Operators and Matrices
Volume 11, Number 4 (2017), 1119–1139
doi:10.7153/oam-2017-11-77

ON THE KERNEL OF A SINGULAR INTEGRAL OPERATOR WITH SHIFT

RUI C. MARREIROS

Dedicated to Professor Viktor G. Kravchenko

(*Communicated by I. M. Spitkovsky*)

Abstract. Some estimates for the dimension of the kernel of the singular integral operator $I - cUP_+ : L_p^n(\mathbb{T}) \to L_p^n(\mathbb{T})$, $p \in (1,\infty)$, with a non-Carleman shift are obtained, where P_+ is the Cauchy projector, U is an isometric shift operator and $c(t)$ is a continuous matrix function on the unit circle $\mathbb{T}$. It is supposed that the shift has a finite set of fixed points and all the eigenvalues of the matrix $c(t)$ at the fixed points, simultaneously belong either to the interior of the unit circle $\mathbb{T}$ or to its exterior. The case of an operator with a general shift is also considered. Some relations between those estimates and the resolvent set of the operator cU are pointed out.

1. Introduction

Let $\mathbb{T}$ denote the unit circle in the complex plane, $\mathbb{T}_+$ and $\mathbb{T}_-$ denote the interior and the exterior (∞ included) of $\mathbb{T}$, respectively. We will also consider the domains $\mathbb{D}_+ = \{z \in \mathbb{C} : |z| < \sin\frac{\pi}{p}\}$ and $\mathbb{D}_- = \{z \in \mathbb{C} : |z| > \sin^{-1}\frac{\pi}{p}\}$; here and bellow we always assume $p \in (1,\infty)$, in correspondence with the Lebesgue space $L_p(\mathbb{T})$. Evidently, $\mathbb{D}_\pm = \mathbb{T}_\pm$, for $p = 2$. On $L_p(\mathbb{T})$ we consider the singular integral operator (SIO) with Cauchy kernel, defined almost everywhere on $\mathbb{T}$ by

$$(S\varphi)(t) = (\pi i)^{-1} \int_{\mathbb{T}} \varphi(\tau)(\tau - t)^{-1} d\tau,$$

where the integral is understood in the sense of its principal value. The operator S is a bounded linear involutive operator ($S^2 = I$, where I is the identity operator on $L_p(\mathbb{T})$). Then it is possible to define in $L_p(\mathbb{T})$ a pair of complementary projection operators,

$$P_\pm = \frac{1}{2}(I \pm S),$$

and to decompose $L_p(\mathbb{T}) = L_p^+(\mathbb{T}) \oplus \overset{\circ}{L}{}_p^-(\mathbb{T})$, with $L_p^+(\mathbb{T}) = \operatorname{im} P_+$ and $\overset{\circ}{L}{}_p^-(\mathbb{T}) = \operatorname{im} P_-$. We also set $L_p^-(\mathbb{T}) = \overset{\circ}{L}{}_p^-(\mathbb{T}) \oplus \mathbb{C}$.

Mathematics subject classification (2010): Primary 47G10, secondary 45P05.
Keywords and phrases: Singular integral operators with shift, kernel dimension, resolvent set.
This research was supported by Fundação para a Ciência e Tecnologia (Portugal) through Centro de Análise Funcional e Aplicações of Instituto Superior Técnico.
The author acknowledges discussions with Professors Viktor G. Kravchenko and Alexei Yu. Karlovich.

Paper OaM-11-77

As usual, $L_\infty(\mathbb{T})$ denotes the space of all essentially bounded functions on $\mathbb{T}$. Let us introduce the concept of matrix function generalized factorization (see, for instance, [3] and [21]). Let $p,q \in (1,\infty)$, with $p^{-1}+q^{-1}=1$; we say that a matrix function $c \in L_\infty^{n\times n}(\mathbb{T})$ admits a (right) generalized factorization in $L_p(\mathbb{T})$, if it can be represented as

$$c = c_-\Lambda c_+, \tag{1}$$

where

$$c_- \in \left[L_p^-(\mathbb{T})\right]^{n\times n},\; c_-^{-1} \in \left[L_q^-(\mathbb{T})\right]^{n\times n},\; c_+ \in \left[L_q^+(\mathbb{T})\right]^{n\times n},\; c_+^{-1} \in \left[L_p^+(\mathbb{T})\right]^{n\times n},$$

$\Lambda(t) = \operatorname{diag}\{t^{\varkappa_j}\}$, $\varkappa_j \in \mathbb{Z}$, $j = \overline{1,n}$, with $\varkappa_1 \geqslant \varkappa_2 \geqslant \ldots \geqslant \varkappa_n$, and $c_-P_+c_-^{-1}I$ represents a bounded linear operator in $L_p^n(\mathbb{T})$. The number $\varkappa = \sum_{j=1}^{n} \varkappa_j$ is called the factorization index of the determinant of the matrix function c. The integers $\varkappa_j$ are uniquely defined by the matrix function c and are called its right partial indices.

Any non-singular continuous matrix function $c \in C^{n\times n}(\mathbb{T})$ admits a generalized factorization (1) in $L_p(\mathbb{T})$ (see, for instance, the above cited [3] and [21]); for our purposes, it will be assumed that

$$c_\pm^{\pm 1} \in C^{n\times n}(\mathbb{T}). \tag{2}$$

For the particular scalar case we note that $\varkappa = \operatorname{ind} c$ if $c \in C(\mathbb{T})$; as usual, $\operatorname{ind}\varphi$ denotes the Cauchy index of a continuous function $\varphi \in C(\mathbb{T})$, i.e.,

$$\operatorname{ind}\varphi = \frac{1}{2\pi}\{\arg\varphi(t)\}_{t\in\mathbb{T}}.$$

Now let ω be a homeomorphism of $\mathbb{T}$ onto itself, which is differentiable on $\mathbb{T}$ and whose derivative does not vanish there. The function $\omega : \mathbb{T} \to \mathbb{T}$ is called a shift function or simply a shift on $\mathbb{T}$. By

$$\omega_k(t) \equiv \omega[\omega_{k-1}(t)], \quad \omega_1(t) \equiv \omega(t), \quad \omega_0(t) \equiv t, \quad t \in \mathbb{T},$$

we denote the k-th iteration of the shift, $k \geqslant 2$, $k \in \mathbb{N}$.

A shift ω is called a (generalized) Carleman shift of order $n \in \mathbb{N}\setminus\{1\}$ if $\omega_n(t) \equiv t$, but $\omega_k(t) \not\equiv t$ for $k = \overline{1,n-1}$. Otherwise, if ω is not a Carleman shift, it is called a non-Carleman shift. In what follows we will consider four different shifts, i.e., $\omega = \zeta,\eta,\alpha,\beta$: ζ and η are general shifts, in the sense Carleman or non-Carleman shifts; α and β are non-Carleman shifts having a finite set of fixed points $\{\tau_1,\tau_2,\ldots,\tau_s\}$, $s \geqslant 1$. Other properties of these shifts will be specified later on whenever necessary.

On $L_p^n(\mathbb{T})$, $p \in (1,\infty)$, associated with a shift ω, we consider a shift operator U_ω defined by

$$(U_\omega\varphi)(t) = u_\omega(t)\varphi[\omega(t)], \quad t \in \mathbb{T},$$

where the function u_ω is chosen in such way that the following properties hold [1]:

[1]Given a shift ω, the property i) is always satisfied taking $u_\omega(t) = |\omega'(t)|^{\frac{1}{p}}$. To verify the property ii) the function u_ω has to be chosen depending on the concrete shift ω (see Section 3.2), which is not always possible.

i) U_ω is isometric, i.e., $\|U_\omega\varphi\|_{L_p^n} = \|\varphi\|_{L_p^n}$, $\omega = \zeta, \eta, \alpha, \beta$.

ii) $U_\omega S = SU_\omega$, where S is the SIO with Cauchy kernel, $\omega = \eta, \beta$.

Let $c \in C^{n\times n}(\mathbb{T})$ be a given continuous matrix function; in this paper, we consider the SIO with shift $T_\omega : L_p^n(\mathbb{T}) \to L_p^n(\mathbb{T})$, $p \in (1,\infty)$, $\omega = \zeta, \eta, \alpha, \beta$, defined by

$$T_\omega = I - cU_\omega P_+. \tag{3}$$

We note that for the SIO with shift of the form

$$T(A_1, A_2) = A_1 P_+ + A_2 P_-, \tag{4}$$

where A_1 and A_2 are the functional operators

$$A_1 = a_1 I + b_1 U_\omega, \quad A_2 = a_2 I + b_2 U_\omega,$$

and $a_1, a_2, b_1, b_2 \in C^{n\times n}(\mathbb{T})$, the Fredholmness conditions and the index formulas are known [13]. The Fredholm criterion can be formulated as follows: the SIO with shift $T(A_1, A_2)$ is Fredholm in $L_p^n(\mathbb{T})$, $p \in (1,\infty)$, if and only if the functional operators A_1 and A_2 are continuously invertible in $L_p^n(\mathbb{T})$. The solvability theory (calculation of the defect numbers, construction of bases for the defect subspaces, spectral properties) of the operator $T(A_1, A_2)$ has been less studied (see [4], [10], [11], and [12]), even for the case of a Carleman shift. For the case of a non-Carleman shift, the question remains open (see [1], [9], [14], [15], and [16]).

We can also write the operator T_ω defined by (3) in the form

$$T_\omega = (I - cU_\omega)P_+ + P_-.$$

So the question of Fredholmness of the SIO with shift T_ω leads to the question of continuous invertibility of the operator $I - cU_\omega$; on the other hand, the invertibility of the operator $I - cU_\omega$ is connected with the description of the resolvent set, and the spectrum, of the operator cU_ω. We also can say that the essential spectrum of the operator $cU_\omega P_+$ is related with the spectrum of the operator cU_ω.

We must say that, in general, in the case of a non-Carleman shift having a finite set of fixed points $\{\tau_1, \tau_2, \ldots, \tau_s\}$, $s \geqslant 1$, the shift α and the corresponding shift operator U_α considered in this paper, the necessary and sufficient conditions of invertibility for the operator $I - cU_\alpha$, can not be expressed in an explicit form. A specificity of the conditions is expressed by a particular choice of a, so-called, α-solutions of the homogeneous functional equation associated with the operator $I - cU_\alpha$ (see Sections 3.4.1–3.4.11, pp. 118–142, in [13], and the Remark 1.1 below). Let us recall some related key concepts. Let $\sigma(g)$, $\rho(g)$ and $\|g\|_2$, denote the spectrum, the spectral radius and the spectral norm of a matrix $g \in \mathbb{C}^{n\times n}$, respectively. Recall that $\rho(g) \equiv \max\{|\lambda| : \lambda$ is an eigenvalue of $g\}$; we also denote $\theta(g) \equiv \min\{|\lambda| : \lambda$ is an eigenvalue of $g\}$. Given a bounded linear operator $A : L_p^n(\mathbb{T}) \to L_p^n(\mathbb{T})$, $\sigma(A)$ and $\rho(A)$, denote the spectrum and the resolvent set of the operator A, respectively; $\sigma(A) \cup \rho(A) = \mathbb{C}$. By $\sigma_{ess}(A) \subset \sigma(A)$ we denote the essential spectrum of A, i.e., the set of those $\lambda \in \mathbb{C}$ for which $A - \lambda I$ is not a Fredholm operator in $L_p^n(\mathbb{T})$.

DEFINITION 1.1. A continuous matrix function $d \in C^{n\times n}(\mathbb{T})$ is called a matrix of normal form on $\mathbb{T}$ if

$$d(t) = \begin{pmatrix} d_1(t) & O^{m\times k} \\ O^{k\times m} & d_2(t) \end{pmatrix}, \tag{5}$$

where $d_1 \in C^{m\times m}(\mathbb{T})$, $d_2 \in C^{k\times k}(\mathbb{T})$, $k+m=n$, $O^{r\times s}$ is a $r\times s$ zero matrix, and

$$\sigma[d_1(\tau_j)] \subset \mathbb{T}_+, \quad \sigma[d_2(\tau_j)] \subset \mathbb{T}_-, \quad j=\overline{1,s}, \quad \det d_2(t) \neq 0, \quad \forall t \in \mathbb{T}.$$

DEFINITION 1.2. A continuous matrix function $c \in C^{n\times n}(\mathbb{T})$ is called α-reducible to the normal on $\mathbb{T}$ if there exists a continuous non-singular matrix function $b(t)$ such that

$$b^{-1}(t)c(t)b[\alpha(t)] = d(t), \tag{6}$$

where $d(t)$ is a matrix of normal form on $\mathbb{T}$.

The following invertibility criterion for the matrix operator $I - cU_\alpha$ takes place in the general case.

THEOREM 1.1. [13] *The operator $I - cU_\alpha$ is continuously invertible in $L_p^n(\mathbb{T})$, $p \in (1,\infty)$, if and only if the matrix $c(t)$ is α-reducible to the normal form on $\mathbb{T}$.*

LEMMA 1.1. [13] *The block triangular matrix $a \in C^{n\times n}(\mathbb{T})$,*

$$a(t) = \begin{pmatrix} d_1(t) & O^{m\times k} \\ f(t) & d_2(t) \end{pmatrix}, \tag{7}$$

where d_1 and d_2 satisfy the conditions of Definition 1.1, and $f \in C^{k\times m}(\mathbb{T})$, is α-reducible to the normal on $\mathbb{T}$.

REMARK 1.1. The α-reducibility of the matrix $c(t)$ to the normal form, i.e., the construction of the non-singular matrix $b(t)$ in (6), is connected with the existence of a certain class of solutions, the α-solutions, of the homogeneous functional equation $\varphi(t) = c(t)\varphi[\alpha(t)]$. It is clear apriori that the α-solutions don't belong to the space $L_p^n(\mathbb{T})$, otherwise the operator $I - cU_\alpha$ would not be invertible in $L_p^n(\mathbb{T})$.

For convenience, we emphasize four cases of explicit sufficient conditions of invertibility for the operator $I - cU_\alpha$:

Case 1. The matrix c satisfies the property $\sigma[c(\tau_j)] \subset \mathbb{T}_+$, $j=\overline{1,s}$;

Case 2. The matrix c satisfies the properties $\sigma[c(\tau_j)] \subset \mathbb{T}_-$, $j=\overline{1,s}$, and $\det c(t) \neq 0$ for all $t \in \mathbb{T}$;

Case 3. The matrix c is a block diagonal matrix of normal form (5).

Case 4. The matrix c is a block triangular matrix of the form (7).

We note that if $n=1$, the scalar case, then the conditions of case 1 and case 2 are not only sufficient but also necessary for the invertibility of the operator $I - cU_\alpha$; i.e., the operator $I - cU_\alpha$ is invertible on $L_p(\mathbb{T})$ if and only if either $|c(\tau_j)| < 1$, $j=\overline{1,s}$, or $|c(\tau_j)| > 1$, $j=\overline{1,s}$, and $c(t) \neq 0$ for all $t \in \mathbb{T}$.

In [16], on the Hilbert space $L_2^n(\mathbb{T})$, we obtained estimates for the defect number $\dim\ker T_\omega$ for the operator $T_\omega = I - cU_\omega P_+$, with matrix and scalar coefficient, satisfying one of the two sets of Fredholmness conditions: the cases 1 ($\omega = \zeta, \alpha$) and 2 ($\omega = \eta, \beta$), above. In the present paper we revisited the mentioned work [16]; we generalize some of the obtained results on the Lebesgue space $L_p^n(\mathbb{T})$, $p \in (1,\infty)$ (Sections 2-6). Then we consider the operator cU_β in the matrix case (Section 7); in this case we can only obtain subsets of the resolvent set of the operator cU_β. We also consider the operator cU_β in the scalar case (Section 8); we write the resolvent set, and the spectrum, of this operator. In both cases, matrix and scalar, we write estimates for the dimension of the kernel of the operator $I - \lambda^{-1}cU_\beta P_+$, where λ belongs to the resolvent set of the operator cU_β. We think we made a small progress on "the very difficult question related to the solvability theory of the SIO of type (4) with a non-Carleman shift" (G. S. Litvinchuk in [20], p. XVI).

2. A SIO with a general shift

In the Sections 2–6 we present some estimates for the dimension of the kernel of the operator (3) on the Lebesgue space $L_p^n(\mathbb{T})$, $p \in (1,\infty)$. We follow the work [16] where this estimates were obtained on the Hilbert space $L_2^n(\mathbb{T})$.

2.1. Estimate one

We begin considering a general shift $\zeta : \mathbb{T} \to \mathbb{T}$, the associated isometric shift operator U_ζ, and the SIO with shift defined by (3) (with $\omega = \zeta$)

$$T_\zeta = I - cU_\zeta P_+. \tag{8}$$

The following results take place.

THEOREM 2.1. [16] *Let T_ζ be the operator defined by* (8) *and*

$$N = I - aU_\zeta P_+, \tag{9}$$

$$M = I - rP_+r^{-1}P_-N^{-1}, \tag{10}$$

where $r \in C^{n\times n}(\mathbb{T})$ is an invertible matrix function satisfying the condition

$$P_+r^{\pm 1}P_+ = r^{\pm 1}P_+, \tag{11}$$

and $a(t) = r(t)c(t)r^{-1}[\zeta(t)]$.

If the operator N is invertible, then the following equality holds

$$\dim\ker T_\zeta = \dim\ker M.$$

PROPOSITION 2.1. [16] *Let M be the operator defined by* (10) *and r a $(n\times n)$ polynomial matrix satisfying the condition* (11)*; let*

$$l_1(r) = \sum_{i=1}^{n} \max_{j=\overline{1,n}} l_{i,j}, \tag{12}$$

where $l_{i,j}$ *is the degree of the element* $r_{i,j}$ *of the polynomial matrix* r. *Then the following inequality holds*

$$\dim\ker M \leqslant l_1(r).$$

We can state the following result.

THEOREM 2.2. *Let* $T_\zeta = I - cU_\zeta P_+$ *be the operator defined by* (8) *and* r *a polynomial matrix satisfying the conditions* (11) *and*

$$\max_{t\in\mathbb{T}} \left\| r(t)c(t)r^{-1}[\zeta(t)] \right\|_2 < \sin\frac{\pi}{p}. \tag{13}$$

Let R_c *be the set of all such matrices* r, $l_1(r)$ *be the number defined by* (12) *for each matrix* r *and*

$$l(c) = \min_{r\in R_c}\{l_1(r)\}. \tag{14}$$

If the set R_c *is not empty, then the following estimate holds*

$$\dim\ker T_\zeta \leqslant l(c).$$

Proof. We set $a(t) = r(t)c(t)r^{-1}[\zeta(t)]$; with (13) we can show that the operator defined by (9) is invertible. Indeed, since $\max_{t\in\mathbb{T}}\|a(t)\|_2 < \sin\frac{\pi}{p}$, $\left\|U_\zeta\right\|_{L_p} = 1$ and $\|P_+\|_{L_p} = \sin^{-1}\frac{\pi}{p}$ (see Corollary 2.5, p. 385, in [5]), it follows that $N = I - aU_\zeta P_+$ is an invertible operator whose inverse is given by the Neumann series

$$N^{-1} = I + aU_\zeta P_+ + (aU_\zeta P_+)^2 + \cdots.$$

Taking into account Theorem 2.1 and Proposition 2.1, the result follows. □

2.2. Estimate two

Consider now a shift η such that the corresponding shift operator U_η satisfies the additional property

$$U_\eta S = SU_\eta;$$

and the SIO with shift (3) (with $\omega = \eta$)

$$T_\eta = I - cU_\eta P_+. \tag{15}$$

Moreover we suppose that the matrix function $c \in C^{n\times n}(\mathbb{T})$ has the property

$$\det c(t) \neq 0, \quad \forall t \in \mathbb{T}. \tag{16}$$

Under condition (16) the continuous matrix function c admits the factorization (1). It is assumed that (2) is satisfied.

We continue with the following result.

THEOREM 2.3. [16] *Let* T_η *be the operator defined by* (15)*, where* $c \in C^{n\times n}(\mathbb{T})$ *satisfies the conditions* (16)*,* (1) *and* (2)*; then the following estimate holds*

$$\dim\ker T_\eta \leqslant \dim\ker(I - \widetilde{c}U_\eta^{-1}P_+) + \sum_{\varkappa_j<0} |\varkappa_j|, \tag{17}$$

where $\widetilde{c} = c_+ c^{-1} c_+^{-1}(\eta_{-1})$.

Now, supposing that the operator $I - \widetilde{c}U_\eta^{-1}P_+$ is under the conditions of Theorem 2.2, we can state the following result.

THEOREM 2.4. *Let* $T_\eta = I - cU_\eta P_+$ *be the operator defined by* (15)*, where* $c \in C^{n\times n}(\mathbb{T})$ *satisfies the conditions* (16)*,* (1) *and* (2)*; and* r *a polynomial matrix satisfying the conditions* (11) *and*

$$\max_{t\in\mathbb{T}} \left\| r(t)\widetilde{c}(t)r^{-1}[\eta(t)] \right\|_2 < \sin\frac{\pi}{p},$$

where $\widetilde{c} = c_+ c^{-1} c_+^{-1}(\eta_{-1})$. *Let* $R_{\widetilde{c}}$ *be the set of all such matrices* r *and* $l(\widetilde{c})$ *the number defined by* (14) *for the matrix* $\widetilde{c}$.

If the set $R_{\widetilde{c}}$ *is not empty, then the following estimate holds*

$$\dim\ker T_\eta \leqslant l(\widetilde{c}) + \sum_{\varkappa_j<0} |\varkappa_j|,$$

where $\varkappa_j \in \mathbb{Z}$, $j = \overline{1,n}$ *are the partial indices of the matrix* c.

Proof. Since the operators U_η and U_η^{-1} verify similar properties, the operator $I - \widetilde{c}U_\eta^{-1}P_+$ satisfies all the conditions of Theorem 2.2; thus

$$\dim\ker(I - \widetilde{c}U_\eta^{-1}P_+) \leqslant l(\widetilde{c}).$$

With (17) the result follows. □

3. A SIO with a non-Carleman shift

The estimate of the dimension of the kernel of the operator T_ω, $\omega = \zeta, \eta, \alpha, \beta$, is related with the construction of the polynomial matrix r (see Theorems 2.2 and 2.4); below we perform this task, in the case of a non-Carleman shift, $\omega = \alpha, \beta$, under certain conditions for the operator T_ω: subcases of the cases 1 and 2 mentioned in the Introduction. Indeed, then we show that the sets R_c, and $R_{\widetilde{c}}$, introduced in Theorem 2.2, and Theorem 2.4, are not empty under those conditions.

3.1. Case 1

Let us consider the SIO with shift defined by (3) (with $\omega = \alpha$)

$$T_\alpha = I - cU_\alpha P_+, \tag{18}$$

with a non-Carleman shift $\alpha : \mathbb{T} \to \mathbb{T}$, which has a finite set of fixed points $\{\tau_1, \tau_2, \dots, \tau_s\}$, $s \geqslant 1$; U_α is the associated isometric shift operator.

The following results take place.

PROPOSITION 3.1. *For every continuous matrix function $d \in C^{n\times n}(\mathbb{T})$ such that*

$$\sigma[d(\tau_j)] \subset \mathbb{D}_+, \quad j = \overline{1,s}, \tag{19}$$

there exists a polynomial matrix r satisfying the conditions

$$\max_{t\in\mathbb{T}} \left\| r(t)d(t)r^{-1}[\alpha(t)] \right\|_2 < \sin\frac{\pi}{p} \tag{20}$$

and

$$P_+ r^{\pm 1} P_+ = r^{\pm 1} P_+. \tag{21}$$

Proof. We consider only the case when $\max_{t\in\mathbb{T}} \|d(t)\|_2 > \sin\frac{\pi}{p}$, because otherwise we have simply $r = E_n$ (E_n is the unit $n\times n$ matrix).

Let

$$\rho_j \equiv \rho[d(\tau_j)], \quad j = \overline{1,s}.$$

Under condition (19) naturally we have that

$$\rho_j < \sin\frac{\pi}{p}, \quad j = \overline{1,s}.$$

Then, for each matrix $d(\tau_j) \in \mathbb{C}^{n\times n}$ satisfying the condition (19), there exists a non-singular matrix $B_j \in \mathbb{C}^{n\times n}$ such that (see, for instance, p. 316 in [6])

$$\left\| B_j d(\tau_j) B_j^{-1} \right\|_2 < \sin\frac{\pi}{p}, \quad j = \overline{1,s}.$$

Now let B be the non-singular polynomial matrix, without zeros on the closure of $\mathbb{T}_+$, defined by (see, for instance, Sections 0.9.11 in [6] and 6.1 in [7])

$$B(t) = B_1 L_1(t) + B_2 L_2(t) + \dots + B_s L_s(t),$$

where

$$L_j(t) = \frac{\prod_{\substack{i=1\\ i\neq j}}^{s} (t-\tau_i)}{\prod_{\substack{i=1\\ i\neq j}}^{s} (\tau_j-\tau_i)}, \quad j = \overline{1,s},$$

are the Lagrange interpolating polynomials.

Then we define the continuous matrix function

$$b(t) = B(t)d(t)B^{-1}[\alpha(t)].$$

We represent the function $b(t)$ in the form

$$b(t) = u(t)v(t),$$

where

$$u(t) \in C^{n\times n}(\mathbb{T}), \quad \max_{t\in\mathbb{T}} \|u(t)\|_2 = \gamma < \sin\frac{\pi}{p},$$

and $v(t)$ is a continuous real valued function on $\mathbb{T}$ such that

$$v(t) \geqslant \delta > 0, \quad t \in \mathbb{T},$$

$$v(\tau_j) < 1, \quad j = \overline{1,s}.$$

Compare with (34)–(36), p. 207, in [16]; from here, doing exactly as in [16], pp. 207–208, in a similar way we obtain the inequality (20). □

THEOREM 3.1. *Let $T_\alpha = I - cU_\alpha P_+$ be the operator defined by (18), where $c \in C^{n\times n}(\mathbb{T})$ satisfies the condition (19). Then the following estimate holds*

$$\dim\ker T_\alpha \leqslant l(c),$$

where $l(c)$ is the number defined by (14) for the matrix c.

Proof. According to Proposition 3.1, there exists a polynomial matrix r such that the conditions (20) and (21) are verified for the matrix c. Taking into account Theorem 2.2, the result follows. □

3.2. Case 2

Now we consider a linear fractional non-Carleman shift preserving the orientation on $\mathbb{T}$

$$\beta(t) = \frac{at+b}{\overline{b}t+\overline{a}}, \quad t \in \mathbb{T},$$

where $a, b \in \mathbb{C}$ are such that $|a|^2 - |b|^2 = 1$. This shift has two fixed points, τ_1 and τ_2, given by the formula

$$\tau_{1,2} = \frac{a - \overline{a} \pm \sqrt{(a+\overline{a})^2 - 4}}{2\overline{b}}.$$

Obviously $\tau_1 \neq \tau_2$ if $|\mathrm{Re}\, a| \neq 1$

The shift $\beta(t)$ admits the factorization

$$\beta(t) = \beta_+(t)t\beta_-(t),$$

where

$$\beta_+(t) = \frac{1}{\overline{b}t+\overline{a}}, \quad \beta_-(t) = \frac{at+b}{t}.$$

We see that the functions $\beta_\pm, \beta_\pm^{-1}$ are analytic in $\mathbb{T}_\pm$ and continuous in the closure of $\mathbb{T}_\pm$, respectively.

For the linear fractional shift $\beta(t)$, it is convenient to consider the isometric shift operator

$$(U_\beta\varphi)(t) = \beta_+(t)\varphi[\beta(t)], \tag{22}$$

because U_β satisfies the additional property

$$U_\beta S = SU_\beta.$$

Then we consider the operator (3) (with $\omega = \beta$)

$$T_\beta = I - cU_\beta P_+, \tag{23}$$

where we suppose now that $c \in C^{n\times n}(\mathbb{T})$ has the properties

$$\sigma[c(\tau_j)] \subset \mathbb{D}_-, \quad j = 1,2, \tag{24}$$

$$\det c(t) \neq 0, \quad \forall t \in \mathbb{T}.$$

The non-singular continuous matrix function c admits the factorization (1) and (2) is assumed. Then we apply Theorem 2.3 to the operator (23); this implies the estimate

$$\dim\ker T_\beta \leqslant \dim\ker(I - \widetilde{c}U_\beta^{-1}P_+) + \sum_{\varkappa_j<0} |\varkappa_j|, \tag{25}$$

where $\widetilde{c} = c_+c^{-1}c_+^{-1}(\beta_{-1})$.

Now we analyze the operator $I - \widetilde{c}U_\beta^{-1}P_+$.

We note that the matrices $\widetilde{c}(t)$ and $c^{-1}(t)$ are similar at the fixed points of the shift; indeed at τ_j, $j = 1,2$,

$$\widetilde{c} = c_+c^{-1}c_+^{-1}.$$

We have that $\sigma[c(\tau_j)] \subset \mathbb{D}_-$; then

$$\sigma[c^{-1}(\tau_j)] = \sigma[\widetilde{c}(\tau_j)] \subset \mathbb{D}_+, \quad j = 1,2.$$

Therefore the operator $I - \widetilde{c}U_\beta^{-1}P_+$ satisfies all the conditions of Theorem 3.1; thus

$$\dim\ker(I - \widetilde{c}U_\beta^{-1}P_+) \leqslant l(\widetilde{c}).$$

Finally, with (25) we get the following estimate.

THEOREM 3.2. *Let $T_\beta = I - cU_\beta P_+$ be the operator defined by (23), where $c \in C^{n\times n}(\mathbb{T})$ satisfies the conditions (24), (16), (1)) and (2). Then the following estimate holds*

$$\dim\ker T_\beta \leqslant l(\widetilde{c}) + \sum_{\varkappa_j<0} |\varkappa_j|,$$

where $l(\widetilde{c})$ is the number defined by (14) for the matrix $\widetilde{c} = c_+c^{-1}c_+^{-1}(\beta_{-1})$ and $\varkappa_j \in \mathbb{Z}$, $j = \overline{1,n}$ are the partial indices of the matrix c.

4. On the scalar case

4.1. The case of a general shift

Let us formulate the obtained results for the operator (8) in the scalar case:

$$T_\zeta = I - cU_\zeta P_+ : L_p(\mathbb{T}) \to L_p(\mathbb{T}). \tag{26}$$

COROLLARY 4.1. *Let T_ζ be the operator defined by (26); if there exists a polynomial r of degree m, with zeros in $\mathbb{T}_-$,*

$$r(t) = \prod_{k=1}^{m}(t-\lambda_k), \quad |\lambda_k| > 1, \quad k = \overline{1,m},$$

such that

$$\left| r(t)c(t)r^{-1}[\zeta(t)] \right| < \sin\frac{\pi}{p}, \quad \forall t \in \mathbb{T}, \tag{27}$$

then

$$\dim\ker T_\zeta \leqslant m.$$

Proof. Follows from Theorem 2.2 with $n = 1$. □

Now we consider the operator (15) in the scalar case:

$$T_\eta = I - cU_\eta P_+ : L_p(\mathbb{T}) \to L_p(\mathbb{T}), \tag{28}$$

where $c \in C(\mathbb{T})$ has the property

$$c(t) \neq 0, \quad \forall t \in \mathbb{T}. \tag{29}$$

The continuous function c admits the factorization (1) and (2) is assumed; in this case

$$c = c_- t^{\varkappa} c_+, \tag{30}$$

where

$$c_- \in L_p^-(\mathbb{T}),\ c_-^{-1} \in L_q^-(\mathbb{T}),\ c_+ \in L_q^+(\mathbb{T}),\ c_+^{-1} \in L_p^+(\mathbb{T}),\ \varkappa = \operatorname{ind} c,$$

and it is assumed that

$$c_\pm^{\pm 1} \in C(\mathbb{T}). \tag{31}$$

Suppose that a polynomial r, satisfying the condition (27) for the function c and the shift η, does not exist, but there exists such one that (27) holds for the function $\widetilde{c} = c_+ c^{-1} c_+^{-1}(\eta_{-1})$. In this case we can state the following result.

COROLLARY 4.2. *Let T_η be the operator defined by (28). Then the following estimate holds*

$$\dim\ker T_\eta \leqslant m + \max(0, -\operatorname{ind} c),$$

where m is the degree of the polynomial r defined in Corollary 4.1 for the function $\widetilde{c} = c_+ c^{-1} c_+^{-1}(\eta_{-1})$ and $\operatorname{ind} c$ is the Cauchy index of the function c.

Proof. Follows from Theorem 2.4 with $n = 1$. □

4.2. The case of a non-Carleman shift

Consider the operator (18) on $L_p(\mathbb{T})$, with $c \in C(\mathbb{T})$,

$$T_\alpha = I - cU_\alpha P_+. \tag{32}$$

COROLLARY 4.3. *For every continuous function $c \in C(\mathbb{T})$ such that*

$$\left|c(\tau_j)\right| < \sin\frac{\pi}{p}, \quad j = \overline{1,s},$$

there exists a polynomial r of degree m, with zeros in $\mathbb{T}_-$,

$$r(t) = \prod_{k=1}^{m}(t-\lambda_k), \quad |\lambda_k| > 1, \quad k = \overline{1,m},$$

such that

$$\left|r(t)c(t)r^{-1}[\alpha(t)]\right| < \sin\frac{\pi}{p}, \quad \forall t \in \mathbb{T}.$$

Moreover

$$\dim\ker T_\alpha \leqslant m,$$

where T_α is the operator defined by (32).

Proof. Follows from Theorem 3.1 with $n = 1$. □

Now consider the operator (23) on $L_p(\mathbb{T})$,

$$T_\beta = I - cU_\beta P_+, \tag{33}$$

where $c \in C(\mathbb{T})$ satisfies the properties (29), (30), (31) and

$$\left|c(\tau_j)\right| > \sin^{-1}\frac{\pi}{p}, \quad j = 1,2.$$

COROLLARY 4.4. *Let T_β be the operator defined by (33). Then the following estimate holds*

$$\dim\ker T_\beta \leqslant m + \max(0, -\operatorname{ind} c),$$

where m is the degree of the polynomial r defined in Corollary 4.3 for the function $\widetilde{c} = c_+ c^{-1} c_+^{-1}(\beta_{-1})$ and $\operatorname{ind} c$ is the Cauchy index of the function c.

Proof. Follows from Theorem 3.2 with $n = 1$. □

5. A SIO with polynomial coefficient relative to the shift operator

Now let us consider the SIO with shift of the form

$$K_\omega = A_\omega P_+ + P_- : L_p(\mathbb{T}) \to L_p(\mathbb{T}), \tag{34}$$

where

$$A_\omega = I + \sum_{i=1}^{n} a_i U_\omega^i,$$

$a_i \in C(\mathbb{T})$, $i = \overline{1,n}$, and U_ω, $\omega = \eta, \beta$, is the shift operator satisfying the property $U_\omega S = S U_\omega$.

Consider also the matrix operator (see [15], [16], [13], and [18])

$$\widetilde{K}_\omega = \widetilde{A}_\omega P_+ + P_- : L_p^n(\mathbb{T}) \to L_p^n(\mathbb{T}), \tag{35}$$

with

$$\widetilde{A}_\omega = I + aU_\omega,$$

where

$$a = \begin{pmatrix} a_1 \; a_2 & \cdots & a_{n-1} & a_n \\ & -E_{n-1} & & O_{(n-1)\times 1} \end{pmatrix}.$$

The following result holds

PROPOSITION 5.1. [16] *Let K_ω and $\widetilde{K}_\omega$ be the operators defined by (34) and (35), respectively. The operator K_ω is a Fredholm operator on $L_p(\mathbb{T})$ if and only if the operator $\widetilde{K}_\omega$ is a Fredholm operator on $L_p^n(\mathbb{T})$. In the affirmative case,* $\dim\ker K_\omega = \dim\ker \widetilde{K}_\omega$ *and* $\dim\operatorname{coker} K_\omega = \dim\operatorname{coker} \widetilde{K}_\omega$.

Obviously the operator $\widetilde{K}_\omega$ is a particular case of the operator T_ω defined by (15) (with $\omega = \eta$) or the operator defined by (23) (with $\omega = \beta$). Then, taking into account Proposition 5.1, Theorems 2.2, 2.4, 3.1 and 3.2, can be used to study the operator K_ω.

6. A SIO with a block triangular matrix coefficient

The results obtained for the matrix cases 1 and 2 (Theorems 3.1 and 3.2, respectively) can be applied to treat the cases 3 and 4 (see Introduction). Let us consider the SIO with non-Carleman shift $T_\beta = I - cU_\beta P_+$, where c is a block diagonal matrix of normal form (5)

$$c(t) = \begin{pmatrix} c_1(t) & O^{m\times k} \\ O^{k\times m} & c_2(t) \end{pmatrix};$$

the operator T_β can be written in the matrix form

$$T_\beta = \begin{pmatrix} T_1 & O^{m\times k} \\ O^{k\times m} & T_2 \end{pmatrix} : L_p^n(\mathbb{T}) \to L_p^n(\mathbb{T}),$$

where

$$T_1 = I - c_1 U_\beta P_+ : L_p^m(\mathbb{T}) \to L_p^m(\mathbb{T}),$$
$$T_2 = I - c_2 U_\beta P_+ : L_p^k(\mathbb{T}) \to L_p^k(\mathbb{T}).$$

Then we have

$$\dim\ker T_\beta = \dim\ker T_1 + \dim\ker T_2. \tag{36}$$

Consider now the operator $T_\beta = I - cU_\beta P_+$, where c is a block triangular matrix of the form (7)

$$c(t) = \begin{pmatrix} c_1(t) & O^{m\times k} \\ f(t) & c_2(t) \end{pmatrix}; \tag{37}$$

in this case the operator T_β can be written in the matrix form

$$T_\beta = \begin{pmatrix} T_1 & O^{m\times k} \\ F & T_2 \end{pmatrix} : L_p^n(\mathbb{T}) \to L_p^n(\mathbb{T}),$$

where T_1 and T_2 are defined above, and

$$F = -fU_\beta P_+ : L_p^m(\mathbb{T}) \to L_p^k(\mathbb{T}).$$

It is not difficult to see that in this case we obtain the inequality [2].

$$\dim\ker T_\beta \leqslant \dim\ker T_1 + \dim\ker T_2. \tag{38}$$

Now, considering the matrix blocks c_1 and c_2 satisfying the properties

$$\sigma[c_1(\tau_j)] \subset \mathbb{D}_+, \quad \sigma[c_2(\tau_j)] \subset \mathbb{D}_-, \quad j = 1,2, \quad \det c_2(t) \neq 0, \quad \forall t \in \mathbb{T};$$

we note that:

a) The operator T_1 satisfies all the conditions of Theorem 3.1. Let $l(c_1)$ be the number defined by (14) for the matrix c_1; then

$$\dim\ker T_1 \leqslant l(c_1).$$

b) The operator T_2 satisfies all the conditions of Theorem 3.2. Let the matrix c_2 satisfy the properties (1) and (2); let $l(\widetilde{c_2})$ be the number defined by (14) for the matrix $\widetilde{c_2} = c_+ c^{-1} c_+^{-1}(\beta_{-1})$, and $\varkappa_j \in \mathbb{Z}$, $j = \overline{1,n}$ be the partial indices of the matrix c_2. Then

$$\dim\ker T_2 \leqslant l(\widetilde{c_2}) + \sum_{\varkappa_j<0} |\varkappa_j|.$$

Taking into account (36) and (38) we can state the following estimate for the matrix cases 3 and 4.

PROPOSITION 6.1. *Let $T_\beta = I - cU_\beta P_+$ be the operator defined by* (23)*, where c is the block triangular matrix defined by* (37)*. Then the following estimate holds*

$$\dim\ker T_\beta \leqslant l(c_1) + l(\widetilde{c_2}) + \sum_{\varkappa_j<0} |\varkappa_j|.$$

---

[2] The equality in (38) can happen when the equation $T_2\phi = -F\varphi_i$ is solved for all φ_i, with $\varphi_i \in \ker T_1$, $i = 1,2,\dots,\dim\ker T_1$; or in particular cases, including when $f = 0$ (the case 3).

7. On the resolvent set of the operator cU and the dimension of the kernel of a SIO with shift – the matrix case

Let us consider on $L_p^n(\mathbb{T})$, $p \in (1,\infty)$, the operators cU and $cU - \lambda I$, where $c \in C^{n\times n}(\mathbb{T})$ is a continuous matrix function, U is the shift operator defined by (22) (i.e., $U := U_\beta$), and $\lambda \in \mathbb{C}$. Consider $\lambda = 0$; the operator $cU - \lambda I$ is invertible if and only if $\det c(t) \neq 0$ for all $t \in \mathbb{T}$. Let $\lambda \neq 0$; the operator $cU - \lambda I$ or, equivalently, the operator $I - \lambda^{-1}cU$ is invertible if and only if $\lambda^{-1}c$ is α-reducible to the normal form on $\mathbb{T}$, according to Theorem 1.1. Then, if $\det c(t) \neq 0$ for all $t \in \mathbb{T}$, the resolvent set and the spectrum of the operator cU are, respectively,

$$\rho(cU) = \{\lambda = 0 \ \vee \ \lambda \in \mathbb{C} \setminus \{0\} : \lambda^{-1}c \text{ is } \alpha\text{-reducible to the normal form on } \mathbb{T}\},$$

$$\sigma(cU) = \mathbb{C} \setminus \rho(cU).$$

Moreover, the essential spectrum of the operator cUP_+ is given by (see Introduction)

$$\sigma_{ess}(cUP_+) = \sigma(cU).$$

If $\det c(t) = 0$ for some $t \in \mathbb{T}$, the resolvent set and the spectrum of the operator cU are, respectively,

$$\rho(cU) = \{\lambda \in \mathbb{C} \setminus \{0\} : \lambda^{-1}c \text{ is } \alpha\text{-reducible to the normal form on } \mathbb{T}\},$$

$$\sigma(cU) = \mathbb{C} \setminus \rho(cU);$$

and

$$\sigma_{ess}(cUP_+) = \sigma(cU).$$

We can obtain concrete subsets of the resolvent set of the operator cU, taking into account the four cases of explicit sufficient conditions of invertibility for the operator $I - cU$ mentioned in the Introduction.

Let $\lambda \neq 0$; the operator $I - \lambda^{-1}cU$ is invertible if one of the two following conditions is fulfilled

a) $\sigma[\lambda^{-1}c(\tau_j)] \subset \mathbb{T}_+$, $j = 1,2$;

b) $\sigma[\lambda^{-1}c(\tau_j)] \subset \mathbb{T}_-$, $j = 1,2$, and $\det[\lambda^{-1}c(t)] \neq 0$ for all $t \in \mathbb{T}$.

Let

$$\mu = \max_{j=1,2} \rho[c(\tau_j)], \qquad \nu = \min_{j=1,2} \theta[c(\tau_j)].$$

The condition a) implies that $|\lambda| > \mu$, and the condition b) implies that $0 < |\lambda| < \nu$.

Then, if $\det c(t) \neq 0$ for all $t \in \mathbb{T}$ the following set belongs to the resolvent set of the operator cU,

$$\{\lambda \in \mathbb{C} : |\lambda| < \nu \ \vee \ |\lambda| > \mu\} \subset \rho(cU). \tag{39}$$

If $\det c(t) = 0$ for some $t \in \mathbb{T}$ the following set belongs to the resolvent set of the operator cU,

$$\{\lambda \in \mathbb{C} : |\lambda| > \mu\} \subset \rho(cU).$$

Now we consider the SIO with shift on $L_p^n(\mathbb{T})$, $p \in (1,\infty)$, defined by

$$T_\lambda = I - \lambda^{-1} c U P_+. \tag{40}$$

and the subsets of the set (39)

$$\mathbb{A} = \left\{\lambda \in \mathbb{C} : |\lambda| > \mu \sin^{-1} \frac{\pi}{p}\right\},$$

$$\mathbb{B} = \left\{\lambda \in \mathbb{C} : 0 < |\lambda| < \nu \sin \frac{\pi}{p}\right\}.$$

The following results take place

PROPOSITION 7.1. *Let $\lambda \in \mathbb{A}$; then there exists a polynomial matrix r satisfying the conditions*

$$\max_{t \in \mathbb{T}} \left\| r(t) \lambda^{-1} c(t) r^{-1}[\beta(t)] \right\|_2 < \sin \frac{\pi}{p}$$

and

$$P_+ r^{\pm 1} P_+ = r^{\pm 1} P_+.$$

Moreover

$$\dim \ker T_\lambda \leqslant l(\lambda^{-1} c), \tag{41}$$

where T_λ is the operator defined by (40)*, and $l(\lambda^{-1}c)$ is the number defined by* (14) *for the matrix $\lambda^{-1}c$.*

Proof. It is easy to see that $\rho[\lambda^{-1} c(\tau_j)] < \sin \frac{\pi}{p}$, i.e., $\sigma[\lambda^{-1} c(\tau_j)] \subset \mathbb{D}_+$, $j = 1,2$; this means that the operator T_λ satisfies all the conditions of Theorem 3.1 and it follows the result. □

PROPOSITION 7.2. *Let $\lambda \in \mathbb{B}$; let T_λ be the operator defined by* (40)*, where the matrix function $c \in C^{n \times n}(\mathbb{T})$ satisfies the properties*

$$\det c(t) \neq 0, \quad \forall t \in \mathbb{T},$$

(1)*, and* (2)*. Then the following estimate holds*

$$\dim \ker T_\lambda \leqslant l(\lambda \widetilde{c}) + \sum_{\varkappa_j < 0} |\varkappa_j|, \tag{42}$$

where $l(\lambda \widetilde{c})$ is the number defined by (14) *for the matrix $\lambda \widetilde{c}$, $\widetilde{c} = c_+ c^{-1} c_+^{-1}(\beta_{-1})$, and $\varkappa_j \in \mathbb{Z}$, $j = \overline{1,n}$ are the partial indices of the matrix c.*

Proof. We have that $\rho[\lambda^{-1} c(\tau_j)] > \sin^{-1} \frac{\pi}{p}$, i.e., $\sigma[\lambda^{-1} c(\tau_j)] \subset \mathbb{D}_-$, $j = 1,2$; and the matrix $\lambda^{-1} c$ admits the factorization $\lambda^{-1} c = \lambda^{-1} c_- \Lambda c_+$. Evidently the partial indices of the matrices $\lambda^{-1} c$ and c are the same. We conclude that the operator T_λ satisfies all the conditions of Theorem 3.2 and the result follows. □

Now let

$$\xi = \max_{t\in\mathbb{T}} \|c(t)\|_2,$$

and the subset of the set $\mathbb{A}$

$$\mathbb{E} = \left\{\lambda \in \mathbb{C} : |\lambda| > \xi \sin^{-1}\frac{\pi}{p}\right\}.$$

The following result takes place

PROPOSITION 7.3. *Let $\lambda \in \mathbb{E}$; then*

$$\dim\ker T_\lambda = 0,$$

and the operator T_λ defined by (40) is invertible.

Proof. Since $\max_{t\in\mathbb{T}} \left\|\lambda^{-1}c(t)\right\|_2 < \sin\frac{\pi}{p}$, $\|U\|_{L_p} = 1$, and $\|P_+\|_{L_p} = \sin^{-1}\frac{\pi}{p}$, it follows that $T_\lambda = I - \lambda^{-1}cUP_+$ is an invertible operator whose inverse is given by the Neumann series

$$T_\lambda^{-1} = I + \frac{c}{\lambda}UP_+ + \left(\frac{c}{\lambda}UP_+\right)^2 + \cdots. \quad \square$$

The operator $I - \lambda^{-1}cU$ is also invertible if the matrix c is the block triangular matrix

$$c(t) = \begin{pmatrix} c_1(t) & O^{m\times k} \\ f(t) & c_2(t) \end{pmatrix}, \tag{43}$$

where $c_1 \in C^{m\times m}(\mathbb{T})$, $c_2 \in C^{k\times k}(\mathbb{T})$, $f \in C^{k\times m}(\mathbb{T})$, $k+m=n$, and

$$\sigma[\lambda^{-1}c_1(\tau_j)] \subset \mathbb{T}_+, \quad \sigma[\lambda^{-1}c_2(\tau_j)] \subset \mathbb{T}_-, \quad j=1,2, \quad \det[\lambda^{-1}c_2(t)] \neq 0, \quad \forall t \in \mathbb{T}.$$

Let

$$\mu_1 = \max_{j=1,2} \rho[c_1(\tau_j)], \qquad \nu_2 = \min_{j=1,2} \theta[c_2(\tau_j)].$$

We have that $\quad \mu_1 < |\lambda| \wedge 0 < |\lambda| < \nu_2$.

Suppose that $\mu_1 < \nu_2$ [3]; then, the following set also belongs to the resolvent set of the operator cU,

$$\{\lambda \in \mathbb{C} : \mu_1 < |\lambda| < \nu_2\} \subset \rho(cU). \tag{44}$$

REMARK 7.1. In general, the subsets, (39) and (44), of the resolvent set of the operator cU are not disjoint.

Let us consider now the SIO with shift defined by (40)

$$T_\lambda = I - \lambda^{-1}cUP_+, \tag{45}$$

where c is the block triangular matrix defined by (43) and the matrix c_2 satisfies the properties (1) and (2); and the subset of the set (44)

$$\mathbb{F} = \left\{\lambda \in \mathbb{C} : \mu_1 \sin^{-1}\frac{\pi}{p} < |\lambda| < \nu_2 \sin\frac{\pi}{p}\right\}.$$

The following result takes place.

[3] Suppose that $\mu_1 > \nu_2$; in this case, $\mu_1 < |\lambda| \wedge 0 < |\lambda| < \nu_2$, defines an empty set.

PROPOSITION 7.4. *Let $\lambda \in \mathbb{F}$ and T_λ be the operator defined by* (45)*. Then the following estimate holds*

$$\dim\ker T_\lambda \leqslant l(\lambda^{-1}c_1) + l(\lambda\widetilde{c_2}) + \sum_{\varkappa_j<0} |\varkappa_j|, \tag{46}$$

where $l(\lambda^{-1}c_1)$ and $l(\lambda\widetilde{c_2})$ are the numbers defined by (14) *for the matrices $\lambda^{-1}c_1$ and $\lambda\widetilde{c_2}$, respectively, $\widetilde{c_2} = c_+c^{-1}c_+^{-1}(\beta_{-1})$, and $\varkappa_j \in \mathbb{Z}$, $j = \overline{1,n}$ are the partial indices of the matrix c_2.*

Proof. The operator T_λ can be written in the matrix form

$$T_\lambda = \begin{pmatrix} T_{\lambda,1} & O^{m\times k} \\ F & T_{\lambda,2} \end{pmatrix} : L_p^n(\mathbb{T}) \to L_p^n(\mathbb{T}),$$

where

$$T_{\lambda,1} = I - \lambda^{-1}c_1UP_+ : L_p^m(\mathbb{T}) \to L_p^m(\mathbb{T}),$$

$$T_{\lambda,2} = I - \lambda^{-1}c_2UP_+ : L_p^k(\mathbb{T}) \to L_p^k(\mathbb{T}),$$

$$F = -\lambda^{-1}fUP_+ : L_p^m(\mathbb{T}) \to L_p^k(\mathbb{T}).$$

The operator $T_{\lambda,1}$ satisfies all the conditions of Proposition 7.1 and the operator $T_{\lambda,2}$ satisfies all the conditions of Proposition 7.2. Taking into account Proposition 6.1, the result follows. □

REMARK 7.2. Since the subsets (39) and (44) of $\rho(cU)$, and the sets $\mathbb{A}$ (or $\mathbb{B}$) and $\mathbb{F}$, are not disjoint in general, we can have two estimates, (41) and (46), or (42) and (46), holding for the same concrete operator T_λ defined by (45).

8. On the resolvent set of the operator cU and the dimension of the kernel of a SIO with shift – the scalar case

Now let us consider on $L_p(\mathbb{T})$, $p \in (1,\infty)$, the operators cU and $cU - \lambda I$, where $c \in C(\mathbb{T})$ is a continuous function, U is the shift operator defined by (22), and $\lambda \in \mathbb{C}$. Consider $\lambda = 0$; the operator $cU - \lambda I$ is invertible if and only if $c(t) \neq 0$ for all $t \in \mathbb{T}$. Let $\lambda \neq 0$; the operator $cU - \lambda I$ or, equivalently, the operator $I - \lambda^{-1}cU$ is invertible if and only if if one of the two following conditions is fulfilled

a) $\left|\lambda^{-1}c(\tau_j)\right| < 1$, $j = 1,2$;

b) $\left|\lambda^{-1}c(\tau_j)\right| > 1$, $j = 1,2$, and $\lambda^{-1}c(t) \neq 0$ for all $t \in \mathbb{T}$.

Let

$$\gamma = \max_{j=1,2} \left|c(\tau_j)\right|, \qquad \delta = \min_{j=1,2} \left|c(\tau_j)\right|.$$

The condition a) implies that $|\lambda| > \gamma$, and the condition b) implies that $0 < |\lambda| < \delta$.

Then, if $c(t) \neq 0$ for all $t \in \mathbb{T}$, the resolvent set and the spectrum of the operator cU are, respectively,

$$\rho(cU) = \{\lambda \in \mathbb{C} : |\lambda| < \delta \vee |\lambda| > \gamma\},$$

$$\sigma(cU) = \{\lambda \in \mathbb{C} : \delta \leqslant |\lambda| \leqslant \gamma\}.$$

The essential spectrum of the operator cUP_+ is given by

$$\sigma_{ess}(cUP_+) = \sigma(cU).$$

If $c(t) = 0$ for some $t \in \mathbb{T}$, the resolvent set and the spectrum of the operator cU are, respectively,

$$\rho(cU) = \{\lambda \in \mathbb{C} : |\lambda| > \gamma\},$$
$$\sigma(cU) = \{\lambda \in \mathbb{C} : |\lambda| \leqslant \gamma\};$$

and

$$\sigma_{ess}(cUP_+) = \sigma(cU).$$

Now we consider the SIO with shift on $L_p(\mathbb{T})$, $p \in (1,\infty)$, defined by

$$T_\lambda = I - \lambda^{-1} cUP_+, \tag{47}$$

and the subsets of $\rho(cU)$

$$\mathbb{G} = \left\{\lambda \in \mathbb{C} : |\lambda| > \gamma \sin^{-1}\frac{\pi}{p}\right\},$$

$$\mathbb{H} = \left\{\lambda \in \mathbb{C} : 0 < |\lambda| < \delta \sin\frac{\pi}{p}\right\}.$$

The following results take place

PROPOSITION 8.1. *Let $\lambda \in \mathbb{G}$; then there exists a polynomial r of degree m, with zeros in $\mathbb{T}_-$,*

$$r(t) = \prod_{k=1}^{m}(t - \lambda_k), \quad |\lambda_k| > 1, \quad k = \overline{1,m},$$

such that

$$\left|r(t)\lambda^{-1}c(t)r^{-1}[\beta(t)]\right| < \sin\frac{\pi}{p}, \quad \forall t \in \mathbb{T}. \tag{48}$$

Moreover

$$\dim\ker T_\lambda \leqslant m,$$

where T_λ is the operator defined by (47).

Proof. We have that $\left|\lambda^{-1}c(\tau_j)\right| < \sin\frac{\pi}{p}$, $j = 1,2$; then the operator T_λ satisfies all the conditions of Corollary 4.3 and it follows the result. □

PROPOSITION 8.2. *Let $\lambda \in \mathbb{H}$; let T_λ be the operator defined by* (47)*, where the function $c \in C(\mathbb{T})$ satisfies the properties*

$$c(t) \neq 0, \quad \forall t \in \mathbb{T},$$

(30), *and* (31). *Then the following estimate holds*

$$\dim\ker T_\lambda \leqslant m + \max(0, -\operatorname{ind} c),$$

where m *is the degree of the polynomial* r *defined in Corollary 4.3 considering the function* $\lambda c_+ c^{-1} c_+^{-1}(\beta_{-1})$ *instead of* $\lambda^{-1}c$ *in* (48), *and* $\operatorname{ind} c$ *is the Cauchy index of the function* c.

Proof. We have that $\left|\lambda^{-1}c(\tau_j)\right| > \sin^{-1}\frac{\pi}{p}$, $j = 1,2$; and the function $\lambda^{-1}c$ admits the factorization $\lambda^{-1}c = \lambda^{-1}c_- t^{\varkappa} c_+$, with $\varkappa = \operatorname{ind} c$. We conclude that the operator T_λ satisfies all the conditions of Corollary 4.4 and the result follows. □

Let

$$\varepsilon = \max_{t\in\mathbb{T}} |c(t)|,$$

and the subset of the set $\mathbb{G}$

$$\mathbb{L} = \left\{\lambda \in \mathbb{C} : |\lambda| > \varepsilon \sin^{-1}\frac{\pi}{p}\right\}.$$

The following result takes place

PROPOSITION 8.3. *Let* $\lambda \in \mathbb{L}$; *then*

$$\dim\ker T_\lambda = 0,$$

and the operator T_λ *defined by* (47) *is invertible.*

Proof. Since $\max_{t\in\mathbb{T}}\left|\lambda^{-1}c(t)\right| < \sin\frac{\pi}{p}$, $\|U\|_{L_p} = 1$, and $\|P_+\|_{L_p} = \sin^{-1}\frac{\pi}{p}$, analogously to the matrix case, it follows that $T_\lambda = I - \lambda^{-1}cUP_+$ is an invertible operator whose inverse is given by the Neumann series

$$T_\lambda^{-1} = I + \frac{c}{\lambda}UP_+ + \left(\frac{c}{\lambda}UP_+\right)^2 + \cdots. \quad \square$$

REFERENCES

[1] A. A. BATUREV, V. G. KRAVCHENKO, AND G. S. LITVINCHUK, *Approximate methods for singular integral equations with a non-Carleman shift*, J. Integral Equations Appl. **8**, 1 (1996), 1–17.

[2] A. C. CONCEIÇÃO AND R. C. MARREIROS, *On the kernel of a singular integral operator with non-Carleman shift and conjugation*, Operators and Matrices **9**, 2 (2015), 433–456.

[3] K. CLANCEY AND I. GOHBERG, *Factorization of Matrix Functions and Singular Integral Operators*, Operator Theory: Advances and Appl., vol. 3. Birkhäuser Verlag, Basel, 1981.

[4] T. EHRHART, *Invertibility theory for Toeplitz plus Hankel operators and singular integral operators with flip*, J. Funct. Anal. **208**, 1 (2004), 64–106.

[5] B. HOLLENBECK AND I. VERBITSKY, *Best Constants for the Riesz Projection*, J. Funct. Anal. **175** (2000), 370–392.

[6] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press, Cambridge, 1990.

[7] R. A. HORN AND C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[8] N. KARAPETIANTS AND S. SAMKO, *Equations with Involutive Operators*, Birkhauser Verlag, Boston, 2001.

[9] YU. I. KARLOVICH AND V. G. KRAVCHENKO, *Singular integral equations with non-Carleman shift on an open contour*, Differential Equations **17**, 12 (1982), 1408–1417.

[10] V. G. KRAVCHENKO, A. B. LEBRE, AND G. S. LITVINCHUK, *Spectrum problems for singular integral operators with Carleman shift*, Math. Nachr. **226** (2001), 129–151.

[11] V. G. KRAVCHENKO, A. B. LEBRE, AND J. S. RODRIGUEZ, *Factorization of singular integral operators with a Carleman shift and spectral problems*, J. Integral Equations Appl. **13**, 4 (2001), 339–383.

[12] V. G. KRAVCHENKO, A. B. LEBRE, AND J. S. RODRIGUEZ, *The kernel of singular integral operators with a finite group of linear-fractional shifts*, Operator Theory 20, 143-154. Theta series in Advanced Mathematics, Bucharest, 2006.

[13] V. G. KRAVCHENKO AND G. S. LITVINCHUK, *Introduction to the Theory of Singular Integral Operators with Shift*, Mathematics and its Appl., vol. 289. Kluwer Academic Publishers, Dordrecht, 1994.

[14] V. G. KRAVCHENKO AND R. C. MARREIROS, *An estimate for the dimension of the kernel of a singular operator with a non-Carleman shift*, Factorization, Singular Operators and Related Problems, 197-204. Kluwer Academic Publishers, Dordrecht, 2003.

[15] V. G. KRAVCHENKO AND R. C. MARREIROS, *On the kernel of some one-dimensional singular integral operators with shift*, The Extended Field of Operator Theory, Operator Theory: Advances and Appl., vol. 171, 245–257, Birkhäuser Verlag, Basel, 2007.

[16] V. G. KRAVCHENKO AND R. C. MARREIROS, *On the dimension of the kernel of a singular integral operator with shift*, Operator Theory, Operator Algebras and Appl., Operator Theory: Advances and Appl., vol. 242, 197–220. Birkhäuser Verlag, Basel, 2014.

[17] V. G. KRAVCHENKO, R. C. MARREIROS, AND J. S. RODRIGUEZ, *An estimate for the number of solutions of an homogenous generalized Riemann boundary value problem with shift*, Recent Progress in Operator Theory and its Appl., Operator Theory: Advances and Appl., vol. 220, 163–178. Birkhäuser Verlag, Basel, 2012.

[18] N. KRUPNIK, *Banach Algebras with Symbol and Singular Integral Operators*, Operator Theory: Advances and Appl., vol. 26. Birkhäuser Verlag, Basel, 1987.

[19] G. S. LITVINCHUK, *Boundary Value Problems and Singular Integral Equations with Shift*, Nauka, Moscow, 1977 (in russian).

[20] G. S. LITVINCHUK, *Solvability Theory of Boundary Value Problems and Singular Integral Equations with Shift*, Mathematics and its Appl., vol. 523. Kluwer Academic Publishers, Dordrecht, 2000.

[21] G. S. LITVINCHUK AND I. M. SPITKOVSKII, *Factorization of Measurable Matrix Functions*, Operator Theory: Advances and Appl., vol. 25. Birkhäuser Verlag, Basel, 1987.

(Received March 7, 2017)

Rui C. Marreiros
Departamento de Matemática
Faculdade de Ciências e Tecnologia, Universidade do Algarve
8005-139 Faro, Portugal
e-mail: rmarrei@ualg.pt