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# D–NORM AND ITS ISOMETRIES ON $\mathfrak{c}_0$ SPACES

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*Abstract.* In this paper, based on the notion of *diameter,* we consider a natural preorder on $\mathfrak{c}_0(I)$ which is said "diametric majorization". Then by using this notion we define a norm on $\mathfrak{c}_0(I)$, where $I$ is assumed to be an infinite set. This norm is equivalent to $\|\cdot\|_\infty$ and is said "d-norm". Finally, the structures of all bounded linear operators on $\mathfrak{c}_0(I)$ preserving diametric majorization and also isometries under the d-norm are both determined. We also give the relation between this isometries and isometries under the usual norm.

## 1. Introduction and preliminaries

Recently, many authors have discussed some various properties and structures of isometries on Banach spaces [5, 8]. For the collections of results in the topics of isometries we refer the reader to the monographs [3, 4].

In the following we point out to some important preliminaries.

DEFINITION 1.1. Let $I$ be an infinite set (equipped with the discrete topology). The point $x_0 \in \mathbb{R}$ is called the limit of $f : I \to \mathbb{R}$ and is denoted by $\lim\limits_{i\in I} f(i) = x_0$ (or more briefly $\lim f = x_0$) if for each neighborhood $V$ of $x_0$ there exists a finite set $F \subseteq I$ such that $f(i) \in V$, for all $i \in I \setminus F$.

It is easily verified that if $\lim f = x_0$, then the set $\{i \in I; f(i) \neq x_0\}$ is at most a countable set. We will use the notation $\mathfrak{c}_0(I)$ for the set of all function $f : I \to \mathbb{R}$ with $\lim f = 0$. It is easily verified that every $f \in \mathfrak{c}_0(I)$ is bounded and $\mathfrak{c}_0(I)$ is a Banach space with the norm defined by $\|f\|_\infty = \sup\limits_{i\in I} |f(i)|$. Each $f \in \mathfrak{c}_0(I)$ can be represented by $\sum\limits_{i\in I} f(i)e_i$, where $e_i : I \to \mathbb{R}$ is defined as $e_i(j) = \delta_{ij}$, the Kronecker delta.

For a subset $C$ of a metric space $(X,d)$ the diameter of $C$ is denoted by $\operatorname{diam}(C)$ and is defined as

$$\operatorname{diam}(C) := \sup\{d(x,y); x,y \in X\}.$$

For a function $f : I \to \mathbb{R}$, to simplify notations, we use $\operatorname{diam}(f)$, $\inf(f)$, and $\sup(f)$, instead of $\operatorname{diam}(\operatorname{Im}(f))$, $\inf\limits_{i\in I}\{f(i)\}$ and $\sup\limits_{i\in I}\{f(i)\}$, respectively. Also, the notation $\operatorname{co}(f)$ will be used for the convex combination of the set $\operatorname{Im}(f) := \{f(i); i \in I\}$.

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The idea of majorization is defined in various forms and on different spaces of finite and infinite dimension, such as $\mathbb{R}^n, M_{m\times n}(\mathbb{R}), \ell^p(I), \ell^1(I)^+, \ell_\infty, \mathfrak{c}_0, \mathfrak{c}$ and etc. See for examples [1, 2, 6, 7, 9]. In this paper, on the basis of the notion of $\operatorname{diam}(f)$, we define a relation on $\mathfrak{c}_0(I)$, that is said to be the diametric majorization. Moreover, we define a norm on $\mathfrak{c}_0(I)$, where $I$ is assumed to be an infinite set, that is equivalent to $\|\cdot\|_\infty$ and discuss on the properties and characterization of all isometries under this norm. All bounded linear operators $T:\mathfrak{c}_0(I)\to\mathfrak{c}_0(I)$ which preserve diametric majorization, and their relations between this isometries are also determined.

## 2. Main results

Diametric majorization defines a relation on $\mathfrak{c}_0(I)$ that compares the distance occurred between the values of $\inf(f)$ and $\sup(f)$. More precisely,

DEFINITION 2.1. For $f,g\in\mathfrak{c}_0(I)$ we say that $f$ is diametrically majorized by $g$ and is denoted by $f\prec_d g$, whenever $\operatorname{diam}(f)\leqslant\operatorname{diam}(g)$.

The comparison under the relation diametric majorization for any arbitrary two elements is possible, i.e., for $f,g\in\mathfrak{c}_0(I)$ we have either $f\prec_d g$ or $g\prec_d f$.

DEFINITION 2.2. For $f\in\mathfrak{c}_0(I)$ define $\|f\|_d:=\operatorname{diam}(f)$ and it is called the d-norm.

It is easily verified that the d-norm is a norm on $\mathfrak{c}_0(I)$, if and only if $I$ is an infinite set. Moreover, since $\|\cdot\|_\infty\leqslant\|\cdot\|_d\leqslant 2\|\cdot\|_\infty$, this norm is a complete norm and is equivalent to the infinity norm. Also, notice that $\|f\|_d$ is equal to the length of (the interval) $\operatorname{co}(f)$, that is $\sup(f)-\inf(f)$.

A bounded linear operator $T:\mathfrak{c}_0(I)\to\mathfrak{c}_0(I)$ is called a diametric majorization preserver if $f\prec_d g$ implies $Tf\prec_d Tg$, for all $f\in\mathfrak{c}_0(I)$. The set of all such operators is denoted by $\mathscr{P}_d$. Also, $T$ is said to be a diameter preserving isometry or (for short) d-isometry if $T$ is an isometry, when $\mathfrak{c}_0(I)$ is equipped with the d-norm. The set of all d-isometries on $\mathfrak{c}_0(I)$ is denoted by $\mathscr{I}_d$.

The next theorem gives the relation between $\mathscr{I}_d$ and $\mathscr{P}_d$.

THEOREM 2.3. *The following statements are equivalent for a bounded linear operator $T:\mathfrak{c}_0(I)\to\mathfrak{c}_0(I)$.*

(i) *$T$ preserves diametric majorization.*

(ii) *$T$ is a scalar multiple of a d-isometry.*

*Proof.* (i)$\Rightarrow$(ii). Suppose that $T\in\mathscr{P}_d$. For each $f,g\in\mathfrak{c}_0(I)$ with $\|f\|_d=\|g\|_d\neq 0$, we have $\|Tf\|_d=\|Tg\|_d$. Hence $\|T\frac{f}{\|f\|_d}\|_d=\|T\frac{g}{\|g\|_d}\|_d$. Thus the value $c:=\|T\frac{f}{\|f\|_d}\|_d$ is constant, independent of chosen $f\in\mathfrak{c}_0(I)$(with $\|f\|_d\neq 0$). Now let

$f \in \mathfrak{c}_0(I)$. If $\|f\|_d = 0$, then $f \prec_d 0$, which implies $Tf \prec_d 0$. Therefore, $\|Tf\|_d = 0$. So, we have obviously

$$\|Tf\|_d = c\,\|f\|_d. \tag{1}$$

Now if $\|f\|_d \neq 0$, then

$$\|Tf\|_d = \|f\|_d \cdot \|T\frac{f}{\|f\|_d}\|_d = c\,\|f\|_d. \tag{2}$$

The conclusion follows by using (1), (2), and considering two cases: $c = 0$ and $c \neq 0$.

(ii)$\Rightarrow$(i). It is evident. □

Theorem 2.3 formulates that the structure of the elements in $\mathscr{P}_d$ on $\mathfrak{c}_0(I)$ is directly related to the structure of d-isometries. For this reason we shall focus on d-isometries on $\mathfrak{c}_0(I)$.

LEMMA 2.4. *Let $T \in \mathscr{I}_d$ and $i_0 \in I$. Then*

$$-1 \leqslant \sum_{j\in I^-} Te_j(i_0) \leqslant 0 \leqslant \sum_{j\in I^+} Te_j(i_0) \leqslant 1,$$

*where $I^- := \{j \in I; Te_j(i_0) < 0\}$ and $I^+ := \{j \in I; Te_j(i_0) > 0\}$.*

*Proof.* To prove $0 \leqslant \sum_{j\in I^+} Te_j(i_0) \leqslant 1$, it is sufficient to show that for each finite subset $F \subseteq I^+$, we have $0 \leqslant \sum_{j\in F} Te_j(i_0) \leqslant 1$. So, we assume that $F \subseteq I^+$. Take $f := \sum_{j\in F} e_j$. Since $\|f\|_d = 1$ and $T \in \mathscr{I}_d$ then $\|Tf\|_d = 1$. Also we have $Tf(i_0) \geqslant 0$, because $Te_j(i_0) \geqslant 0$ for all $j \in I^+$. Now if $Tf(i_0) > 1$, then by using the fact that 0 is a limit point of $\mathrm{Im}(f)$, we have $1 = \|Tf\|_d = \sup(Tf) - \inf(Tf) \geqslant Tf(i_0) - 0 = Tf(i_0) > 1$, which leads to a contradiction. So, $Tf(i_0) \in [0,1]$. A similar arguments shows $-1 \leqslant \sum_{j\in I^-} Te_j(i_0) \leqslant 0$. □

The next result concerns the limits of $\left(\inf Te_j\right)_{j\in I}$ and $\left(\sup Te_j\right)_{j\in I}$.

LEMMA 2.5. *Let $T \in \mathscr{I}_d$, $j_0 \in I$, and $\lambda := \sup Te_{j_0}$. Then*

$$\lambda = \lim_{j\in I} \lambda_j, \quad \textit{and} \quad \lambda - 1 = \lim_{j\in I} \eta_j,$$

*where $\lambda_j := \sup Te_j$, and $\eta_j := \inf Te_j$.*

*Proof.* Because $\|Te_{j_0}\|_d = 1$, it follows that $\lambda \in [0,1]$. By replacing $-T$ by $T$ (if necessary), we may assume that $0 < \lambda \leqslant 1$. Since $\lambda = \sup Te_{j_0} > 0$, we have $\lambda = \max Te_{j_0}$. So, there exists $i_1 \in I$ with $Te_{j_0}(i_1) = \lambda$. Now let $0 < \varepsilon < \lambda$ is arbitrary. Then $\inf Te_{j_0} = \lambda - 1$ and therefore there is $i_2 \in I$ such that

$$Te_{j_0}(i_2) \in [\lambda - 1, \lambda - 1 + \varepsilon).$$

Assume that $\varepsilon_0 := \min\{\varepsilon, 1 - \varepsilon\}$. Since $\lim_{i\in I} Te_{j_0}(i) = 0$, there exists a finite subset $F \subseteq I$ such that

$$\forall i \in I \setminus F, \quad |Te_{j_0}(i)| < \frac{\varepsilon_0}{2}.$$

It is easy to see that $i_1 \in F$. Moreover, without loss of generality, we may assume that $i_1 \in F$.

Since $F$ is a finite set, there is a finite set $G \subseteq I$ such that

$$\forall j \in I \setminus G, \ \forall i \in F \quad |Te_j(i)| < \frac{\varepsilon_0}{2}.$$

Therefore, we have

$$\lambda - 1 - \frac{\varepsilon_0}{2} \leqslant Te_{j_0}(i_2) + Te_j(i_2) \leqslant (\lambda - 1 + \varepsilon) + \frac{\varepsilon_0}{2}$$
$$\leqslant \lambda - \frac{\varepsilon_0}{2} \leqslant Te_{j_0}(i_1) + Te_j(i_1) \leqslant \lambda + \frac{\varepsilon_0}{2} + \varepsilon.$$

Let $j \in I \setminus G$ be a fixed element. According to the previous relations, if there exists $i \in I$ such that

$$Te_{j_0}(i) + Te_j(i) > \lambda + \frac{\varepsilon_0}{2} + \varepsilon,$$

or

$$Te_{j_0}(i) + Te_j(i) < \lambda - 1 - \frac{\varepsilon_0}{2},$$

then we have $1 = \|Te_{j_0} + Te_j\|_d > 1$. This contradiction implies that for all $i \in I$

$$\lambda - 1 - \frac{\varepsilon_0}{2} \leqslant Te_{j_0}(i) + Te_j(i) \leqslant \lambda + \frac{\varepsilon_0}{2} + \varepsilon,$$

or

$$\lambda - 1 - \frac{\varepsilon_0}{2} - Te_{j_0}(i) \leqslant Te_j(i) \leqslant \lambda + \frac{\varepsilon_0}{2} + \varepsilon - Te_{j_0}(i).$$

So, if $i \in I \setminus F$, then

$$\lambda - 1 - \varepsilon \leqslant \lambda - 1 - \varepsilon_0 \leqslant Te_j(i) \leqslant \lambda + \varepsilon_0 + \varepsilon \leqslant \lambda + 2\varepsilon, \tag{3}$$

and if $i \in F$, then

$$\frac{-\varepsilon}{2} \leqslant Te_j(i) \leqslant \frac{\varepsilon}{2}. \tag{4}$$

By using (3) and (4) we have

$$\lambda - 1 - \varepsilon \leqslant \min\left\{\lambda - 1 - \varepsilon, \frac{-\varepsilon}{2}\right\} \leqslant \inf Te_j \leqslant \sup Te_j \leqslant \max\left\{\lambda + 2\varepsilon, \frac{\varepsilon}{2}\right\} \leqslant \lambda + 2\varepsilon.$$

If $\sup Te_j < \lambda - \varepsilon$, then $\sup Te_j - \inf Te_j < 1$, which contradicts because $\sup Te_j - \inf Te_j = 1$. Therefore,

$$\lambda - \varepsilon \leqslant \lambda_j = \sup Te_j \leqslant \lambda + 2\varepsilon.$$

A similar argument shows

$$\lambda - 1 - \varepsilon \leqslant \eta_j = \inf Te_j \leqslant \lambda - 1 + 2\varepsilon.$$

Thus, we proved that

$$\forall\, 0 < \varepsilon < \lambda, \ \exists G \subseteq I (\text{finite set}) \ \text{s.t.}\ \forall j \in I \setminus G \quad |\lambda_j - \lambda| \leqslant 2\varepsilon\ ,\ |\eta_j - (\lambda - 1)| \leqslant 2\varepsilon,$$

i.e. $\lambda = \lim_{j \in I} \lambda_j$ and $\lambda - 1 = \lim_{j \in I} \eta_j$. $\square$

REMARK 2.6. Let $T \in \mathscr{I}_d$. Then the value of $\inf Te_j$ is independent of $j$ because according to the previous lemma

$$\sup Te_{j_1} = \lim_{j \in I} \left( \sup Te_j \right) = \sup Te_{j_2},$$

holds for each $j_1, j_2 \in I$. A similar argument also holds for $\inf Te_j$.

THEOREM 2.7. *Suppose that $T \in \mathscr{I}_d$. Then one of the following conditions hold.*

(i) *For each $j \in I$, $\inf Te_j = -1$ and $\sup Te_j = 0$; or*

(ii) *For each $j \in I$, $\inf Te_j = 0$ and $\sup Te_j = 1$.*

*Proof.* By the previous remark the value of $\lambda := \sup Te_j$ is constant. It is clear that $\lambda \in [0,1]$. To show that $\lambda = 0$ or $\lambda = 1$, suppose on the contrary $\lambda \in (0,1)$. Since $\inf Te_j = \lambda - 1 < 0 < \lambda = \sup Te_j$, we conclude that $\lambda - 1 = \min Te_j = Te_{j_0}(i_1)$ and $\lambda = \max Te_j = Te_{j_0}(i_2)$, for some $i_1, i_2 \in I$. Now for each $0 < \varepsilon < \min\{\lambda, 1-\lambda\}$, there exists a finite set $F \subseteq I$ such that $|Te_{j_0}(i)| < \frac{\varepsilon}{2}$ for all $i \in I \setminus F$. It is clear that $i_1, i_2 \in F$. Moreover, there exists a finite set $G \subseteq I$ such that for all $i \in F$, and $j \in I \setminus G$, we have $|Te_j(i)| < \frac{\varepsilon}{2}$, because $F$ is finite and $\lim_{j \in I} Te_j(i) = 0$ for all $i \in F$. Suppose that $j \in I \setminus G$. Then we have

$$a_1 := Te_{j_0}(i_1) - Te_j(i_1) = (\lambda - 1) - Te_j(i_1) \in \mathrm{Im}(Te_{j_0} - Te_j), \tag{5}$$

and

$$a_2 := Te_{j_0}(i_2) - Te_j(i_2) = \lambda - Te_j(i_2) \in \mathrm{Im}(Te_{j_0} - Te_j). \tag{6}$$

Now, there exist $i_3, i_4 \in I$ with $Te_j(i_3) = \lambda - 1$ and $Te_j(i_4) = \lambda$, since $\min Te_j = \lambda - 1$ and $\max Te_j = \lambda$. It is clear that $i_3, i_4 \in I \setminus F$. Also we have

$$\forall i \in F, \quad |Te_j(i)| < \frac{\varepsilon}{2} < \min\{\lambda, 1-\lambda\}.$$

Hence $Te_j(i) \notin \{\lambda, \lambda - 1\}$, and we also have

$$a_3 := Te_{j_0}(i_3) - Te_j(i_3) = Te_{j_0}(i_3) - \lambda \in \mathrm{Im}(Te_{j_0} - Te_j), \tag{7}$$

and

$$a_4 := Te_{j_0}(i_4) - Te_j(i_4) = Te_{j_0}(i_4) - \lambda \in \mathrm{Im}(Te_{j_0} - Te_j). \tag{8}$$

Relations (5)–(8) imply $a_1, a_2, a_3, a_4 \in \mathrm{Im}(Te_{j_0} - Te_j)$ and furthermore,

$$a_1 = \lambda - 1 - Te_j(i_1) \leqslant \lambda - 1 + \frac{\varepsilon}{2}\lambda - \frac{\varepsilon}{2} \leqslant \lambda - Te_j(i_2) = a_2.$$

So we have $a_1 \leqslant a_2$. A similar method implies $a_3 \leqslant a_4$. Thus

$$\mathrm{co}(Te_{j_0} - Te_j) \subseteq \left[ \min\left(\lambda - 1 - \frac{\varepsilon}{2}, -\lambda - \frac{\varepsilon}{2}\right), \max\left(\lambda + \frac{\varepsilon}{2}, 1 - \lambda + \frac{\varepsilon}{2}\right) \right] \tag{9}$$

Now if $\lambda \leqslant 1-\lambda$, then the length of the interval

$$\left[\min\left(\lambda-1-\frac{\varepsilon}{2},-\lambda-\frac{\varepsilon}{2}\right),\max\left(\lambda+\frac{\varepsilon}{2},1-\lambda+\frac{\varepsilon}{2}\right)\right],$$

used in the previous relation, is equal to $2-2\lambda+\varepsilon$. So, according to (9) we have

$$2=\|Te_{j_0}-Te_j\|_d \leqslant 2-2\lambda+\varepsilon.$$

Thus $\lambda \leqslant \frac{\varepsilon}{2}$, which contradicts the choice $\varepsilon$, since $\varepsilon$ was selected such that $0<\varepsilon<\min\{\lambda,1-\lambda\}$. In the case $\lambda \geqslant 1-\lambda$, using again (9), we obtain $2=\|Te_{j_0}-Te_j\|_d \leqslant 2\lambda+\varepsilon$. This case also contradicts the choice of $\varepsilon$. □

In the following theorem, we obtain the structure of diameter preserving isometries on $\mathfrak{c}_0(I)$.

THEOREM 2.8. *Suppose that $T:\mathfrak{c}_0(I)\to\mathfrak{c}_0(I)$ is a bounded linear operator. Then $T\in\mathscr{I}_d$ if and only if one of the following conditions hold.*

(i) *For each $j\in I$, $\min Te_j=0$ and $\max Te_j=1$, and for each $i\in I$, $0\leqslant \sum_{k\in I} Te_k(i)\leqslant 1$;*
*or*

(ii) *For each $j\in I$, $\min Te_j=-1$ and $\max Te_j=0$, and for each $i\in I$, $-1\leqslant \sum_{k\in I} Te_k(i)\leqslant 0$.*

*Proof.* Using Theorem 2.7 for the constant value $\lambda:=\sup Te_j$, we have $\lambda=0$, or $\lambda=1$. By replacing $-T$ by $T$, we may assume that $\lambda=1$. So, $\inf Te_j=0$ and $\sup Te_j=1$, for all $j\in I$. Now suppose $j_0\in I$. Since $Te_{j_0}\in\mathfrak{c}_0(I)$ and $\sup Te_{j_0}>0$, the value of $\max Te_{j_0}$ exists and $\max Te_{j_0}=\sup Te_{j_0}=1$.

A similar argument for $j_1\neq j_0$, leads to $\max Te_{j_1}=1$. Thus there exists $i_1\in I$ with $Te_{j_1}(i_1)=1$. On the other hand $0\leqslant Te_{j_0}(i_1)\leqslant 1$. If $Te_{j_0}(i_1)>0$, then

$$1<Te_{j_0}(i_1)+Te_{j_1}(i_1)\leqslant \sum_{j\in I^+} Te_j(i_1),$$

which contradicts to Lemma 2.4. Thus $Te_{j_0}(i_1)=0$, and therefore $\min Te_{j_0}$ exists and is equal to $0$. On the other hand,

$$\min Te_k=0\leqslant Te_k(i)\leqslant \max Te_k=1.$$

Thus using Lemma 2.4, again we have

$$0\leqslant \sum_{k\in I} Te_k(i)=\sum_{k\in I^+} Te_k(i)\leqslant 1.$$

Conversely, suppose that (i) satisfies and $f=(f_j)_{j\in I}\in\mathfrak{c}_0(I)$. For each $j_0\in I$, there is $i_0\in I$ such that $Te_{j_0}(i_0)=1$ because $\max Te_{j_0}=1$. So, we can imply that $Te_j(i_0)=0$ for all $j\neq j_0$ since $\sum_{j\in I} Te_j(i_0)=1$, $0\leqslant Te_j(i_0)\leqslant 1$, and $Te_{j_0}(i_0)=1$. Therefore

$$Tf(i_0)=\sum_{j\in I} Te_j(i_0)f_j=Te_{j_0}(i_0)f_{j_0}=f_{j_0},$$

which implies $\mathrm{Im}(f) \subseteq \mathrm{Im}(Tf)$. Thus

$$\|f\|_d \leqslant \|Tf\|_d. \tag{10}$$

On the other hand, for $f \in \mathfrak{c}_0(I)$, $\inf(f) \leqslant 0 \leqslant \sup(f)$ since $I$ is an infinite set. So, we have

$$\inf(f) \leqslant \inf(f) \sum_{j\in I} Te_j(i) \leqslant Tf(i) = \sum_{j\in I} Te_j(i) f_j \leqslant \sup(f) \sum_{j\in I} Te_j(i) \leqslant \sup(f).$$

This implies

$$\|Tf\|_d \leqslant \|f\|_d. \tag{11}$$

From (10) and (11), $T$ is a d-isometry. Now, if condition (ii) holds, then the operator $-T$ satisfies (i). Therefore, the previous part of this proof shows that $-T$ is a d-isometry. Then $T$ is also a d-isometry. □

Note that the previous theorem says that any d-isometry $T : \mathfrak{c}_0(I) \to \mathfrak{c}_0(I)$ is either positive (i.e. $Tf \geqslant 0$, for all $f \geqslant 0$) or negative operator (i.e. $Tf \leqslant 0$, for all $f \geqslant 0$).

The following remark compares the relation between isometries under the usual norm on $\mathfrak{c}_0(I)$ and d-isometries.

REMARK 2.9. It can be proved that, the operator $T : \mathfrak{c}_0(I) \to \mathfrak{c}_0(I)$ is an isometry(in the usual sense), if and only if $T$ satisfies the following conditions.

(i) For each $j \in I$, $\|Te_j\|_\infty = 1$,

(ii) For each $i \in I, \sum_{j\in I} |Te_j(i)| \leqslant 1$.

Thus according to Theorem 2.8, every d-isometry $T : \mathfrak{c}_0(I) \to \mathfrak{c}_0(I)$ is an isometry, but the converse need not be true in general. For example, if $T : \mathfrak{c}_0 \to \mathfrak{c}_0$, is defined for each $f = (f_1, f_2, \ldots) \in \mathfrak{c}_0$ by $T(f) = (\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} f_n, f_1, f_2, \ldots)$, then $T$ is an isometry however, $T \notin \mathscr{I}_d$, because $\|Te_1\|_d = \|(\frac{-1}{2}, 1, 0, 0, 0, \ldots)\|_d = \frac{3}{2} \neq 1 = \|e_1\|_d$.

REMARK 2.10. Let $I$ be a finite set with $n$ elements. Lemma 2.4, Theorems 2.7 and 2.8 do not hold. Towards a counterexample for these results we define $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix $T = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$. Note that $T$ satisfies:

$$\|Tf\|_d = |(x+3y) - (2x+2y)| = |x-y| = \|f\|_d,$$

for each $f = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$, which shows that $T \in \mathscr{I}_d$.

In fact, an easy computation shows that if $n = 1$, then every linear map belongs to $\mathscr{P}_d$ and if $n = 2$, then $T \in \mathscr{I}_d$ if and only if $T$ has the matrix form $T = \begin{bmatrix} a & b+c \\ a+c & b \end{bmatrix}$, for some $a, b, c \in \mathbb{R}$. But for $n \geqslant 3$ without being able to characterize $\mathscr{P}_d$, we give a large class of matrices in $\mathscr{P}_d$. In fact, if $\theta : \mathbb{R}^n \to \mathbb{R}$ is a linear operator and $P : \mathbb{R}^n \to \mathbb{R}^n$ is a permutation, then for $T : \mathbb{R}^n \to \mathbb{R}^n$ which is defined by

$$Tf = \theta(f)\mathbf{e} + \alpha P(f),$$

we have $T \in \mathscr{P}_d$, where $\mathbf{e} := \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. This claim can be proved easily.

Remark 2.10 shows a significant difference for the diametric majorization preservers when $I$ is finite and infinite. More precisely, we give examples of diametric majorization preservers, but a complete characterization for these operators remains open.

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REFERENCES

[1] F. BAHRAMI, A. BAYATI ESHKAFTAKI, S. M. MANJEGANI, *Majorization on $\ell_\infty$ and on its closed linear subspace $\mathfrak{c}$, and their linear preservers*, Linear Algebra Appl. **437** (2012) 2340–2358.
[2] A. BAYATI ESHKAFTAKI, N. EFTEKHARI, *Convex majorization on discerete $\ell^p(I)$ spaces*, Linear Algebra Appl. **474** (2015) 124–140.
[3] R. J. FLEMING, J. E. JAMISON, *Isometries on Banach Spaces: Function Spaces*, CRC Press, Boca Raton, FL, 2003.
[4] R. J. FLEMING, J. E. JAMISON, *Isometries on Banach Spaces: Vector-Valued Function Spaces and Operator Spaces*, CRC Press, Boca Raton, FL, 2007.
[5] N. J. GAL, J. JAMISON, *Isometries and isometric equivalence of hermitian operators on $A^{1,p}(X)$*, J. Math. Anal. Appl. **339** (2008) 225–239.
[6] A. M. HASANI, M. RADJABALIPOUR, *On linear preservers of (right) matrix majorization*, Linear Algebra Appl. **423** (2007) 255–261.
[7] F. KHALOOEI, A. SALEMI, *The structure of linear preservers of left matrix majorizations on $\mathbb{R}^p$*, Electronic Journal of Linear Algebra **18** (2009) 88–97.
[8] L. LI, R. WANG, *Surjective isometries on the vector-valued differentiable function spaces*, J. Math. Anal. Appl. **427** (2015) 547–556.
[9] M. LJUBENOVIC, D. S. DJORDJEVIC, *Linear preservers of weak majorization on $\ell^1(I)^+$ when $I$ is an infinite set*, Linear Algebra Appl. **517** (2017) 177–198.

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