 implies

$$\emptyset = \mathbb{C}_I \cap \Sigma[\mathbb{F}_{\overline{\mathbb{C}}_-}] = (\mathbb{C}_I \cup \mathbb{C}_{II}) \cap \Sigma[\mathbb{F}_{\overline{\mathbb{R}}}] \tag{5.9}$$

and that  $\Sigma[\mathbb{F}_{\overline{\mathbb{R}}}]$  contains the set  $i\mathbb{R}$  produced by the case  $N = 1$  of Example 3.1 (i). Similarly,  $\Sigma[\mathbb{F}_{\overline{\mathbb{C}}_-}]$  contains the set  $i\mathbb{C}_+ \cup i\mathbb{R}$  of Example 3.1 (ii). For the minimal decay function  $\tau_{\min}$  over  $\mathbb{F}_{\overline{\mathbb{C}}_-}$  one has  $\tau_{\min}(f; \mathbb{F}_{\overline{\mathbb{C}}_-}) = 0$  for all  $f \leq 0$ .

### 6. Extremal resonances over $\mathbb{A} = \overline{\mathbb{R}}$ and $\mathbb{A} = \overline{\mathbb{C}}_-$

We study first the boundary  $\text{bd} \Sigma[\mathbb{F}_{\overline{\mathbb{C}}_-}]$ , and then from this study obtain results on resonances of minimal decay over  $\mathbb{F}_{\overline{\mathbb{R}}}$ . The idea behind this is that there are more possible perturbations of the parameter tuple  $a$  inside of  $\overline{\mathbb{C}}_-^N$ , than in the case  $a \in \overline{\mathbb{R}}^N$ . So the restrictions on the possible perturbations of  $k$  over  $\mathbb{F}_{\overline{\mathbb{C}}_-}$  are stronger. However, it occurs that for every  $k \in \text{bd} \Sigma[\mathbb{F}_{\overline{\mathbb{C}}_-}]$ , there exists  $a \in \overline{\mathbb{R}}^N$  that generates  $k$  in the sense that  $k \in \Sigma_{\text{res}}(a, Y)$ . As a result the resonances of minimal decay over  $\mathbb{F}_{\overline{\mathbb{R}}}$  inherit the stronger necessary conditions of extremity derived for  $\mathbb{A} = \overline{\mathbb{C}}_-$ .

In the simplest form, our main abstract result states that if  $a \in \mathbb{R}^N$  generates the resonance  $k$  of minimal decay over  $\mathbb{F}_{\overline{\mathbb{R}}}$ , then there exists  $\xi \in [-\pi, \pi)$  such that

$$\text{the ray } e^{i\xi}[0, +\infty) \text{ contains all first minors } \det \Gamma_{a, Y}^{[j]}(k) \text{ (see Theorem 6.3).} \tag{6.1}$$

(Note that the ray  $e^{i\xi}[0, +\infty) \subset \mathbb{C}$  includes its vertex at 0.)

To formulate the result in the general form that includes the possibility of  $a_j = \infty$  for some  $j$  and the case  $\mathbb{A} = \overline{\mathbb{C}}_-$ , we take the convention of Section 4, which assumes that the centers  $y_j$  are enumerated in such a way that  $a_j \neq \infty$  for  $1 \leq j \leq n$  and  $a_j = \infty$  for  $n < j \leq N$ , and use the regularized determinant  $D = D_a$  defined by (4.3) and depending on the modified parameters  $\beta_j$ .

**THEOREM 6.1.** *Assume that  $a \in \overline{\mathbb{C}}_-^N$  and  $k \in \Sigma(a, Y)$  are such that  $k \in \text{bd} \Sigma[\mathbb{F}_{\overline{\mathbb{C}}_-}]$ . Then the following statements hold:*

(i) *There exists  $\xi \in [-\pi, \pi)$  such that*

$$\partial_{\beta_j} D(b; k) \text{ belongs to the ray } e^{i\xi}[0, +\infty) \text{ for all } j = 1, \dots, N. \tag{6.2}$$

(ii) *If  $a_i \in \mathbb{C}_-$  for some  $i$ , then  $\partial_{\beta_i} D(b; k) = 0$  and*

$$k \in \Sigma(a', Y) \text{ for } a' \text{ defined by } a'_j = \begin{cases} c, & j = i \\ a_j, & j \neq i \end{cases} \text{ with arbitrary } c \in \overline{\mathbb{C}}_-. \tag{6.3}$$

(iii) *There exists  $a' \in \overline{\mathbb{R}}^N$  such that  $k \in \Sigma(a', Y)$  and  $n(a') = n_{\min}(k; \mathbb{F}_{\overline{\mathbb{C}}_-})$ . (Recall that  $n_{\min}$  is defined by (4.2).)*

The proof is given in Section 6.1. Note that Statement (iii) and Theorem 4.1 imply

$$\text{bd} \Sigma[\mathbb{F}_{\overline{\mathbb{C}}_-}] \subset \text{bd} \Sigma[\mathbb{F}_{\overline{\mathbb{R}}}] \subset \Sigma[\mathbb{F}_{\overline{\mathbb{R}}}] \subset \Sigma[\mathbb{F}_{\overline{\mathbb{C}}_-}]. \tag{6.4}$$

On the other hand, the  $m$ -accretivity statement of Proposition 2.2 implies  $\mathbb{C}_1 \cap \Sigma[\mathbb{F}_{\overline{\mathbb{C}}_-}] = \emptyset$ . Combining this with (6.4), we obtain the next corollary.

**COROLLARY 6.2.** *Let  $f \geq 0$ . Then the frequency  $f$  is achievable over  $\mathbb{F}_{\overline{\mathbb{C}}_-}$  exactly when it is achievable over  $\mathbb{F}_{\overline{\mathbb{R}}}$ . For such frequencies  $f$ , one has  $\tau_{\min}(f; \overline{\mathbb{R}}) = \tau_{\min}(f; \overline{\mathbb{C}}_-)$ .*

With the use of Corollary 6.2 and Theorem 6.1 we will prove in Section 6.1 the following necessary conditions over  $\mathbb{F}_{\overline{\mathbb{R}}}$ .

**THEOREM 6.3.**

(i) *If  $k$  and  $a \in \overline{\mathbb{R}}^N$  are of minimal decay (over  $\mathbb{F}_{\overline{\mathbb{R}}}$ ) for the frequency  $\text{Re} k$ , then there exists  $\xi \in [-\pi, \pi)$  such that (6.2) hold.*

(ii) *If  $a \in \overline{\mathbb{R}}^N$  and  $k \in \Sigma_{\text{res}}(a, Y)$  are such that  $k \in \text{bd} \Sigma_{\text{res}}[\mathbb{F}_{\overline{\mathbb{R}}}]$ , then there exists  $\xi \in [-\pi, \pi)$  such that  $\partial_{\beta_j} D(b; k)$  belongs to the line  $e^{i\xi} \mathbb{R}$  for all  $j = 1, \dots, N$ .*

### 6.1. Proofs of Theorems 6.1 and 6.3

Recall that by  $\text{Cone}W$  we denote the nonnegative convex cone generated by a subset  $W$  of a linear space. Let  $S = \overline{\mathbb{C}}_-^N$  or  $S = \overline{\mathbb{R}}^N$ , and let  $\mathbb{F}$  be defined by (3.2).

Let  $D(\beta; z) = D_a(\beta; z)$  and  $b \in \mathbb{C}^N$  be defined as in Section 4. The change from  $\alpha$ -coordinates to  $\beta$ -coordinates of Section 4 maps  $S$  onto  $\tilde{S}$ . Let  $\tilde{S} := S \cap \mathbb{C}^N$  (this excludes all infinite points). We would like to consider  $\beta \in \tilde{S}$  and the sets  $\Sigma_D(\beta)$  of zeroes of  $D(\beta, \cdot)$  generated by such  $\beta$ . Obviously,

$$\text{the set of all such zeros, } \Sigma_D[\tilde{S}] := \bigcup_{\beta \in \tilde{S}} \Sigma_D(\beta), \text{ is a subset of } \Sigma[\mathbb{F}]. \quad (6.5)$$

For  $v \in \mathbb{C}^N$ , let us consider the directional derivatives  $\frac{\partial D(b; z)}{\partial \beta}(v) := \partial_\zeta D(b + v\zeta; z)$ , where  $\zeta \in \mathbb{R}$ . Put  $\frac{\partial D(b; k)}{\partial \beta}[\tilde{S} - b] := \{ \frac{\partial D(b; z)}{\partial \beta}(v) : v \in \tilde{S} - b \}$ . When  $v$  belongs to the convex set  $\tilde{S} - b$ , the linear perturbations  $b + v\zeta$  for  $\zeta \in [0, 1]$  remain in  $\tilde{S}$ . Hence, [27, Theorem 4.1 and Proposition 4.2] imply that if  $\text{Cone} \frac{\partial D(b; k)}{\partial \beta}[\tilde{S} - b] = \mathbb{C}$ , then  $k$  is an interior point of  $\Sigma_D[\tilde{S}]$ . Taking (6.5) into account we obtain our main technical lemma.

**LEMMA 6.4.** *If  $k \in \text{bd}\Sigma[\mathbb{F}]$ , then there exists a closed half-plane  $e^{i\xi}(\mathbb{C}_+ \cup \mathbb{R})$  that contains  $\text{Cone} \frac{\partial D(b; k)}{\partial \beta}[\tilde{S} - b]$ .*

Let us prove statement (ii) of Theorem 6.1. Assume that  $k \in \text{bd}\Sigma[\mathbb{F}_{\overline{\mathbb{C}}_-}]$  and  $a_i \in \mathbb{C}_-$ . So  $b_i = a_i \in \mathbb{C}_-$ . Then every  $v \in \mathbb{C}^N$  such that  $v_j = 0$  for  $j \neq i$  belongs to  $\tilde{S} - b$  for small enough  $v_i \in \mathbb{C}$ . Note that  $\frac{\partial D(b; k)}{\partial \beta}(v) = v_i \partial_{\beta_i} D(b; k)$ . Lemma 6.4 implies that there exists a half-plane  $e^{i\xi}(\mathbb{C}_+ \cup \mathbb{R})$  that contains  $\frac{\partial D(b; k)}{\partial \beta}(v)$  for all  $v_i \in \mathbb{C}$ . This implies  $\partial_{\beta_i} D(b; k) = 0$  and, in turn, due to Lemma 4.3, implies (6.3).

Now, statement (iii) of Theorem 6.1 follows from statement (ii).

Let us prove statement (i) of Theorem 6.1. For all  $j$  such that  $b_j \in \mathbb{C}_-$ , statement (ii) implies (6.2) with arbitrary  $\xi$ . For  $j$  such that  $b_j \in \mathbb{R}$ , it is easy to see that  $v = (c\delta_{ji})_{i=1}^N$  is contained in  $\tilde{S} - b$  for arbitrary  $c \in \overline{\mathbb{C}}_- \setminus \{\infty\}$ . The corresponding derivatives  $\frac{\partial D(b; k)}{\partial \beta}(v) = c\partial_{\beta_j} D(b; k)$  are contained in one complex half-plane only if (6.2) holds. Thus, (i) follows from Lemma 6.4. This completes the proof of Theorem 6.1.

The above arguments and Lemma 6.4 allows one to obtain easily Theorem 6.3 (ii). Theorem 6.3 (i) follows immediately from Theorem 6.1 (iii), Corollary 6.2, and (5.7).

## 7. Equidistant case

In this section an example where the configuration of location of the interactions is symmetric is studied in order to obtain explicit formulas for optimizers. Namely, we consider the case where  $N = 4$  and  $y_j$  are vertices of a regular tetrahedron with edges of length  $L$ . We denote  $n_{\min}(k) := n_{\min}(k; \mathbb{F}_{\overline{\mathbb{R}}})$  (see (4.2)).

**THEOREM 7.1.** *Let  $N = 4$ . Assume that  $|y_j - y_{j'}| = L$  for all  $j \neq j'$ . Then a frequency  $f$  is achievable over  $\mathbb{F}_{\mathbb{R}}$  exactly when  $f \neq \pm l\pi/L$ ,  $l \in \mathbb{N}$ . The minimal decay function  $\tau(f) = \tau(f; \mathbb{F}_{\mathbb{R}})$  is given by the following explicit formulas:*

$$\begin{aligned} \tau(0) &= 0, & \tau(f) &= \frac{1}{L} \ln \frac{Lf}{\sin(Lf)} \quad \text{for } f \in \pm \bigcup_{l=0}^{\infty} \left( \frac{2l\pi}{L}, \frac{(2l+1)\pi}{L} \right), \\ \text{and } \tau(f) &= \frac{1}{L} \ln \frac{-Lf}{3\sin(Lf)} \quad \text{for } f \in \pm \bigcup_{l \in \mathbb{N}} \left( \frac{(2l-1)\pi}{L}, \frac{2l\pi}{L} \right). \end{aligned}$$

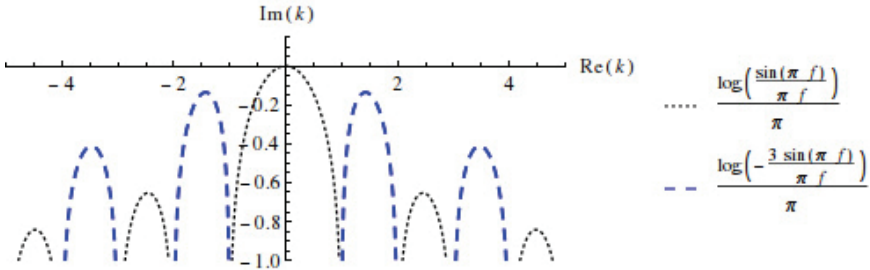


Figure 1: Resonances  $k = f - i\tau(f)$  of minimal decay over  $\mathbb{F}_{\mathbb{R}}$  in the equidistant case  $L = \pi$ ,  $N = 4$ ; --- marks the case  $n_{\min}(k) = 4$ , ..... marks the case  $n_{\min}(k) = 2$  (see Lemmas 7.2 and 7.4).

The rest of this subsection is devoted to the proof of Theorem 7.1.

Let  $n(a) = 4$ ,  $A_j := 4\pi L a_j$  and  $\varkappa := Lz$  (for the definition of  $n(a)$  see (4.1)). Then for arbitrary  $z$  and  $a \in \mathbb{R}^4$ ,

$$(-4\pi L)^4 \det \Gamma_{a,Y}(z) = \prod_{j=1}^4 (i\varkappa - A_j - e^{i\varkappa}) + e^{i\varkappa} \sum_{j=1}^4 \prod_{\substack{1 \leq j' \leq 4 \\ j' \neq j}} (i\varkappa - A_{j'} - e^{i\varkappa}) \quad \text{and} \quad (7.1)$$

$$(-4\pi L)^4 \det \Gamma_{a,Y}(z) = (-4\pi L)^3 (i\varkappa - A_i - e^{i\varkappa}) \partial_{\alpha_i} \det \Gamma_{a,Y}(z) + e^{i\varkappa} \prod_{j' \neq i} (i\varkappa - A_{j'} - e^{i\varkappa}). \quad (7.2)$$

Assume now that  $z = k \in \Sigma_{\text{res}}(a, Y)$ . Then  $\det \Gamma_{a,Y}(k) = 0$ , and so, (7.1) implies

$$0 = \prod_{j=1}^4 (i\varkappa - A_j - e^{i\varkappa}) + e^{i\varkappa} \sum_{j=1}^4 \prod_{j' \neq j} (i\varkappa - A_{j'} - e^{i\varkappa}). \quad (7.3)$$

**LEMMA 7.2.** *Assume that  $n(a) = n_{\min}(k) = 4$  and  $k \in \Sigma_{\text{res}}(a, Y)$  is of minimal decay for  $f \in \mathbb{R}$ . Then  $f \in \pm \bigcup_{l \in \mathbb{N}} ((2l-1)\pi/L, 2l\pi/L)$ , all  $a_j$  are equal to each other, and*

$$a_1 = \dots = a_4 = \frac{1}{4\pi L} \ln \frac{-Lf}{3\sin(Lf)} - \frac{f \cot(Lf)}{4\pi}, \quad k = f + i \frac{1}{L} \ln \frac{3\sin(Lf)}{-Lf}. \quad (7.4)$$

*Proof.* Since  $n(a) = 4$ , we see that  $a_j \in \mathbb{R}$ ,  $\beta_j = \alpha_j$ , for  $j = 1, \dots, 4$ , and  $D_a(\alpha; z) = \det \Gamma_{\alpha, Y}(z)$ . By Example 3.1,  $n_{\min}(k) = 1$  for all  $k \in i\mathbb{R}$ . So  $n_{\min}(k) = 4$  yields that  $f = \operatorname{Re} k \neq 0$ . Formula (7.2) implies that for each  $i$  either  $i\kappa - A_i - e^{i\kappa} = 0$ , or

$$(-4\pi L)^3 \partial_{\alpha_i} \det \Gamma_{\alpha, Y}(k) = -\frac{e^{i\kappa} \prod_{j' \neq i} (i\kappa - A_{j'} - e^{i\kappa})}{(i\kappa - A_i - e^{i\kappa})}. \tag{7.5}$$

Let us show that in the case  $n_{\min}(k) = 4$ , one has

$$i\kappa - A_i - e^{i\kappa} \neq 0 \text{ for all } i. \tag{7.6}$$

Assume that  $i\kappa - A_i - e^{i\kappa} = 0$  holds for certain  $i$ . Then (7.3) yields that there exists  $i' \neq i$  such that  $i\kappa - A_{i'} - e^{i\kappa} = 0$ . Hence,  $A_{i'} = A_i$  and, for  $a'$  defined by  $a'_j = \begin{cases} a_j, & j = i, i' \\ \infty, & j \neq i, i' \end{cases}$ , we have (taking into account the convention of Section 3)

$$\det \Gamma_{a', Y}(k) = \det \Gamma_{\tilde{a}, \tilde{Y}}(k) = (-4\pi L)^{-2} (i\kappa - A_i - e^{i\kappa})(i\kappa - A_i + e^{i\kappa}) = 0. \tag{7.7}$$

This means  $n_{\min}(k) \leq 2$ , a contradiction.

Thus, we see that (7.5) and  $\partial_{\alpha_i} \det \Gamma_{\alpha, Y}(z) \neq 0$  hold for all  $i = 1, \dots, 4$ . To combine these conditions with Theorem 6.3 (i), assume now that  $k$  is of minimal decay for the frequency  $\operatorname{Re} k$ . Then (6.2) and (7.5) imply that for arbitrary  $i \neq j$ ,

$$\frac{\partial_{\alpha_i} \det \Gamma_{\alpha, Y}(k)}{\partial_{\alpha_j} \det \Gamma_{\alpha, Y}(k)} = \frac{(i\kappa - A_j - e^{i\kappa})^2}{(i\kappa - A_i - e^{i\kappa})^2} \in \mathbb{R}_+. \tag{7.8}$$

Let us show that  $a_j = a_{j'}$  for all  $j, j' = 1, \dots, 4$ . Assume that the converse is true. Then  $A_i \neq A_{i'}$  for certain  $i$  and  $i'$ . However, (7.8) implies that  $\frac{i\kappa - A_j - e^{i\kappa}}{i\kappa - A_{j'} - e^{i\kappa}} \in \mathbb{R} \setminus \{0\}$  for all  $j$  and  $j'$ . So there exists  $\tilde{\xi} \in \mathbb{R}$  such that  $i\kappa - A_j - e^{i\kappa} = c_j e^{i\tilde{\xi}}$  with  $c_j \in \mathbb{R} \setminus \{0\}$  for all  $j$ . Since  $A_i, A_{i'} \in \mathbb{R}$  and  $0 \neq A_i - A_{i'} = (c_{i'} - c_i) e^{i\tilde{\xi}}$ , we see that  $e^{i\tilde{\xi}} \in \mathbb{R}$ , and, in turn,  $i\kappa - e^{i\kappa} \in \mathbb{R}$ . Combining this with (7.3) and (7.6), one gets  $e^{i\kappa} \in \mathbb{R}$ , and in turn, gets  $i\kappa \in \mathbb{R}$  from  $i\kappa - e^{i\kappa} \in \mathbb{R}$ . Finally, note that  $i\kappa \in \mathbb{R}$  contradicts  $\operatorname{Re} k \neq 0$ .

Summarizing, we have proved that if  $k$  is of minimal decay and  $n_{\min}(k) = 4$ , then all  $A_j$  are equal to the same number, which we denote by  $c$ . Due to (7.6), equality (7.3) turns into  $c = i\kappa + 3e^{i\kappa}$ . Since  $c \in \mathbb{R}$ , taking  $\operatorname{Re}(\cdot)$  and  $\operatorname{Im}(\cdot)$  of the last equality we can derive an explicit relation between  $\varkappa_1 := \operatorname{Re} \kappa \neq 0$ ,  $\varkappa_2 := \operatorname{Im} \kappa \geq 0$ , and  $c$  using the arguments similar to that of the example in [3, Section II.1.1] (see also [5, 44]). Indeed, taking  $\operatorname{Im}(\cdot)$  one obtains  $\varkappa_1 + 3e^{-\varkappa_2} \sin \varkappa_1 = 0$  and, in turn, that  $\varkappa_1 \in \pm \cup_{l \in \mathbb{N}} ((2l - 1)\pi, 2l\pi)$  and  $\varkappa_2 = \ln \frac{3 \sin \varkappa_1}{-\varkappa_1}$ . This gives the second part of (7.4). The value of  $c = 4\pi L a_j$  is found by taking  $\operatorname{Re}(\cdot)$ .  $\square$

LEMMA 7.3. Assume that  $k \in \Sigma_{\operatorname{res}}(a, Y)$  is of minimal decay for  $f \in \mathbb{R}$ ,  $a \in \overline{\mathbb{R}}^4$ , and  $n(a) = 3$ . Then  $n_{\min}(k) \leq 2$ .

*Proof.* Assume that  $n(a) = n_{\min}(k) = 3$ . Let us enumerate  $y_j$  such that  $a_j \in \mathbb{R}$  and  $\beta_j = \alpha_j$  for  $j = 1, 2, 3$ . Then  $a_4 = \infty$  and  $\beta_4 = -1/\alpha_4$ . Since  $D_a((\beta_1, \beta_2, \beta_3, 0); z) = \det \Gamma_{\tilde{\alpha}, \tilde{Y}}(z)$ , we obtain analogously to (7.3) and (7.2) that for  $i = 1, 2, 3$ ,

$$0 = \prod_{j=1}^3 (i\mathcal{I} - A_j - e^{i\mathcal{I}z}) + e^{i\mathcal{I}z} \sum_{j=1}^3 \prod_{\substack{1 \leq j' \leq 3 \\ j' \neq j}} (i\mathcal{I} - A_{j'} - e^{i\mathcal{I}z}) \quad \text{and} \quad (7.9)$$

$$0 = (-4\pi L)^2 (i\mathcal{I} - A_i - e^{i\mathcal{I}z}) \partial_{\beta_i} D_a(k) + e^{i\mathcal{I}z} \prod_{\substack{1 \leq j' \leq 3 \\ j' \neq i}} (i\mathcal{I} - A_{j'} - e^{i\mathcal{I}z}).$$

In the same way as in the proof of Lemma 7.2, one can show that  $n(a) = n_{\min}(k) = 3$  implies  $i\mathcal{I} - A_j - e^{i\mathcal{I}z} \neq 0$  and for  $j = 1, 2, 3$ ,

$$(-4\pi L)^2 \partial_{\beta_j} D_a(k) = - \frac{e^{i\mathcal{I}z} \prod_{\substack{1 \leq j' \leq 3 \\ j' \neq j}} (i\mathcal{I} - A_{j'} - e^{i\mathcal{I}z})}{(i\mathcal{I} - A_j - e^{i\mathcal{I}z})} \neq 0 \quad (7.10)$$

For  $j = 4$ , (4.6) with  $c = (4\pi L)^{-1} e^{i\mathcal{I}z}$  implies

$$-(4\pi L)^4 \partial_{\beta_4} D_a(k) = e^{i\mathcal{I}z} \prod_{j=1}^3 (i\mathcal{I} - A_j - e^{i\mathcal{I}z}) \neq 0. \quad (7.11)$$

Since  $k$  is of minimal decay, we obtain from Theorem 6.3 (i) that for  $j = 1, 2, 3$ ,

$$(4\pi L)^2 \frac{\partial_{\beta_4} D_a(k)}{\partial_{\beta_j} D_a(k)} = (i\mathcal{I} - A_j - e^{i\mathcal{I}z})^2 \in \mathbb{R}_+ \text{ and so } i\mathcal{I} - A_j - e^{i\mathcal{I}z} \in \mathbb{R} \setminus \{0\}.$$

Hence,  $i\mathcal{I} - e^{i\mathcal{I}z} \in \mathbb{R}$  and, like in Lemma 7.2, one obtains from (7.9) that  $e^{i\mathcal{I}z}$  and  $i\mathcal{I}$  are real. The latter implies  $k \in i\mathbb{R}$  and, in turn,  $n_{\min}(k) = 1$ , a contradiction.  $\square$

LEMMA 7.4. (i) Assume that  $k \in \Sigma_{\text{res}}(a, Y)$  is of minimal decay for  $f \in \mathbb{R}$ ,  $a \in \overline{\mathbb{R}^4}$ , and  $n(a) = n_{\min}(k) = 2$ . Then:

(i.a)  $f \in \pm \bigcup_{l=0}^{\infty} (\frac{2l\pi}{L}, \frac{(2l+1)\pi}{L})$  and  $k = f + \frac{i}{L} \ln \frac{\sin(Lf)}{Lf}$ ;

(i.b) it is possible to enumerate  $y_j$  so that  $a_1 = a_2 = \frac{1}{4\pi L} \ln \frac{Lf}{\sin(Lf)} - \frac{f}{4\pi} \cot(Lf)$  and  $a_3 = a_4 = \infty$ .

(ii) If  $f$ ,  $k$ , and  $a$  satisfy (i.a)-(i.b), then  $k$  and  $a$  are of minimal decay for  $f$ .

*Proof.* (i) As before, let us enumerate  $y_j$  such that  $a_j \in \mathbb{R}$ ,  $j = 1, 2$ . Using the fact that  $n_{\min}(k) = 1$  for  $k \in i\mathbb{R}$ , one shows in a way similar to the proof of Lemma 7.3, that  $i\mathcal{I} \notin \mathbb{R}$ ,  $A_1 = A_2$ ,  $i\mathcal{I} - A_1 - e^{i\mathcal{I}z} \in \mathbb{R}$ , and that  $i\mathcal{I} - A_1 + e^{i\mathcal{I}z} \neq 0$ . Then (7.7) implies that  $i\mathcal{I} - A_1 - e^{i\mathcal{I}z} = 0$ . Taking imaginary and real parts of  $A_1 = i\mathcal{I} - e^{i\mathcal{I}z}$  in a way similar to the proof of Lemma 7.2, one gets (i.a-b).

(ii) Suppose (i.a) and (i.b). It is obvious that  $k \in \Sigma(a, Y)$  and so  $f$  is an achievable frequency. By Theorem 4.1, there exists resonance  $k_0$  of minimal decay for  $f$ . The facts that  $n_{\min}(k_0) \neq 1, 3, 4$  follow, resp., from  $f \neq 0$ , Lemma 7.3, and the facts that  $f$



is not in the frequency range of Lemma 7.2. Thus,  $n_{\min}(k_0) = 2$ , and statement (i) of the lemma implies  $k_0 = k$ .  $\square$

Combining arguments of the proof of Lemma 7.4 (ii) with Lemmas 7.2, 7.3 and 7.4 (i), it is easy to show that  $\mathbb{R} \setminus \text{Re}\Sigma_{\text{res}} [\mathbb{F}_{\mathbb{R}}] = \frac{\pi}{L}(\mathbb{Z} \setminus \{0\})$  and the fact that

$$\text{if } f, k, \text{ and } a \text{ are as in Lemma 7.2, then } k \text{ and } a \text{ are of minimal decay for } f. \tag{7.12}$$

This completes the proof of Theorem 7.1.

### 8. Additional remarks and discussion

**Achievable frequencies.** Generically, if  $N \geq 2$ , the set of achievable frequencies  $\text{Re}\Sigma_{\text{res}} [\mathbb{F}_{\mathbb{R}}]$  takes the whole line  $\mathbb{R}$ . More precisely, the example with two centers at the end of [3, Section II.1.1] easily implies the following statement.

**PROPOSITION 8.1.** *Let  $N \geq 2$ . If  $f_0 \in \mathbb{R} \setminus \text{Re}\Sigma_{\text{res}} [\mathbb{F}_{\mathbb{R}}]$ , then for all  $1 \leq j, j' \leq N$ ,  $j \neq j'$ , there exist nonzero integers  $l_{j,j'}$  such that  $f_0 = \frac{\pi l_{j,j'}}{|y_j - y_{j'}|}$ . In particular, the set  $\mathbb{R} \setminus \text{Re}\Sigma_{\text{res}} [\mathbb{F}_{\mathbb{R}}]$  either consists of isolated points, or is empty.*

**Minimization of the resonance width  $\varepsilon$ .** The interpretation of resonances  $k$  from the point of view of the Schrödinger equation  $i\partial_t u = H_{\alpha} u$  is usually done in another system of parameters. Namely,  $E = \text{Re}k^2$  is interpreted as *the energy of the resonance*  $k$  and  $\varepsilon = 2|\text{Im}k^2|$  is *the width of the resonance* (see e.g. [43]). For nonnegative potentials with constraints on their  $\mathcal{L}^p$ -norms and compact supports, the problem of finding local and global minimizers of  $\varepsilon$  was considered in [23, 45].

The results of previous sections can be easily adapted to the problem of minimization of resonance width. The analogue of the problem of [23] for point interactions can be addressed in the following way.

**COROLLARY 8.2.** *Let  $0 \leq E_1 \leq E_2 \leq +\infty$  and  $N \geq 2$ . Then:*

- (i) *There exists  $a \in \mathbb{R}^N$  and  $k_0 \in \Sigma_{\text{res}}(a, Y)$  such that  $|\text{Im}k_0^2| = \inf_{\substack{k \in \Sigma_{\text{res}} [\mathbb{F}_{\mathbb{R}}] \\ E_1 \leq \text{Re}k^2 \leq E_2}} |\text{Im}k^2|$ .*
- (ii) *For any  $a$  and  $k_0$  satisfying (i), the necessary condition (i) of Theorem 6.1 holds.*

*Proof of statement (i).* It follows from Example 3.1 and the example in [3, Section II.1.1] that for any  $E \geq 0$  and any two-point set  $Y' = \{y'_1, y'_2\}$  there exist a tuple  $\alpha' \in \mathbb{R}^2$  and a resonance  $k \in \Sigma_{\text{res}}(\alpha', Y')$  such that  $E = \text{Re}k^2$  (see also Section 7). So, when  $N \geq 2$ , the existence of minimizer follows in the case  $E_2 < +\infty$  from Theorem 4.1, and in the case  $E_2 = +\infty$  from Theorem 4.1 and the uniform bound (5.2).  $\square$

The statement (ii) of Corollary 8.2 follows immediately from the following strengthened version of Theorem 6.3 (i).

**THEOREM 8.3.** Denote by  $\Omega_+$  the path-connected components of the open set  $\mathbb{C} \setminus \Sigma_{\text{res}}[\mathbb{F}_{\mathbb{R}}]$  that contain  $\mathbb{C}_+$ . If  $k \in \text{bd}\Omega_+$  and  $a \in \overline{\mathbb{R}}^N$  are such that  $k \in \Sigma_{\text{res}}(a, Y)$ , then there exists  $\xi \in [-\pi, \pi)$  such that (6.2) holds.

The proof of Theorem 8.3 follows the same lines as that of Theorem 6.3 (i).

**Symmetries and non-uniqueness of extremizers.** If there exists a unique operator  $H$  that generates an extremal (in any sense) resonance, then  $H$  preserves all the symmetries of this optimization problem.

This obvious principle can be illustrated by the tetrahedron equidistant case of Theorem 7.1 if we consider a frequency  $f \in \pm \bigcup_{l \in \mathbb{N}} ((2l-1)\pi/L, 2l\pi/L)$  and operators  $H_a \in \mathbb{F}_{\mathbb{R}}$  of minimal decay for  $f$ . Indeed, the tuple  $a$  found in Lemma 7.2 is the unique tuple of minimal decay for  $f$ . The corresponding optimal operator  $H_a$  possesses all the symmetries of the symmetry group  $T_a$  of a regular tetrahedron. The corresponding resonance of minimal decay is simple (i.e., of multiplicity 1).

**EXAMPLE 8.4.** In the case  $f \in \pm \bigcup_{l=0}^{\infty} (\frac{2l\pi}{L}, \frac{(2l+1)\pi}{L})$  (with  $N = 4$  and  $|y_j - y_{j'}| = L$  for all  $j \neq j'$ ), the situation is different since an infinite family of generic  $H_\alpha$  of minimal decay preserves only one of the symmetries. Let us consider in more details operators  $H_a$  that generate the resonance  $k$  of minimal decay over  $\mathbb{F}_{\mathbb{R}}$  for such  $f$  (this  $k$  is calculated in Lemma 7.4).

It is easy to see that a 4-tuple  $a \in \overline{\mathbb{R}}^4$  is of minimal decay for  $f$  if and only if two of the parameters  $a_1, \dots, a_4$  are equal to  $a_* := \frac{1}{4\pi L} \ln \frac{Lf}{\sin(Lf)} - \frac{f}{4\pi} \cot(Lf)$ .

(8.1)

Indeed, in the case (8.1), one can see that at least two of the numbers  $A_j$  satisfy  $i\chi - A_j - e^{i\chi} = 0$ . So (7.1) and Lemma 4.2 imply  $k \in \Sigma_{\text{res}}(a, Y)$ . On the other hand, assume that  $a$  does not satisfy (8.1). Then (7.3), (7.9), and (7.7) imply  $i\chi - A_j - e^{i\chi} \neq 0$  for all  $j = 1, \dots, 4$ . Applying the arguments of Lemmas 7.2 - 7.4 (i), it is not difficult to see that  $n(a) \neq 4, 3, 2, 1$ , a contradiction.

We see that, in the case of vertices of a regular tetrahedron and  $\mathbb{F} = \mathbb{F}_{\mathbb{R}}$ ,

- (a) each operator  $H_a$  of minimal decay has at least one of the symmetries (of the symmetry group) of  $\mathbb{F}$ ,
- (b) there exists one  $H_a$  of minimal decay that possesses all the symmetries of  $\mathbb{F}$ .

The question to what extent the above observations (a) and (b) remain true for other feasible sets  $\mathbb{F}$  and other resonance optimization problems [10, 9, 25] seems to be natural. We would like to note that related questions often appear in numerical and engineering studies [25, 34, 39].

**REMARK 8.1.** It worth to note that an example of a 1-D resonance optimization problem that possesses two different optimizers generating the same resonance of minimal decay has been constructed recently in [29]. This example involves the equation of an inhomogeneous string and uses essentially the specific effects for its resonances on  $i\mathbb{R}_-$ .

**Multiple resonances of minimal decay.** In many reasonable settings generic resonances are simple [13]. Resonances of minimal decay are very specific ones. Section 7 shows that they can be multiple (i.e., of multiplicity  $\geq 2$ ).

EXAMPLE 8.5. In the settings of Example 8.4, formula (7.1) implies that the resonance of minimal decay  $k$  for  $f \in \pm \bigcup_{l=0}^{\infty} (\frac{2l\pi}{L}, \frac{(2l+1)\pi}{L})$  is

- (i) of multiplicity 2 for  $H_a$  if and only if exactly three of parameters  $a_1, \dots, a_4$  are equal to  $a_*$ ;
- (ii) of multiplicity 3 for  $H_a$  exactly when  $a_1 = \dots = a_4 = a_*$ .

It seems that the above effect with existence of multiple resonances of minimal decay is new. The explicitly computed 1-D resonances of minimal decay in [27, 29] are simple. However, it was noticed in numerical optimization experiments of [25] that, in the 2-D case with upper and lower constraints on the index of refraction, the gradient ascent iterative procedure stopped when it encountered a multiple resonance because it was not able to determine which resonance branch to follow. In our opinion the Schrödinger operators with a finite number of point interactions is a good choice of a model for the study of the phenomena behind this numerical difficulty.

REMARK 8.2. As it was pointed out by the referee, the optimization technique of this paper can be applied to other types of non-selfadjoint spectral problems, e.g., to resonances of non-compact quantum graphs with finitely many edges [11, 31, 32, 42].

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## REFERENCES

- [1] S. ALBEVERIO, J. E. FENSTAD, R. HØEGH-KROHN, *Singular perturbations and nonstandard analysis*, Trans. Amer. Math. Soc. **252** (1979), 275–295.
- [2] S. ALBEVERIO, F. GESZTESY, R. HØEGH-KROHN, *The low energy expansion in nonrelativistic scattering theory*, in Annales de l’IHP Physique théorique, vol. **37**, no. 1, 1982, 1–28.
- [3] S. ALBEVERIO, F. GESZTESY, R. HØEGH-KROHN, H. HOLDEN, *Solvable models in quantum mechanics*, 2nd edition, with an appendix by P. Exner, AMS Chelsea Publishing, Providence, RI, 2005.
- [4] S. ALBEVERIO, F. GESZTESY, R. HØEGH-KROHN, L. STREIT, *Charged particles with short range interactions*, in Annales de l’IHP Physique théorique, vol. **38**, no. 3, 1983, 263–293.
- [5] S. ALBEVERIO, R. HØEGH-KROHN, *Perturbation of resonances in quantum mechanics*, J. Math. Anal. Appl. **101** (1984), 491–513.
- [6] S. ALBEVERIO, P. KURASOV, *Singular perturbations of differential operators: solvable Schrödinger-type operators*, Cambridge University Press, 2000.
- [7] C. A. BERENSTEIN, R. GAY, *Complex analysis and special topics in harmonic analysis*, Springer Science & Business Media, 2012.
- [8] S. BOYD, L. VANDENBERGHE, *Convex optimization*, Cambridge university press, Cambridge, 2004.
- [9] S. J. COX, M. L. OVERTON, *Perturbing the critically damped wave equation*, SIAM J. Appl. Math. **56** (1996), 1353–1362.
- [10] S. COX, E. ZUAZUA, *The rate at which energy decays in a string damped at one end*, Indiana Univ. Math. J. **44** (1995), 545–573.
- [11] E. B. DAVIES, P. EXNER, J. LIPOVSKÝ, *Non-Weyl asymptotics for quantum graphs with general coupling conditions*, Journal of Physics A **43** (47) (2010), 474013.
- [12] U. P. DHARANIPATHY, M. MINKOV, M. TONIN, V. SAVONA, R. HOUDRÉ, *High-Q silicon photonic crystal cavity for enhanced optical nonlinearities*, Applied Physics Letters **105** (2014), 101101.
- [13] S. DYATLOV, M. ZWORSKI, *Mathematical theory of scattering resonances*, Book in progress, <http://math.mit.edu/dyatlov/res/>.
- [14] P. EXNER, *An isoperimetric problem for point interactions*, J. Phys. A **38** (2005), 4795.
- [15] P. EXNER, *Open quantum systems and Feynman integrals*, Springer Science & Business Media, Berlin, 2012.
- [16] P. EXNER, M. FRAAS, E. M. HARRELL, *On the critical exponent in an isoperimetric inequality for chords*, Physics Letters A **368** (2007), 1–6.
- [17] P. EXNER, E. M. HARRELL, M. LOSS, *Inequalities for means of chords, with application to isoperimetric problems*, Lett. Math. Phys. **75** (2006), 225–233.
- [18] P. EXNER, V. LOTOREICHIK, *A spectral isoperimetric inequality for cones*, Lett. Math. Phys. **107** (2017), 717–732.
- [19] R. FROESE, *Asymptotic distribution of resonances in one dimension*, J. Differential Equations **137** (1997), 251–272.
- [20] F. GESZTESY, *Perturbation theory for resonances in terms of Fredholm determinants*, in Resonances—Models and Phenomena, edited by S. Albeverio, L. S. Ferreira and L. Streit, Springer, Berlin–Heidelberg, 1984, 78–104.
- [21] YIQI GU, XIAOLIANG CHENG, *A Numerical Approach for Defect Modes Localization in an Inhomogeneous Medium*, SIAM J. Appl. Math. **73** (2013), 2188–2202.
- [22] E. M. HARRELL, *General lower bounds for resonances in one dimension*, Comm. Math. Phys. **86** (1982), 221–225.
- [23] E. M. HARRELL, R. SVIRSKY, *Potentials producing maximally sharp resonances*, Trans. Amer. Math. Soc. **293** (1986), 723–736.
- [24] P. HEIDER, D. BEREBICHEZ, R. V. KOHN, AND M. I. WEINSTEIN, *Optimization of scattering resonances*, Struct. Multidisc. Optim. **36** (2008), 443–456.
- [25] C.-Y. KAO, F. SANTOSA, *Maximization of the quality factor of an optical resonator*, Wave Motion **45** (2008), 412–427.
- [26] I. M. KARABASH, *Optimization of quasi-normal eigenvalues for 1-D wave equations in inhomogeneous media; description of optimal structures*, Asymptotic Analysis **81** (2013), 273–295.
- [27] I. M. KARABASH, *Pareto optimal structures producing resonances of minimal decay under  $L^1$ -type constraints*, J. Differential Equations **257** (2014), 374–414.
- [28] I. M. KARABASH, O. M. LOGACHOVA, I. V. VERBYTSKYI, *Nonlinear bang-bang eigenproblems*

- and optimization of resonances in layered cavities*, Integr. Equ. Oper. Theory **88** (1) (2017), 15–44.
- [29] I. M. KARABASH, O. M. LOGACHOVA, I. V. VERBYTSKYI, *Overdamped modes and optimization of resonances in layered cavities*, to appear in Methods of Functional Analysis and Topology.
- [30] T. KATO, *Perturbation theory for linear operators*, Springer Science & Business Media, Berlin, 2013.
- [31] V. KOSTRYKIN, R. SCHRADER, *Kirchhoff's rule for quantum wires*, Journal of Physics A **32** (4), (1999), 595.
- [32] T. KOTTOS, U. SMILANSKY, *Quantum graphs: a simple model for chaotic scattering*, Journal of Physics A **36** (12), (2003), 3501–3524.
- [33] P. D. LAX, R. S. PHILLIPS, *A logarithmic bound on the location of the poles of the scattering matrix*, Arch. Ration. Mech. Anal. **40** (1971), 268–280.
- [34] X. LIANG, S. G. JOHNSON, *Formulation for scalable optimization of microcavities via the frequency-averaged local density of states*, Optics express **21** (2013), 30812–30841.
- [35] R. E. LANGER, *On the zeros of exponential sums and integrals*, Bulletin of the American Mathematical Society **37** (4), (1931), 213–239.
- [36] V. LOTOREICHIK, *Spectral isoperimetric inequalities for  $\delta$ -interactions on open arcs and for the Robin Laplacian on planes with slits*, (2016), arXiv preprint arXiv:1609.07598.
- [37] B. MAES, J. PETRÁČEK, S. BURGER, P. KWIECIEN, J. LUKSCH, I. RICHTER, *Simulations of high- $Q$  optical nanocavities with a gradual 1D bandgap*, Optics express **21** (2013), 6794–6806.
- [38] G. MORA, J. M. SEPULCRE, T. VIDAL, *On the existence of exponential polynomials with prefixed gaps*, Bulletin of the London Mathematical Society **45** (6), (2013), 1148–1162.
- [39] M. NOTOMI, E. KURAMOCHI, H. TANIYAMA, *Ultra-high- $Q$  nanocavity with 1D photonic gap*, Optics Express **16** (15) (2008), 11095–11102.
- [40] KAI OGASAWARA, *The analysis of resonant states in a one-dimensional potential of quantum dots*, Honours Physics Bachelor thesis, University of British Columbia, 2014.
- [41] B. OSTING, M. I. WEINSTEIN, *Long-lived scattering resonances and Bragg structures*, SIAM J. Appl. Math. **73** (2013), 827–852.
- [42] V. PIVOVARCHIK, H. WORACEK, *Eigenvalue asymptotics for a star-graph damped vibrations problem*, Asymptotic Analysis **73** (3), (2011), 169–185.
- [43] M. REED, B. SIMON, *Analysis of Operators*, vol. IV of Methods of Modern Mathematical Physics, Academic Press, New York, 1978.
- [44] A. A. SHUSHKOV, *Structure of resonances for symmetric scatterers*, Theoretical and Mathematical Physics **64** (1985), 944–949.
- [45] R. SVIRSKY, *Maximally resonant potentials subject to  $p$ -norm constraints*, Pacific J. Math. **129** (1987), 357–374.
- [46] B. R. VAINBERG, *Eigenfunctions of an operator that correspond to the poles of the analytic continuation of the resolvent across the continuous spectrum*, (Russian) Mat. Sb. (N. S.) **87** (129) (1972), 293–308, English transl.: in Mathematics of the USSR-Sbornik, **16** (2) (1972).
- [47] M. ZWORSKI, *Sharp polynomial bounds on the number of scattering poles*, Duke Math. J. **59** (1989), 311–323.

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