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INVERSE STURM–LIOUVILLE PROBLEM FOR A STAR GRAPH BY THREE SPECTRA

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Abstract. A three spectra problem for a star graph of three edges is solved. The given data are 1) the spectrum of a boundary value problem on the whole graph with the Dirichlet boundary conditions at the pendant vertices, continuity and Kirchhoff's conditions at the interior vertex, 2) the spectrum of the Dirichlet-Neumann problem on one of the edges, 3) the spectrum of the Dirichlet-Dirichlet problem on the union of two other edges. The aim is to find the potentials on the edges. Conditions on three sequences of numbers are found sufficient to be the spectra of these three problems.

1. Introduction

Usually the term 'quantum graphs' means metric graphs considered as quasi-one-dimensional domains with differential operations defined on these domains, see e. g. [20], [3]. In quantum mechanics the Sturm-Liouville or the Dirac equation are considered on the edges of a graph subject to matching conditions at the interior vertices and boundary conditions at the pendant vertices. These are Dirichlet, Neumann or Robin conditions at pendant vertices and continuity conditions together with Kirchhoff's conditions at interior vertices. These conditions correspond to selfadjoint operators see [19]. Such models are often used in problems of free-electron theory of conjugate molecules in chemistry and in the theory of quantum wires and thin wave-guides. The differential operations together with the matching and boundary conditions define an operator which is usually called continuous Laplacian. Since the literature on this topic is vast we refer just to some of the authors: [1], [6], [7], [10], [11], [12], [19], [21].

Finite-dimensional analogues of the above boundary value problems appear in the theory of vibrations of nets and graphs of the so called Stieltjes strings or Sturmian systems (see [13], [14], [15], [26]). Another source of boundary value problems on graphs is synthesis of electrical circuits (see, e.g. [32]).

Let us summarize the main results known for inverse problems on graphs.

1. In case of commensurable lengths of the edges and zero potentials the spectrum does not determine uniquely the form of the graph [16]. In case of non-commensurable

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lengths of the edges and zero potentials the spectrum determines uniquely the form of the graph [2].

2. If we consider two spectral problems with Dirichlet boundary condition at a pendant vertex the first one and with Neumann boundary condition at this vertex the second one, then the spectra of these two problems uniquely determine the potential on the edge incident with this pendant vertex (see [5] and [35]).

3. Inverse problem for a star graph where the given data consists of the spectrum on the whole graph and the spectra of the Dirichlet-Dirichlet problems on the edges was solved in [29]–[31].

4. Estimates on maximal possible multiplicity of an eigenvalue depends only on the form of the graph [25], [17]. Estimates on all possible multiplicities of eigenvalues for the problem on a star graph see in [4].

5. Ambarzumian's theorem is true for trees [8] and some other graphs [33], [18].

Our goal is to solve an inverse problem on a star graph with different potentials on the edges. We use classical results of I. Gel'fand, B.M. Levitan [24], V.A. Marchenko [27] as a tools in our investigation.

In Sec. 2 we give physical motivation for the problem under consideration. We show that such problems occur in description of transverse vibrations of a star graph of smooth strings as well as in quantum theory of waveguides. In Sec.3 we consider the direct problem, i.e. we describe location of the spectra (including the asymptotics) of the boundary value problem on the whole star graph, of the Dirichlet-Neumann problem on one of the edges and of the Dirichlet-Dirichlet problem on the union of the rest two edges. In Sec. 4 we consider the corresponding inverse problem. We find conditions sufficient for three sequences of numbers to be the spectra of these three problems.

2. Physical motivation

Let us consider a plane star graph of three smooth inhomogeneous stretched strings (labeled by subscripts $1,2,3$) each having one end joined at the interior vertex of the graph and the other end fixed. Small transverse vibrations of such a graph with the edges denoted by e_j $(j=1,2,3)$ are described by the following system of equations

$$\frac{\partial^2}{\partial s^2}u_j(s,t)-\rho_j(s)\frac{\partial^2}{\partial t^2}u_j(s,t)=0,\quad j=1,2,3,\quad s\in[0,l], \tag{1}$$

$$u_j(0,t)=0, \tag{2}$$

$$u_1(l,t)=u_2(l,t)=u_3(l,t), \tag{3}$$

$$\sum_1^3\frac{\partial}{\partial s}u_j(s,t)\Big|_{s=l}=0. \tag{4}$$

The strings are supposed to be of the same length l. Here $u_j(s,t)$ stands for the transverse displacement of the j-th string at position s and time t, $\rho_j(s)$ is the density of the j-th string. Conditions (2) mean that the pendant vertices are fixed, conditions (3) mean continuity of the net at the interior vertex, condition (4) describes the balance of forces at the interior vertex.

Substituting $u_j(s,t) = v_j(\lambda,s)e^{i\lambda t}$ into (1)–(4) we obtain

$$\frac{\partial^2}{\partial s^2}v_j(\lambda,s)+\lambda^2\rho_j(s)v_j(\lambda,s)=0,\ \ (j=1,2,3),\ \ s\in[0,l],$$

$$v_j(\lambda,0)=0,$$
$$v_1(\lambda,l)=v_2(\lambda,l)=v_3(\lambda,l),$$
$$\sum_1^3 \frac{\partial}{\partial s}v_j(\lambda,s)\bigg|_{s=l}=0.$$

If the densities $\rho_j(s)$ belong to the Sobolev space $W_2^2(0,l)$ and $\rho_j(s)>0$ for $s\in[0,l]$, then we write $\rho_j[x_j]\overset{def}{=}\rho_j(s(x_j))$, apply the Liouville transformation [9] (p. 292), see also [28] (p. 47):

$$x_j(s)=\int_0^s \rho_j(s')^{1/2}ds',$$

$$y_j(\lambda,x_j)=\rho_j[x_j]^{1/4}v_j(\lambda,s(x_j))$$

and obtain

$$y_j''+\lambda^2 y_j-q_j(x)y_j=0,\ \ j=1,2,3,\ \ x\in[0,a_j], \tag{5}$$

$$y_j(\lambda,0)=0,\ \ j=1,2,3, \tag{6}$$

$$\rho_1[a_1]^{-1/4}y_1(\lambda,a_1)=\rho_2[a_2]^{-1/4}y_2(\lambda,a_2)=\rho_3[a_3]^{-1/4}y_3(\lambda,a_3), \tag{7}$$

$$\sum_{j=1}^3 y_j'(\lambda,a_j)+\beta y_1(\lambda,a_1)=0, \tag{8}$$

where primes denote x-differentiation and

$$q_j(x_j)=\rho_j[x_j]^{-1/4}\frac{d^2}{dx_j^2}\left(\rho_j[x_j]^{1/4}\right), \tag{9}$$

$$a_j=\int_0^l \rho_j(s)^{1/2}ds,$$

$$\beta=-\frac{1}{4}\sum_{j=1}^3\rho_j[a_j]^{-1}\ \frac{d\rho_j[x_j]}{dx_j}\bigg|_{x_j=a_j}.$$

Thus, we consider problem (5)–(8). It is known (see, i.e. [28], Theorem 8.4.1) that the spectrum of problem (5)–(8) obtained this way consists of real nonzero normal (isolated Fredholm) eigenvalues.

This problem occurs also in quantum mechanics when one considers a quantum particle subject to the Schrödinger equation moving in a quasi-one-dimensional star-shaped wave-guide. In this case q_j are not obtained from (9) but are real $L_2(0,a_j)$-functions and problem (5)–(8) may have a finite number of pure imaginary eigenvalues located symmetrically with respect to the origin (see [28], Theorem 8.4.1). In the sequel we consider this general case.

3. Direct problem

For the sake of simplicity we suppose in what follows $a_1 = a_2 = a_3 \overset{def}{=} a$, $\rho_1[a_1] = \rho_2[a_2] = \rho_3[a_3]$. Thus, we deal with the *main problem*:

$$-y_j'' + q_j(x_j)y = \lambda^2 y_j, \;\; x_j \in [0,a], \;\; j = 1,2,3, \tag{10}$$

$$y_j(0) = 0, \;\; j = 1,2,3, \tag{11}$$

$$y_1(a) = y_2(a) = y_3(a), \tag{12}$$

$$y_1'(a) + y_2'(a) + y_3'(a) + \beta y_1(a) = 0 \tag{13}$$

with real potentials $q_j \in L_2(0,a)$ and a real constant β. The spectrum of the main problem we denote by $\{\lambda_k\}_{-\infty,\, k\neq 0}^{\infty}$ $(\lambda_{-k} = -\lambda_k)$.

Simultaneously we consider the Dirichlet-Neumann problem on the edge e_1

$$-y_1'' + q_1(x)y_1 = \lambda^2 y_1, \;\; x \in [0,a] \tag{14}$$

$$y_1(0) = y_1'(a) = 0, \tag{15}$$

with the spectrum denoted by $\{\mu_k\}_{-\infty,\, k\neq 0}^{\infty}$ $(\mu_{-k} = -\mu_k)$, and the Dirichlet-Dirichlet problem on the union of edges $e_2 \cup e_3$

$$-y_j'' + q_j(x)y_j = \lambda^2 y_j, \;\; x_j \in [0,a] \;\; j = 2,3, \tag{16}$$

$$y_j(0) = 0, \;\; j = 2,3, \tag{17}$$

$$y_2(a) = y_3(a), \tag{18}$$

$$y_2'(a) + y_3'(a) + \beta y_2(a) = 0 \tag{19}$$

with the spectrum denoted by $\{\nu_k\}_{-\infty,\, k\neq 0}^{\infty}$ $(\nu_{-k} = -\nu_k)$.

Let us denote by $s_j(\lambda,x)$ $(j = 1,2,3)$ the solution of (10) which satisfies the conditions

$$s_j(\lambda,0) = s_j'(\lambda,0) - 1 = 0.$$

Looking for a solution of (10)–(13) in the form $y_j = C_j s_j(\lambda,x)$ where C_j are constants we find that the spectrum of problem (10)–(13) coincides with the set of zeros of the function

$$\phi(\lambda) := s_1'(\lambda,a)s_2(\lambda,a)s_3(\lambda,a) + s_1(\lambda,a)s_2'(\lambda,a)s_3(\lambda,a) + s_1(\lambda,a)s_2(\lambda,a)s_3'(\lambda,a) + \beta s_1(\lambda,a)s_2(\lambda,a)s_3(\lambda,a). \tag{20}$$

In the same way, the spectrum of problem (16)–(19) coincides with the set of zeros of the function

$$\psi(\lambda) := s_2'(\lambda,a)s_3(\lambda,a) + s_2(\lambda,a)s_3'(\lambda,a) + \beta s_2(\lambda,a)s_3(\lambda,a). \tag{21}$$

The spectrum of problem (14)–(15) coincides with the set of zeros of $s_1'(\lambda,a)$.

DEFINITION 1. (see Definition 12.2.2 in [28] or Section 2.5 in [34]). An entire function ω of exponential type $\leqslant \sigma$ is said to belong to the Paley-Wiener class $\mathscr{L}^{\sigma}$ if its restriction to the real axis belongs to $L_2(-\infty,\infty)$.

We will use the following representations (see e.g. [28] Corollary 12.2.10):

$$s_j(\lambda,a)=\frac{\sin\lambda a}{\lambda}-B_j\frac{\cos\lambda a}{\lambda^2}+\frac{f(\lambda)}{\lambda^2}, \tag{22}$$

$$s_j'(\lambda,a)=\cos\lambda a+B_j\frac{\sin\lambda a}{\lambda}+\frac{f(\lambda)}{\lambda}, \tag{23}$$

where $B_j\in\mathbb{R}$. Here and in the sequel we use the same notation f for different functions from $\mathscr{L}^{a}$.

Substituting (22) and (23) into (21) we obtain

$$\psi(\lambda)=\frac{\sin 2\lambda a}{\lambda}-\frac{(B_2+B_3)\cos 2\lambda a}{\lambda^2}+\frac{F(\lambda)}{\lambda^2} \tag{24}$$

where $F\in\mathscr{L}^{2a}$.

DEFINITION 2. ([28], Definition. 5.1.20). The function θ is said to be a Nevanlinna function, or $\mathscr{N}$-function if:

(i) θ is analytic in the half-planes $\operatorname{Im}\lambda>0$ and $\operatorname{Im}\lambda<0$;

(ii) $\theta(\overline{\lambda})=\overline{\theta(\lambda)}$ if $\operatorname{Im}\lambda\neq 0$;

(iii) $\operatorname{Im}\lambda\operatorname{Im}\theta(\lambda)\geqslant 0$ for $\operatorname{Im}\lambda\neq 0$.

DEFINITION 3. ([28], Definition. 5.1.26).

1. The class $\mathscr{N}^{ep}$ of essentially positive Nevanlinna functions is the set of all functions $\theta\in\mathscr{N}$ which are analytic in $\mathbb{C}\backslash[0,\infty)$ with the possible exception of finitely many poles.

2. The class $\mathscr{N}^{ep}_{+}$ is the set of all functions $\theta\in\mathscr{N}^{ep}$ such that for some $\gamma\in\mathbb{R}$ we have $\theta(\lambda)>0$ for all $\lambda\in(-\infty,\gamma)$.

LEMMA 1. *1.* $\frac{\psi(\sqrt{z})}{s_2'(\sqrt{z},a)s_3'(\sqrt{z},a)}\in\mathscr{N}^{ep}_{+}$,

2. $\frac{s_2(\sqrt{z},a)s_3(\sqrt{z},a)}{\psi(\sqrt{z})}\in\mathscr{N}^{ep}_{+}$,

3. $\frac{\phi(\sqrt{z})}{\psi(\sqrt{z})s_1'(\sqrt{z},a)}\in\mathscr{N}^{ep}_{+}$,

Proof. 1. Since $\frac{s_j(\sqrt{z},a)}{s_j'(\sqrt{z},a)}\in\mathscr{N}^{ep}_{+}$ we conclude that

$$\frac{\psi(\sqrt{z})}{s_2'(\sqrt{z},a)s_3'(\sqrt{z},a)}=\frac{s_2(\sqrt{z},a)}{s_2'(\sqrt{z},a)}+\frac{s_3(\sqrt{z},a)}{s_3'(\sqrt{z},a)}\in\mathscr{N}^{ep}_{+}.$$

2. We evaluate

$$\frac{s_2(\sqrt{z},a)s_3(\sqrt{z},a)}{\psi(\sqrt{z})}=\left(\left(\frac{s_2(\sqrt{z},a)}{s_2'(\sqrt{z},a)}\right)^{-1}+\left(\frac{s_3(\sqrt{z},a)}{s_3'(\sqrt{z},a)}\right)^{-1}+\beta\right)^{-1}.$$

Since $\frac{s_j(\sqrt{z},a)}{s'_j(\sqrt{z},a)} \in \mathcal{N}_+^{ep}$ we conclude by Lemma 5.1.22 in [28] that

$$\frac{s_2(\sqrt{z},a)s_3(\sqrt{z},a)}{\psi(\sqrt{z})} \in \mathcal{N}.$$

It is clear from (22) and (23) that $\frac{s_j(\sqrt{z})}{s'_j(\sqrt{z},a)} \underset{z\to-\infty}{\to} +0$ and therefore statement 2 is true.

3. Since

$$\frac{\phi(\sqrt{z})}{s'_1(\sqrt{z},a)\psi(\sqrt{z})} = \frac{s_2(\sqrt{z},a)s_3(\sqrt{z},a)}{\psi(\sqrt{z})} + \frac{s_1(\sqrt{z},a)}{s'_1(\sqrt{z},a)}$$

we arrive at statement 3. □

COROLLARY 1. *1. The sequence $\{\lambda_k^2\}_{k\neq1}^{\infty}$ of zeros of $\phi(\sqrt{z})$ interlace with the union $\{\xi_k^2\}_{k=1}^{\infty} := \{\mu_k^2\}_{k=1}^{\infty} \cup \{\nu_k^2\}_{k=1}^{\infty}$ of sequences of zeros of the functions $s'_1(\sqrt{z},a)$ and $\psi(\sqrt{z})$:*

$$-\infty < \xi_1^2 \leqslant \lambda_1^2 \leqslant \xi_2^2 \leqslant \lambda_2^2 \leqslant \ldots \leqslant \xi_k^2 \leqslant \lambda_k^2 \leqslant \ldots$$

2. Multiplicity of any λ_k^2 and any ξ_k^2 does not exceed 2.
3. If $\lambda_k^2 = \lambda_{k+1}^2$ is a double zero then $\xi_k^2 < \lambda_k^2 = \xi_{k+1}^2 = \lambda_{k+1}^2 < \xi_{k+2}^2$.

Proof. Statement 1 follows from statement 3 of Lemma 1. Statement 2 for λ_k^2 is a consequence of the general result in [17] (Theorem 4.3) while for ξ_k^2 it follows from simplicity of μ_k^2 and ν_k^2. To prove statement 3 we notice that if $\lambda_k^2 = \lambda_{k+1}^2$ then interlacing in statement 1 implies $\lambda_k^2 = \xi_{k+1}^2 = \lambda_{k+1}^2$. In this case $s_j(\lambda_k,a) = 0$ for $j = 1,2,3$. Assume that ξ_{k+1} is a double zero, then $\xi_k = \lambda_k = \mu_p$ for some p. That means $s'_1(\lambda_k,a) = s'_1(\mu_p,a) = 0$ what contradicts $s_1(\lambda_k,a) = 0$. □

NOTATION. We set $B_j = \frac{1}{2\pi}\int_0^a q_j(x)dx$ and here and in the sequel use the same notation $\{\beta_k\}_{-\infty,k\neq0}^{\infty}$ (or $\{\beta_k\}_{-\infty}^{\infty}$) for all sequences which belong to l_2.

DEFINITION 4. We call the indexing of the sequence $\{\lambda_k\}$ of real numbers *proper* if:

(i) $\lambda_{-k} = -\lambda_k$;
(ii) $\lambda_k \geqslant \lambda_p$ if $k > p > 0$;
(iii) the multiplicities are taken into account;
(iv) the index set is $\mathbb{Z}$ if $0 \in \{\lambda_k\}$ and is $\mathbb{Z}\setminus\{0\}$ if $0 \notin \{\lambda_k\}$.

LEMMA 2. ([28], Theorem 7.4.7 with $\tilde{\nu} = 0$).
1. The set $\{\mu_k\}_{-\infty,k\neq0}^{\infty}$ of zeros of $s'_1(\lambda,a)$ which being indexed properly ($\mu_{-k} = -\mu_k$, $(\mu_k)^2 < (\mu_{k+1})^2$ for $k \in \mathbb{N}$) behave asymptotically as follows:

$$\mu_k = \frac{\pi(k-1/2)}{a} + \frac{B_1}{k} + \frac{\beta_k}{k}.$$

2. The set $\{\nu_k\}_{-\infty,k\neq 0}^{\infty}$ *of zeros of* $\psi(\lambda)$ *can be represented as a union of sequences* $\left\{\nu_k^{(2)}\right\}_{-\infty,k\neq 0}^{\infty}$ *and* $\left\{\nu_k^{(3)}\right\}_{-\infty,k\neq 0}^{\infty}$ *which being enumerated properly behave asymptotically as follows:*

$$\nu_k^{(2)} = \frac{\pi k}{a} + \frac{B_2 + B_3}{2k} + \frac{\beta_k}{k},$$

$$\nu_k^{(3)} = \frac{\pi(k-1/2)}{a} + \frac{B_2 + B_3 + \frac{\beta}{\pi}}{2k} + \frac{\beta_k}{k}.$$

LEMMA 3. ([28], Theorem 7.4.7 with $\tilde{\nu} = 0$). *The set* $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$ *of zeros of* $\varphi(\lambda)$ *can be represented as the union of three subsequences* $\bigcup_{j=1}^{3}\left\{\rho_k^{(j)}\right\}_{-\infty,k\neq 0}^{\infty}$ *which being enumerated properly behave asymptotically as follows:*

$$\rho_k^{(j)} = \frac{\pi k}{a} + \frac{M_j}{k} + \frac{\beta_k}{k}, \quad j = 1,2, \tag{25}$$

$$\rho_k^{(3)} = \frac{\pi(k-\frac{1}{2})}{a} + \frac{1}{3k}\left(\sum_{j=1}^{3} B_j + \frac{\beta}{\pi}\right) + \frac{\beta_k}{k}, \tag{26}$$

where M_j $(j = 1,2)$ *are the solutions (all real but not necessarily different) of the equation*

$$P(M) \stackrel{def}{=} 3M^2 - 2(B_1 + B_2 + B_3)M + (B_1B_2 + B_1B_3 + B_2B_3) = 0.$$

LEMMA 4. ([28], Corollary 7.4.8) *Let* $q_j(x)$ *belong to the Sobolev space* $W_2^1(0,a)$. *Then*

1. The sequences $\{\mu_k\}_{-\infty,k\neq 0}^{\infty}$ *behave asymptotically as follows*

$$\mu_k = \frac{\pi(k-1/2)}{a} + \frac{B_1}{(k-\frac{1}{2})} + \frac{\beta_k}{k^2}.$$

2. The sequences $\left\{\nu_k^{(j)}\right\}_{-\infty,k\neq 0}^{\infty}$ *behave asymptotically as follows:*

$$\nu_k^{(2)} = \frac{\pi k}{a} + \frac{B_2 + B_3}{2k} + \frac{\beta_k}{k^2},$$

$$\nu_k^{(3)} = \frac{\pi(k-1/2)}{a} + \frac{B_2 + B_3 + \frac{\beta}{\pi}}{2k-1} + \frac{\beta_k}{k^2}.$$

3. Instead of (25)-(26) we have

$$\rho_k^{(j)} = \frac{\pi k}{a} + \frac{M_j}{k} + \frac{\beta_k}{k^2}, \quad j = 1,2,$$

$$\rho_k^{(3)} = \frac{\pi(k-\frac{1}{2})}{a} + \frac{1}{3(k-\frac{1}{2})}\left(\sum_{j=1}^{3} B_j + \frac{\beta}{\pi}\right) + \frac{\beta_k}{k^2}.$$

REMARK 1. If $q_j(x) \in W_2^2(0,a)$ for $j=2,3$ then

$$\nu_k^{(3)} = \frac{\pi(k-1/2)}{a} + \frac{B_2+B_3+\frac{\beta}{\pi}}{2k-1} + \frac{B}{k^3} + \frac{\beta_k}{k^3}, \tag{27}$$

where B is a real constant.

In the next section we find sufficient conditions on three sequences of real numbers to be the spectra of the three above problems and give a method of recovering q_1, q_2, q_3 using $\{\lambda_k\}_{-\infty,\, k\neq 0}^{\infty}$, $\{\mu_k\}_{-\infty,\, k\neq 0}^{\infty}$, $\{\nu_k\}_{-\infty,\, k\neq 0}^{\infty}$.

4. Inverse problem

Here we deal with the problem of recovering the potentials $\{q_j(x)\}$ $(j=1,2,3)$ and the parameter β from the spectral data.

DEFINITION 5. (see, e.g. Definition 11.2.15 in [28] or [23], Sec. 1) An entire function ω of positive exponential type is said to be a sine type function if

(i) there is $h>0$ such that all zeros of ω lie in the strip $\{\lambda\in\mathbb{C} : |\mathrm{Im}\lambda|<h\}$,

(ii) there are $h_1\in\mathbb{R}$ and positive numbers $m<M$ such that $m\leqslant|\omega(\lambda)|\leqslant M$ holds for $\lambda\in\mathbb{C}$ with $\mathrm{Im}\lambda=h_1$,

(iii) the exponential type of ω in the lower half-plane coincides with the exponential type of ω in the upper half-plane.

The next theorem is the version of Theorem 8.4.1 in [28] adapted for the case of a star graph of two edges, i.e. for the graph P_2 which consists of the edges e_2 and e_3. Also we omit requirement $\nu_1^2>0$ and admit q_j be arbitrary real functions from $L_2(0,a)$. It can be achieved shifting $\lambda^2\to\lambda^2+c$ where $c\in\mathbb{R}$ in Theorem 8.4.1 of [28].

THEOREM 1. *Let three properly indexed sequences* $(\tilde{\nu}_k^{(j)})_{k-\infty,k\neq 0}^{\infty}$, $j=2,3$, *and* $(\nu_k)_{k=-\infty,k\neq 0}^{\infty}$ *of real numbers be given, satisfying the following conditions:*

1. *The sequences* $(\tilde{\nu}_k^{(j)})_{k=-\infty,k\neq 0}^{\infty}$, $j=2,3$, *are such that:*

(i) $\tilde{\nu}_1^{(j)}>0$*;*

(ii) $\tilde{\nu}_k^{(j)}\neq\tilde{\nu}_{k'}^{(j')}$ *whenever* $(k,j)\neq(k',j')$*;*

(iii)

$$\tilde{\nu}_k^{(j)} = \frac{\pi k}{a} + \frac{\tilde{B}_j}{k} + \frac{\beta_k}{k^2},\ j=2,3,\ k\in\mathbb{N}, \tag{28}$$

where the $\tilde{B}_j$ *are real constants,* $\tilde{B}_2\neq\tilde{B}_3$.

2. *The sequence* $(\nu_k)_{k=-\infty,k\neq 0}^{\infty}$ *can be represented as the union of two properly indexed subsequences* $(\tilde{\rho}_k^{(j)})_{k=-\infty,k\neq 0}^{\infty}$, $k=2,3$, *which behave asymptotically*

$$\tilde{\rho}_k^{(2)} = \frac{\pi k}{a} + \frac{\tilde{M}_2}{k} + \frac{\beta_k}{k^2}, \quad k\in\mathbb{N},$$

$$\tilde{\rho}_k^{(3)} = \frac{\pi(k-\frac{1}{2})}{a} + \frac{\tilde{B}_0}{k} + \frac{\beta_k}{k^2}, \quad k \in \mathbb{N},$$

where $\tilde{B}_0 \in \mathbb{R}$ *and* $\tilde{M}_2 = \frac{\tilde{B}_2+\tilde{B}_3}{2}$

3. *The properly indexed sequences of real numbers* $(\nu_k)_{k=-\infty,k\neq 0}^{\infty}$ *and* $(\tilde{\xi}_k)_{k=-\infty,k\neq 0}^{\infty}$ *interlace, where the sequence* $(\tilde{\xi}_k)_{k=-\infty,k\neq 0}^{\infty}$ *is the union of the sequences* $(\tilde{\nu}_k^{(j)})_{k=-\infty,k\neq 0}^{\infty}$, $j=2,3$*:*

$$\nu_1^2 < \tilde{\xi}_1^2 < \nu_2^2 < \tilde{\xi}_2^2 < \dots .$$

Then there exists a unique set (q_2, q_3, β) *where real functions* $q_j \in L_2(0,a)$ *and* $\beta \in \mathbb{R}$ *such that the sequence* $(\nu_k)_{k=-\infty,k\neq 0}^{\infty}$ *coincides with the spectrum of problem (16)–(19), where*

$$\beta = \pi\left(2\tilde{B}_0 - \tilde{B}_2 - \tilde{B}_3\right),$$

and such that the sequences $(\tilde{\nu}_k^{(j)})_{k=-\infty,\, k\neq 0}^{\infty}$, $j=2,3$, *coincide with the spectra of problems*

$$-y_j'' + q_j(x)y_j = \lambda^2 y_j, \;\; x_j \in [0,a] \;\; j=2,3, \tag{29}$$

$$y_j(0) = y_j(a) = 0, \;\; j=2,3. \tag{30}$$

Now we are ready to state the main result of this paper.

Denote by Q the class of sets $\{\{q_j(x)\}_{j=1}^3, \beta\}$, which satisfy the following conditions: the real-valued functions $q_j(x)$ $(j \in 1,2,3)$ belong to $L_2(0,a)$, $\beta \in \mathbb{R}$.

THEOREM 2. *Let three properly indexed sequences be given denoted by* $\{\mu_k\}_{-\infty,k\neq 0}^{\infty}$, $\{\nu_k\}_{-\infty,k\neq 0}^{\infty}$ *and* $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$ *satisfying the following conditions:*

1. $\{\nu_k\}_{-\infty,k\neq 0}^{\infty} = \{\nu_k^{(2)}\}_{-\infty,k\neq 0}^{\infty} \bigcup \{\nu_k^{(3)}\}_{-\infty,k\neq 0}^{\infty}$, $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty} = \bigcup_{j=1}^{3} \left\{\rho_k^{(j)}\right\}_{-\infty,k\neq 0}^{\infty}$, *where*

$$\mu_k = \frac{\pi(k-1/2)}{a} + \frac{D_1}{(k-\frac{1}{2})} + \frac{\beta_k}{k^2}, \; k \in \mathbf{N}, \tag{31}$$

$$\nu_k^{(2)} = \frac{\pi k}{a} + \frac{D_2}{2k} + \frac{\beta_k}{k^2}, \; k \in \mathbf{N}, \tag{32}$$

$$\nu_k^{(3)} = \frac{\pi(k-1/2)}{a} + \frac{D_3}{2k-1} + \frac{T}{k^3} + \frac{\beta_k}{k^3}, \; k \in \mathbf{N},$$

$$\rho_k^{(j)} = \frac{\pi k}{a} + \frac{M_j}{k} + \frac{\beta_k}{k^2}, \;\; j=1,2, \; k \in \mathbf{N},$$

$$\rho_k^{(3)} = \frac{\pi(k-\frac{1}{2})}{a} + \frac{B_0}{(k-\frac{1}{2})} + \frac{\beta_k}{k^2}, \;\; k \in \mathbf{N}.$$

Here D_j $(j=1,2,3)$, M_j $(j=1,2)$, B_0 *and* T *are real constants and*

$$B_0 = \frac{D_1}{3} + \frac{2D_3}{3},$$

$$M_1 + M_2 = \frac{2D_1}{3} + \frac{2D_2}{3},$$

$$D_2^2 + 4D_1D_2 > 12M_1M_2. \tag{33}$$

2. The sequences $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$ *and* $\{\xi_k\}_{-\infty}^{\infty} \stackrel{def}{=} \{\mu_k\}_{-\infty,k\neq 0}^{\infty} \bigcup \{\nu_k\}_{-\infty,k\neq 0}^{\infty}$ *(*$\xi_{-k} = -\xi_k$, $\xi_k < \xi_{k+1}$*) interlace in the strict sense:*

$$\xi_1^2 < \lambda_1^2 < \xi_2^2 < \lambda_2^2 < \ldots \tag{34}$$

Then there exists a set $\left\{\{q_j(x)\}_{j=1}^3, \beta\right\} \in Q$ *such that the sequence* $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$ *coincides with the spectrum of problem (10)–(13), where*

$$\beta = \pi(D_3 - D_2)$$

and the sequence $\{\nu_k\}_{-\infty,\,k\neq 0}^{\infty}$, *coincides with the spectrum of problem (16)–(19) and* $\{\mu_k\}_{-\infty,\,k\neq 0}^{\infty}$ *coincides with the spectrum of problem (14)–(15).*

Proof. Let us construct the functions

$$\phi_j(\lambda) = a \prod_{k=1}^{\infty} \left(\frac{a^2}{\pi^2 k^2} \left((\rho_k^{(j)})^2 - \lambda^2 \right) \right), \; j = 1, 2, \tag{35}$$

$$\phi_3(\lambda) = \prod_{k=1}^{\infty} \left(\frac{a^2}{\pi^2 (k - \frac{1}{2})^2} \left((\rho_k^{(3)})^2 - \lambda^2 \right) \right), \tag{36}$$

$$\psi_2(\lambda) = a \prod_{k=1}^{\infty} \left(\frac{a^2}{\pi^2 k^2} \left((\nu_k^{(2)})^2 - \lambda^2 \right) \right), \tag{37}$$

$$\psi_3(\lambda) = \prod_{k=1}^{\infty} \left(\frac{a^2}{\pi^2 (k - \frac{1}{2})^2} \left((\nu_k^{(3)})^2 - \lambda^2 \right) \right), \tag{38}$$

$$\tau(\lambda) = \prod_{k=1}^{\infty} \left(\frac{a^2}{\pi^2 (k - \frac{1}{2})^2} \left((\mu_k)^2 - \lambda^2 \right) \right).$$

It is known (see Lemma 12.3.4 in [28]) that

$$\phi_j(\lambda) = \frac{\sin \lambda a}{\lambda} - \frac{\pi M_j \cos \lambda a}{\lambda^2} + \frac{E_j \sin \lambda a}{\lambda^3} + \frac{f(\lambda)}{\lambda^3}, \; j = 1, 2, \tag{39}$$

$$\phi_3(\lambda) = \cos \lambda a + \frac{\pi B_0 \sin \lambda a}{\lambda} + \frac{E_3 \cos \lambda a}{\lambda^2} + \frac{f(\lambda)}{\lambda^2}, \tag{40}$$

$$\psi_2(\lambda) = \frac{\sin \lambda a}{\lambda} - \frac{\pi D_2 \cos \lambda a}{\lambda^2} + \frac{N_2 \sin \lambda a}{\lambda^3} + \frac{f(\lambda)}{\lambda^3}, \tag{41}$$

$$\psi_3(\lambda) = \cos \lambda a + \frac{\pi D_3 \sin \lambda a}{\lambda} + \frac{N_3 \cos \lambda a}{\lambda^2} + \frac{f(\lambda)}{\lambda^2}, \tag{42}$$

$$\tau(\lambda) = \cos\lambda a + \frac{\pi D_1 \sin\lambda a}{\lambda} + \frac{E\cos\lambda a}{\lambda^2} + \frac{f(\lambda)}{\lambda^2}, \tag{43}$$

where $E_J \in \mathbb{R}$ $(j = 1,2,3)$, $N_j \in \mathbb{R}$ $(j = 2,3)$.

It is clear in view of (39)–(43) that the functions $\varphi(\lambda) = \lambda^2\phi_1(\lambda)\phi_2(\lambda)\phi_3(\lambda)$, $\lambda\psi_1(\lambda)\psi_2(\lambda)$ and $\tau(\lambda)$ are of sine type.

Consider the functional equation

$$\tau(\lambda)X(\lambda) + 2\psi_2(\lambda)\psi_3(\lambda)Y(\lambda) = 3\phi_1(\lambda)\phi_2(\lambda)\phi_3(\lambda) \tag{44}$$

with a pair of unknown functions (X,Y). It is clear that due to $\tau(\mu_k) = 0$ we have

$$Y(\mu_k) = \frac{3\phi_1(\mu_k)\phi_2(\mu_k)\phi_3(\mu_k)}{2\psi_2(\mu_k)\psi_3(\mu_k)} \tag{45}$$

where the denominator is nonzero because $\{\mu_k\}_{-\infty,\ k\neq 0}^{\infty} \cap \{\nu_k\}_{-\infty,\ k\neq 0}^{\infty} = \emptyset$ due to (34).

By Lemma 1.4.3 in [27] or Lemma 12.2.1 in [28] we know that for any Paley-Wiener function f the sequence $\{f(\mu_k)\}_{-\infty,k\neq 0}^{\infty} \in l_2$. Substituting (31) into (39)–(42) we obtain

$$\begin{aligned}\phi_j(\mu_k) &= \frac{\sin\mu_k a}{\mu_k} - \frac{\pi M_j \cos\mu_k a}{\mu_k^2} + \frac{E_j \sin\mu_k a}{\mu_k^3} + \frac{f(\mu_k)}{\mu_k^3} \\ &= \frac{(-1)^{k-1}a}{\pi(k-1/2)} + O(k^{-3}),\end{aligned} \tag{46}$$

$$\begin{aligned}\phi_3(\mu_k) &= \cos\mu_k a + \frac{\pi B_0 \sin\mu_k a}{\mu_k} + \frac{E_3 \cos\mu_k a}{\mu_k^2} + \frac{f(\mu_k)}{\mu_k^2} \\ &= (-1)^k\frac{aD_1}{k} + (-1)^{k-1}\frac{aB_0}{k} + \frac{\beta_k}{k^2},\end{aligned} \tag{47}$$

$$\begin{aligned}\psi_2(\mu_k) &= \frac{\sin\mu_k a}{\mu_k} - \frac{\pi D_2 \cos\mu_k a}{\mu_k^2} + \frac{N_1 \sin\mu_k a}{\mu_k^3} + \frac{f(\mu_k)}{\mu_k^3} \\ &= \frac{(-1)^{k-1}a}{\pi(k-1/2)} + O(k^{-3}),\end{aligned} \tag{48}$$

$$\begin{aligned}\psi_3(\mu_k) &= \cos\mu_k a + \frac{\pi D_3 \sin\mu_k a}{\mu_k} + \frac{N_2 \cos\mu_k a}{\mu_k^2} + \frac{f(\mu_k)}{\mu_k^2} \\ &= (-1)^k\frac{aD_1}{k} + (-1)^{k-1}\frac{aD_3}{k} + \frac{\beta_k}{k^2}.\end{aligned} \tag{49}$$

We look for a solution (X,Y) of equation to (44) in the form where

$$Y(\lambda) = \frac{\sin\lambda a}{\lambda} - \frac{\pi D_1 \cos\lambda a}{\lambda^2} + \frac{\tilde{Y}(\lambda)}{\lambda^2}. \tag{50}$$

Then using (45) we obtain from (50)

$$\tilde{Y}(\mu_k) = -\mu_k \sin \mu_k a + \pi D_1 \cos \mu_k a + \frac{3\mu_k^2 \phi_1(\mu_k)\phi_2(\mu_k)\phi_3(\mu_k)}{2\psi_2(\mu_k)\psi_3(\mu_k)}.$$

In view of (46)–(49) this implies

$$\tilde{Y}(\mu_k) = (-1)^k \frac{\pi(k-1/2)}{a} + (-1)^{k-1} \frac{3\pi(k-1/2)}{2a} \frac{(B_0 - D_1)}{(D_3 - D_1)} + \beta_k = \beta_k.$$

Since τ is sine-type we conclude (see [23], Theorem A or [28], Theorem 11.3.14) that the series

$$\tilde{Y}(\lambda) = \tau(\lambda) \sum_{-\infty,\ k\neq 0}^{\infty} \frac{\tilde{Y}(\mu_k)}{\left.\frac{d\tau(\lambda)}{d\lambda}\right|_{\lambda=\mu_k} (\lambda - \mu_k)}$$

converges uniformly to a function from $\mathscr{L}^a$ on any compact domain of the complex plane and on the real axis in the norm of $L_2(-\infty,\infty)$. Thus if we denote $\{\sigma_k\}_{-\infty,k\neq 0}^{\infty}$ the sequence of zeros of Y then due to (50) Lemma 3.4.2 in [26] or Lemma 12.3.2 in [28] implies

$$\sigma_k = \frac{\pi k}{a} + \frac{\pi D_1}{k} + \frac{\beta_k}{k}. \tag{51}$$

Now let us consider equation (45). If $\mu_k = \xi_p$, then due to (34) there are $p-k$ squares of zeros of the function $\psi_2(\lambda)\psi_3(\lambda)$ and $p-1$ squares of zeros of $\phi_1(\lambda)\phi_2(\lambda)\phi_3(\lambda)$ located on the interval $(-\infty, \mu_k^2)$. Since due to (35)–(38) $\psi_j(\lambda) \underset{\lambda^2\to-\infty}{\longrightarrow} +\infty$ and $\phi_j(\lambda) \underset{\lambda^2\to-\infty}{\longrightarrow} +\infty$ we arrive at $\psi_2(\lambda)\psi_3(\lambda) \underset{\lambda^2\to-\infty}{\longrightarrow} +\infty$ and $\phi_1(\lambda)\phi_2(\lambda)\phi_3(\lambda) \underset{\lambda^2\to-\infty}{\longrightarrow} +\infty$. Thus, $(-1)^{p-k}\psi_2(\mu_k)\psi_3(\mu_k) > 0$ and $(-1)^{p-1}\phi_1(\mu_k)\phi_2(\mu_k)\phi_3(\mu_k) > 0$ and, consequently,

$$Y(\mu_k)(-1)^{k-1} = \frac{3\phi_1(\mu_k)\phi_2(\mu_k)\phi_3(\mu_k)}{2\psi_2(\mu_k)\psi_3(\mu_k)}(-1)^{k-1} > 0, \ \ k \geqslant 1.$$

Therefore, taking into account (51), (31) we conclude that $\{\sigma_k\}_{-\infty,k\neq 0}^{\infty}$ interlace with $\{\mu_k\}_{-\infty,k\neq 0}^{\infty}$:

$$\mu_1^2 < \sigma_1^2 < \mu_2^2 < \sigma_2^2 < \dots$$

Using Theorem 3.4.1 in [27] or Theorem 12.6.2 in [28] we conclude that there exists a unique real function $q_1 \in L_2(0,a)$ which generates the Dirichlet-Neumann problem (14), (15) with the spectrum $\{\mu_k\}_{-\infty,\ k\neq 0}^{\infty}$ and Dirichlet-Dirichlet problem

$$-y_1'' + q_1(x)y = \lambda^2 y_1, \ \ x \in [0,a]$$

$$y_1(0) = y_1(a) = 0.$$

with the spectrum $\{\sigma_k\}_{-\infty,\ k\neq 0}^{\infty}$. The method of recovering q_1 is described, e. g., in [27], Theorem 3.3.1 or [28], Proposition 12.4.8.

We are looking for a solution (X,Y) to (44) with X of the form

$$X(\lambda) = \frac{\sin^2\lambda a}{\lambda^2} - \pi D_2 \frac{\sin\lambda a\cos\lambda a}{\lambda^3} + 3V\frac{\sin^2\lambda a}{\lambda^4} + \pi^2(3M_1M_2 - D_1D_2)\frac{\cos^2\lambda a}{\lambda^4} + \frac{\tilde{X}(\lambda)}{\lambda^4}, \tag{52}$$

where

$$V = -R_1 - R_2 - R_3 + S,$$
$$R_j = \frac{D_3^2}{8} + \frac{aD_3}{2\pi} + \frac{aD_3M_j}{2} - \frac{a^2E_j}{\pi^2} \quad (j=1,2),$$
$$R_3 = T + \frac{D_3^3}{48} + \frac{D_3^2}{8} + \frac{D_3}{2} + \frac{E_3D_3}{2},$$
$$S = T + \frac{D_3^3}{48} + \frac{D_3^2}{8} + \frac{D_3}{2} + \frac{ED_3}{2}.$$

Substituting $\lambda = \nu_k^{(2)}$ into (44) we obtain

$$X(\nu_k^{(2)}) = \frac{3\phi_1(\nu_k^{(2)})\phi_2(\nu_k^{(2)})\phi_3(\nu_k^{(2)})}{\tau(\nu_k^{(2)})}. \tag{53}$$

Here $\tau(\nu_k^{(2)}) \neq 0$ because of (34). Substituting (32) into (39), (40) and (43) we arrive at

$$\phi_j(\nu_k^{(2)}) = \frac{(-1)^k a^2}{\pi k^2}\left(\frac{D_2}{2} - M_j\right) + \frac{\beta_k}{k^2}, \quad j=1,2, \tag{54}$$

$$\phi_3(\nu_k^{(2)}) = (-1)^k + O(k^{-2}), \tag{55}$$

$$\tau(\nu_k^{(2)}) = (-1)^k + O(k^{-2}). \tag{56}$$

Using (54)–(56) we obtain from (53) that

$$X(\nu_k^{(2)}) = \frac{3\phi_1(\nu_k^{(2)})\phi_2(\nu_k^{(2)})\phi_3(\nu_k^{(2)})}{\tau(\nu_k^{(2)})} = \frac{a^4}{\pi^2k^4}\left(-\frac{D_2^2}{4} - D_1D_2 + 3M_1M_2\right) + \frac{\beta_k}{k^4}. \tag{57}$$

On the other hand, (52) implies

$$X(\nu_k^{(2)}) = \frac{a^4}{\pi^2(k-1/2)^4}\left(-\frac{D_2^2}{4} - D_1D_2 + 3M_1M_2\right) + \frac{\tilde{X}(\nu_k^{(2)})}{k^4}. \tag{58}$$

Comparing (57) with (58) we obtain

$$\{\tilde{X}(\nu_k^{(2)})\}_{-\infty,k\neq 0}^{\infty} \in l_2. \tag{59}$$

Substituting $\lambda = \nu_k^{(3)}$ into (44) we obtain

$$X(\nu_k^{(3)}) = \frac{3\phi_1(\nu_k^{(3)})\phi_2(\nu_k^{(3)})\phi_3(\nu_k^{(3)})}{\tau(\nu_k^{(3)})}. \tag{60}$$

Here $\tau(\nu_k^{(3)}) \neq 0$ because of (34). Since

$$\phi_j(\nu_k^{(3)}) = \frac{(-1)^{k-1}a}{\pi(k-1/2)}\left(1 - \frac{R_j}{k^2}\right) + \frac{\beta_k}{k^3}, \quad j = 1,2,$$

$$\phi_3(\nu_k^{(3)}) = (-1)^k \frac{a}{k-1/2}\left(-\frac{D_3}{2} + B_0 - \frac{R_3}{k^2}\right) + \frac{\beta_k}{k^3}, \tag{61}$$

and

$$\tau(\nu_k^{(3)}) = (-1)^k \frac{a}{k-1/2}\left(\frac{-D_3}{2} + D_1 - \frac{S}{k^2}\right) + \frac{\beta_k}{k^3}$$

we conclude that

$$\begin{aligned}\frac{3\phi_1(\nu_k^{(3)})\phi_2(\nu_k^{(3)})\phi_3(\nu_k^{(3)})}{\tau(\nu_k^{(3)})} &= \frac{3a^2}{\pi^2(k-1/2)^2}\left(\frac{B_0 - \frac{D_3}{2}}{D_1 - \frac{D_3}{2}} + \frac{V}{k^2}\right) + \frac{\beta_k}{k^4} \\ &= \frac{a^2}{\pi^2(k-1/2)^2} + \frac{3V}{k^4} + \frac{\beta_k}{k^4}.\end{aligned} \tag{62}$$

On the other hand, (52) implies

$$X(\nu_k^{(3)}) = \frac{a^2}{\pi^2(k-1/2)^2} + \frac{3V}{k^4} + \frac{\tilde{X}(\nu_k^{(3)})}{k^4}. \tag{63}$$

Substituting (62) into (60) and comparing the result with (63) we obtain

$$\{\tilde{X}(\nu_k^{(3)})\}_{-\infty,k\neq 0}^{\infty} \in l_2$$

and with account of (59)

$$\{\tilde{X}(\nu_k)\}_{-\infty,k\neq 0}^{\infty} \in l_2.$$

Since $\lambda\psi_2(\lambda)\psi_3(\lambda)$ is a sine-type function with simple zeros ($\nu_k^{(j)} = \nu_p^{(i)}$ if and only if $k = p$ and $j = i$) we conclude (see [23], Theorem A or [28], Theorem 11.3.14) that the series

$$\tilde{X}(\lambda) = \lambda\psi_2(\lambda)\psi_3(\lambda) \sum_{-\infty,\ k\neq 0}^{\infty} \frac{\tilde{X}(\nu_k)}{\left.\frac{d\lambda\psi_2(\lambda)\psi_3(\lambda)}{d\lambda}\right|_{\lambda=\nu_k}(\lambda - \nu_k)} \tag{64}$$

converges uniformly to a function from $\mathscr{L}^{2a}$ on any compact domain of the complex plane and on the real axis in the norm of $L_2(-\infty,\infty)$. Here we set $\tilde{X}(0) = 0$ by definition. Substituting (64) into (52) we find $X(\lambda)$.

If we denote by $\{\chi_k\}_{-\infty,k\neq 0}^{\infty}$ the sequence of zeros of X then we conclude that this sequence can be given as the union of two subsequences $\{\chi_k\}_{-\infty,k\neq 0}^{\infty} = \{\chi_k^{(2)}\}_{-\infty,k\neq 0}^{\infty} \cup \{\chi_k^{(3)}\}_{-\infty,k\neq 0}^{\infty}$ where

$$\chi_k^{(j)} = \frac{\pi k}{a} + \frac{T_j}{k} + \frac{\beta_k}{k}, \tag{65}$$

$$T_2 = \frac{\pi}{2}(D_2 + \sqrt{D_2^2 + 4D_1 D_2 - 12M_1 M_2}),$$
$$T_3 = \frac{\pi}{2}(D_2 - \sqrt{D_2^2 + 4D_1 D_2 - 12M_1 M_2})$$

(see Proposition 4.5 in [22]). Here $T_j \in \mathbb{R}$ due (33). The sequences $\{\chi_k\}_{-\infty,k\neq 0}^{\infty} = \{\chi_k^{(2)}\}_{-\infty,k\neq 0}^{\infty} \cup \{\chi_k^{(3)}\}_{-\infty,k\neq 0}^{\infty}$ and $\{\nu_k\}_{-\infty,k\neq 0}^{\infty} = \{\nu_k^{(2)}\}_{-\infty,k\neq 0}^{\infty} \cup \{\nu_k^{(3)}\}_{-\infty,k\neq 0}^{\infty}$ satisfy the condition 1 of Theorem 1 with $\tilde{\nu}_k^{(j)} = \chi_k^{(j)}$, $\tilde{B}_j = T_j$, $\tilde{\rho}_k^{(j)} = \nu_k^{(j)}$, $\tilde{M}_2 = D_2$ and $\tilde{B}_0 = D_3$.

Let us prove that they satisfy condition 2 in Theorem 1. To this end we nitice that by (44) we have

$$X(\nu_k) = \frac{3\phi_1(\nu_k)\phi_2(\nu_k)\phi_3(\nu_k)}{\tau(\nu_k)}$$

where $\tau(\nu_k) \neq 0$ for all k due to (34). If $\nu_k = \xi_p$, then there are $p-k$ squares of zeros of the function $\tau(\lambda)$ and $p-1$ squares of zeros of $\phi_1(\lambda)\phi_2(\lambda)\phi_3(\lambda)$ on the interval $(-\infty, \nu_k^2)$. Since $\tau(\lambda) \underset{\lambda^2\to-\infty}{\to} = +\infty$ and $\phi_j(\lambda) \underset{\lambda^2\to-\infty}{\to} = +\infty$ we conclude that $\tau(\lambda) \underset{\lambda^2\to-\infty}{\to} = +\infty$ and $\phi_1(\lambda)\phi_2(\lambda)\phi_3(\lambda) \underset{\lambda^2\to-\infty}{\to} = +\infty$. Thus, $(-1)^{p-k}\tau(\nu_k) > 0$ and $(-1)^{p-1}\phi_1(\nu_k)\phi_2(\nu_k)\phi_3(\nu_k) > 0$ and, consequently,

$$X(\nu_k)(-1)^{k-1} = \frac{3\phi_1(\nu_k)\phi_2(\nu_k)\phi_3(\nu_k)}{\tau(\nu_k)}(-1)^{k-1} > 0.$$

Therefore, taking into account (52), (28) we conclude that $\{\nu_k\}_{-\infty,k\neq 0}^{\infty}$ interlace with $\{\chi_k\}_{-\infty,k\neq 0}^{\infty}$:

$$\nu_1^2 < \chi_1^2 < \nu_2^2 < \chi_2^2 < \dots$$

Thus, by Theorem 1 there exists a pair q_2, q_3 of real functions from $L_2(0,a)$ and a real constant β such that the spectrum of problem (16)–(19) is $\{\nu_k\}_{-\infty,k\neq 0}^{\infty} = \{\nu_k^{(2)}\}_{-\infty,k\neq 0}^{\infty} \cup \{\nu_k^{(3)}\}_{-\infty,k\neq 0}^{\infty}$ and the spectra of problems (29), (30) are $\{\chi_k^{(2)}\}_{-\infty,k\neq 0}^{\infty}$ and $\{\chi_k^{(3)}\}_{-\infty,k\neq 0}^{\infty}$. For the method of recovering of $\{q_2, q_3, \beta\}$ see the proof of Theorem 8.4.1 in [28], in particular, in our terms $\beta = \pi(2D_3 - T_2 - T_3)$.

We have already seen that obtained q_1 generates problem (14), (15) with the spectrum $\{\mu_k\}_{-\infty,k\neq 0}^{\infty}$. It remains to prove that the spectrum of problem (10)–(13) generated by the found q_1, q_2, q_3 and β coincides with $\{\lambda_k\}_{-\infty,k\neq 0}^{\infty}$. Indeed, for $j = 1,2,3$, let

$S_j(\lambda,x)$ be the solution to (29) with the potential q_j which satisfies $S_j(\lambda,0)=0$ and $S_j'(\lambda,0)=1$. By (20) the characteristic function ϕ of problem (10)–(13) is given by

$$\begin{aligned}\phi(\lambda) := S_1'(\lambda,a)S_2(\lambda,a)S_3(\lambda,a)+S_1(\lambda,a)(S_2'(\lambda,a)S_3(\lambda,a)\\ +S_2(\lambda,a)S_3'(\lambda,a)+\beta S_2(\lambda,a)S_3(\lambda,a)).\end{aligned} \tag{66}$$

We already know that the sequence of zeros of $S_1'(\lambda,a)$ coincides with the sequence of zeros of $\tau(\lambda)$ while the sequence of zeros of $(S_2'(\lambda,a)S_3(\lambda,a)+S_2(\lambda,a)S_3'(\lambda,a)+\beta S_2(\lambda,a)S_3(\lambda,a))$ coincides with the sequence of zeros of $\psi_1(\lambda)\psi_2(\lambda)$. Comparing (23) with (43) we see that $S_1'(\lambda,a)$ and $\tau(\lambda)$ have the same leading terms. Since $S_1'(\lambda,a)$ and $\tau(\lambda)$ are sine type functions with the same set of zeros and the same leading term it follows from Lemma 11.2.29 in [28] that

$$S_1'(\lambda,a)\equiv\tau(\lambda). \tag{67}$$

We have seen that the set of zeros of

$$(S_2'(\lambda,a)S_3(\lambda,a)+S_2(\lambda,a)S_3'(\lambda,a)+\beta S_2(\lambda,a)S_3(\lambda,a))$$

coincides with $\{\nu_k\}_{-\infty,k\neq0}^{\infty}$. Using (21) and (24) we obtain

$$S_2'(\lambda,a)S_3(\lambda,a)+S_2(\lambda,a)S_3'(\lambda,a)+\beta S_2(\lambda,a)S_3(\lambda,a)=\frac{\sin 2\lambda a}{\lambda}-\frac{f(\lambda)}{\lambda}.$$

Comparing (24) with the formula

$$2\psi_2(\lambda)\psi_3(\lambda)=\frac{\sin 2\lambda a}{\lambda}-\frac{f(\lambda)}{\lambda}$$

which is a consequence of (41) and (42) we conclude that the sine type functions $2\lambda\psi_2(\lambda)\psi_3(\lambda)$ and $\lambda(S_1(\lambda,a)(S_2'(\lambda,a)S_3(\lambda,a)+S_2(\lambda,a)S_3'(\lambda,a)+\beta S_2(\lambda,a)S_3(\lambda,a))$ have the same leading terms and the same set of zeros and by Lemma 11.2.29 in [28]

$$S_2'(\lambda,a)S_3(\lambda,a)+S_2(\lambda,a)S_3'(\lambda,a)+\beta S_2(\lambda,a)S_3(\lambda,a)\equiv 2\psi_2(\lambda)\psi_3(\lambda). \tag{68}$$

As we have seen the set of zeros $\{\chi_k\}_{-\infty,k\neq0}^{\infty}$ is the union of the spectra of problems (29), (30) with $j=2$ and with $j=3$, i.e. the union of the sets of zeros of $S_2(\lambda,a)$ and $S_3(\lambda,a)$. Comparing (52) with the representations (22) for $S_2(\lambda,a)$ and $S_3(\lambda,a)$ we see that $X(\lambda)$ and $S_2(\lambda,a)S_3(\lambda,a)$ have the same main term and therefore

$$X(\lambda)=S_2(\lambda,a)S_3(\lambda,a). \tag{69}$$

The comparison of (50) with (22) for $j=1$ implies

$$Y(\lambda)\equiv S_1(\lambda,a). \tag{70}$$

Substituting (67)–(70) into (66) we obtain

$$\phi(\lambda)=\tau(\lambda)X(\lambda)+2\psi_2(\lambda)\psi_3(\lambda)Y(\lambda).$$

Comparing this equation with (44) we arrive at $\phi(\lambda)=3\phi_1(\lambda)\phi_2(\lambda)\phi_3(\lambda)$. $\square$

Comparing Theorem 2 with Lemmas 1–3 and Remark 1 we see that the sufficient conditions of Theorem 2 are close to the necessary conditions.

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