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ON THE OPEN BALL CENTERED AT AN INVERTIBLE ELEMENT OF A BANACH ALGEBRA

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Abstract. Let A be a complex unital Banach algebra. Since the set of invertible elements is open, there is an open ball around every invertible element. In this article, we investigate the Banach algebras for which the radius given by the Neumann series is optimal.

1. Introduction

Let A be a complex unital Banach algebra with unit e. The sets $G(A)$ and $Sing(A)$ denote the set of invertible and singular elements of A respectively. The spectrum, spectral radius and the resolvent of an element a in A, are denoted by $\sigma(a)$, $r(a)$ and $\rho(a)$ respectively.

It is well known that $G(A)$ is open in A, as the open ball centered at any invertible element a with radius $\frac{1}{\|a^{-1}\|}$, denoted by $B\left(a, \frac{1}{\|a^{-1}\|}\right)$, is contained in $G(A)$ ([2] Theorem 2.11). It is a natural to ask if there is a bigger open ball centered at a inside $G(A)$? It will be convenient to have the following definition.

DEFINITION 1. An element $a \in G(A)$ is said to satisfy condition (B) (or belongs to the B class) if the biggest open ball centered at a, contained in $G(A)$, is of radius $\frac{1}{\|a^{-1}\|}$ i.e

$$\overline{B\left(a, \frac{1}{\|a^{-1}\|}\right)} \cap Sing(A) \neq \phi.$$

We say a Banach algebra A satisfies condition (B) if every $a \in G(A)$ satisfies condition (B).

Condition (B) was first encountered by Kulkarni and Sukumar in [7] where Corollary 2.21 says that in a Banach algebra A satisfying condition (B), every member of the ε-condition spectrum of an element a in A (denoted by $\sigma_\varepsilon(a)$), is a spectral value of a perturbed a. Further in [5] Theorem 3.3 states that if A is Banach algebra satisfying condition (B), and $a \in A$, then for every open set Ω containing $\sigma(a)$, there exists

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$0 < \varepsilon < 1$ such that $\sigma_\varepsilon(a) \subset \Omega$. For examples and properties of ε-condition spectrum see [7]. The ε-condition spectrum is a handy tool in the numerical solutions of operator equations. Like in the case where if X is a Banach space and $T : X \to X$ is a bounded linear map, then $\lambda \notin \sigma_\varepsilon(T)$ means that the operator equation $Tx - \lambda x = y$ has a stable solution for every $y \in X$.

A similar but a more particular class of operators on a Hilbert space, called the G_1 class of operators have received considerable attention in literature earlier (See [9] and related references therein).

In this article we investigate the classical Banach algebras that satisfy condition (B), the ones which do not, and discuss their basic properties. The basic approach adopted to show if $a \in G(A)$ satisfies condition (B) is calculating $\|a^{-1}\|$ and producing an element $s \in Sing(A)$ such that $\|a - s\| = \frac{1}{\|a^{-1}\|}$.

Firstly we give a sufficient condition [Theorem 1] for a Banach algebra to satisfy condition (B). Then we characterize all commutative Banach algebras in which condition (B) holds [Theorem 2] and also see that such algebras are isomorphic to a uniform algebra [Theorem 3]. We study linear maps that preserve condition (B) and also the algebras where the same fails to hold. Next we prove that every C*-algebra satisfies condition (B) [Theorem 5]. Later we provide some sufficient conditions for the same to be satisfied in other algebras.

2. Main results

A will denote a complex unital Banach algebra throughout and the fact that for any $a \in A$, $r(a) = \|a\|$ if and only if $\|a^2\| = \|a\|^2$ will be used frequently.

We begin by giving a sufficient condition for an element $a \in G(A)$ to satisfy condition (B).

THEOREM 1. *Let $a \in G(A)$ such that $\|(a^{-1})^2\| = \|a^{-1}\|^2$, then a satisfies condition (B).*

Proof. Since $\|(a^{-1})^2\| = \|a^{-1}\|^2$, by the compactness of spectrum there exists $\lambda_0 \in \sigma(a)$ such that

$$\frac{1}{\|a^{-1}\|} = \frac{1}{r(a^{-1})} = \inf\{|\lambda| : \lambda \in \sigma(a)\} = |\lambda_0|.$$

The element $s = a - \lambda_0 \in A$ can be taken as a singular element in the boundary of $B\left(a, \frac{1}{\|a^{-1}\|}\right)$ with the required property. $\square$

Now we will see that the sufficient condition in Theorem 1 turns out to be necessary for commutative Banach algebras.

THEOREM 2. *Let A be a commutative Banach algebra. Then $a \in G(A)$ satisfies condition (B) if and only if $\|(a^{-1})^2\| = \|a^{-1}\|^2$.*

Proof. If a satisfies (B), there exists $s \in Sing(A)$ such that

$$\|a^{-1}\|^2 = \frac{1}{\|a-s\|^2} \leqslant \frac{1}{\|(a-s)^2\|} = \frac{1}{\|a^2-(sa+as-s^2)\|} \leqslant \|(a^{-1})^2\|,$$

where $sa+as-s^2 \in Sing(A)$ as A is commutative. Thus we have $\|a^{-1}\|^2 = \|(a^{-1})^2\|$. □

COROLLARY 1. *Let A be a finite dimensional Banach algebra that satisfies condition (B). Then A is commutative if and only if $\|a^2\| = \|a\|^2$ for every $a \in A$.*

Proof. The proof follows from the fact that invertible elements are dense in a finite dimensional Banach algebra and Corrolary 15.8 in [2]. □

REMARK 1. The converse of Theorem 1 may not be true if A is non-commutative. For this, we will see later (Theorem 5) that any invertible operator on a Hilbert space satisfies condition (B), but if we take J to be a complex valued invertible matrix such that J^{-1} is a Jordan matrix with $r(J^{-1}) < 1$, then $r(J^{-1}) \neq \|J^{-1}\|$.

REMARK 2. In the commutative case, as (B) is solely dependent on the spectral radius of $a^{-1} \in A$, a will satisfy (B) even in the smallest Banach subalgebra containing e, a and a^{-1}.

REMARK 3. It can be shown that in a commutative unital Banach algebra, the elements satisfying condition (B) form a monoid under multiplication.

Next we see how a commutative unital Banach algebra that satisfies condition (B) is isomorphic to a uniform algebra.

A *uniform algebra*, U is a Banach subalgebra of $C(X)$ with the uniform norm such that U separates the points of X (a compact Hausdorff space) and contains the constants. Since $r(a) = \|a\|$ for every a in a uniform algebra, by Theorem 1, U satisfies condition (B). A (unital) *Banach function algebra* satisfies the same axioms as a uniform algebra except that the complete norm on the algebra need not be the uniform norm. A Banach function algebra may not satisfy condition (B) as seen in the this example.

EXAMPLE 1. Let $C^{(1)}[0,1]$ be the space of all complex valued functions on [0,1] with continuous first order derivative equipped with the norm

$$\|f\| = \|f\|_\infty + \|f^{(1)}\|_\infty \quad \text{for all } f \in C^{(1)}[0,1].$$

Then $(C^{(1)}[0,1], \|.\|)$ is a commutative semi simple Banach function algebra. Consider the function $f(x) = e^x$ for all $x \in [0,1]$ and notice that $\|(f^{-1})^2\| \neq \|f^{-1}\|^2$.

THEOREM 3. *Let A be a commutative Banach algebra that satisfies condition (B), then A is isomorphic to a uniform algebra.*

Proof. As A satisfies condition (B), by Theorem 2 we have $r(a) = \|a\|$ for every $a \in G(A)$, and hence for every $a \in A$

$$r((a-\lambda)^{-1}) = \|(a-\lambda)^{-1}\| \quad (\lambda \in \rho(a))$$

i.e,

$$\|(a-\lambda)^{-1}\| = \frac{1}{dist(\lambda, \sigma(a))} \quad (\lambda \in \rho(a)).$$

Hence from Corollary 3.13 of [6] we get

$$\|a\| \leqslant exp(1)r(a) \text{ for every } a \in A. \tag{1}$$

Let $\mathscr{M}_A$ denote the character space of A, which is a compact Hausdorff space. From equation (1), and by using the Gelfand transform, A is isomorphic onto a subalgebra of $C(\mathscr{M}_A)$ which is closed, point separating and contains the constants. □

The next example shows that the converse of Theorem 3 may not hold.

EXAMPLE 2. Let $A = \mathbb{C}^2$, with coordinate wise multiplication. A with the norm $\|(a,b)\|_\infty = \max\{|a|,|b|\}$, is a uniform algebra. A is also a Banach algebra with the norm $\|(a,b)\|_1 = |a|+|b|$ and $(A, \|.\|_1)$ is isomorphic to $(A, \|.\|_\infty)$. But $(A, \|.\|_1)$ does not satisfy condition (B), as $r(a,b) < \|(a,b)\|_1$ if and only if (a,b) is invertible in $(A, \|.\|_1)$.

The mere definition of condition (B) results into the following theorem.

THEOREM 4. *Let A and B be unital Banach algebras. Let $\phi : A \to B$ be an isometric Banach algebra isomorphism. Then ϕ preserves condition (B).*

In the following example we see that Theorem 4 may not hold if ϕ is not onto.

EXAMPLE 3. Consider the complex field $\mathbb{C}$. Then the identity map is an isometric homomorphism from $\mathbb{C}$ into $(C^{(1)}[0,1], \|.\|)$ but $(C^{(1)}[0,1], \|.\|)$ does not satisfy (B) (See Example 1).

Further if we drop the isometry condition, Theorem 4 may not work.

EXAMPLE 4. Let X be a locally compact Hausdorff space and X^∞ denote the one point compactification of X. Then X^∞ is a compact Hausdorff space (See [4], Chapter 5, Theorem 21). $C(X^\infty)$, being a uniform algebra satisfies condition (B). Let $C_0(X)$ denote the vector space of all continuous functions on X that vanish at infinity. Then $C_0(X)$ is a Banach algebra with point wise multiplication and the sup-norm $\|f\|_X = \sup_{x \in X} |f(x)|$. $C_0(X)$ is unital if and only if X compact. Let $C_0(X)^e$ denote the *unitization* of $C_0(X)$ (See [3] section 2.3). In particular, we will take X to be the open interval $(1,\infty)$. Then by Proposition 16.5 in [2], $\left(\frac{1}{x^2}, 1\right)$ has the inverse $\left(\frac{-1}{1+x^2}, 1\right)$ in $C_0((1,\infty))^e$. But in view of Theorem 2, $\left(\frac{1}{x^2}, 1\right)$ does not satisfy condition (B). Define the map $\psi : C_0((1,\infty))^e \to C((1,\infty)^\infty)$ by

$$\psi(f, \lambda) = f + \lambda e,$$

where $e(x) = 1$ for every $x \in (1,\infty)^\infty$ and each $f \in C_0((1,\infty))$ is extended by assigning zero to the point ∞. It can be proved that ψ is a Banach algebra isomorphism, but not an isometry (See [3] Lemma 2.3.2).

From the next example we see that finite dimensional Banach algebras may fail to satisfy condition (B).

EXAMPLE 5. Consider $L^1(\mathbb{Z}_2) = \{f | f : \mathbb{Z}_2 \longrightarrow \mathbb{C}\}$ with the norm $\|f\| = |f(0)| + |f(1)|$ and multiplication defined as

$$(f * g)(0) = f(0)g(0) + f(1)g(1)$$
$$(f * g)(1) = f(0)g(1) + f(1)g(0).$$

Here the identity element being $(e(0), e(1)) = (1,0)$. From Theorem 2 it is easy to verify that $f = (1,0)$ and $g = (0,i)$ satisfies condition (B) but $f+g$ does not. In particular, as in this case, $\alpha f + \beta e$ may not satisfy condition (B) for some $\alpha, \beta \in \mathbb{C}$.

Now we use polar decomposition of invertible elements in a C*-algebra to prove condition (B) for the same.

THEOREM 5. *Let A be any C*-algebra, then A satisfies condition (B).*

Proof. Let $a \in G(A)$, then a has a unique decomposition $a = bu$ where $b \geqslant 0$ and u is a unitary element in A . Moreover, $b = (aa^*)^{\frac{1}{2}}$ (by Corollary 6.40 in [1]). Continuous functional calculus implies the invertibility of b from the invertibility of a, moreover $b^{-1} \geqslant 0$ and

$$\|b^{-1}\|^2 = \|b^{-1}b^{-1^*}\| = \|a^{-1}a^{-1^*}\| = \|a^{-1}\|^2.$$

Hence $\|b^{-1}\| = \|a^{-1}\|$. Since b^{-1} is self adjoint,

$$\frac{1}{\|b^{-1}\|} = \frac{1}{\sup\{|\lambda| : \lambda \in \sigma(b^{-1})\}} = \{\inf|\lambda| : \lambda \in \sigma(b)\} = |\lambda_0| \text{ say.}$$

As $a - \lambda_0 u = bu - \lambda_0 u = (b - \lambda_0 e)u$, non invertibility of $b - \lambda_0 e$ implies $a - \lambda_0 u$ is not invertible and

$$\|a - (a - \lambda_0 u)\| = \|\lambda_0 u\| = |\lambda_0| = \frac{1}{\|b^{-1}\|} = \frac{1}{\|a^{-1}\|}. \quad \square$$

Let X be a complex Banach space. Recall that $B(X)$ is the algebra of all bounded linear operators on X under product as composition of operators and the operator norm as norm. Since $B(H)$ is a C*- algebra, by Theorem 5, it satisfies condition (B). If we consider a Banach space instead of a Hilbert space, we have a sufficient condition. An operator $T \in B(X)$ is called norm attaining if there exists an element $x \in X$ with $\|x\| = 1$, such that $\|Tx\| = \|T\|$.

THEOREM 6. *Let $T \in G(B(X))$ such that T^{-1} is norm attaining, then T satisfies condition (B).*

Proof. We have that T^{-1} is norm attaining, hence there exist $x, y \in X$, $\|x\| = \|y\| = 1$ such that

$$T^{-1}x = \|T^{-1}\|y.$$

Using Hahn Banach theorem there exist $f \in X^*$ for which $\|f\| = f(y) = 1$. Consider $A \in B(X)$ defined as

$$Au = -\|T^{-1}\|^{-1}f(u)x \text{ for every } u \in X.$$

In particular $Ay = -\|T^{-1}\|^{-1}x$. Using the fact that $T^{-1}x = \|T^{-1}\|y$, we get $\|T^{-1}\|^{-1}x = Ty$ and hence $Ay = -Ty$, implying $0 \in \sigma(T+A)$. Observe that $\|A\| \leqslant \|T^{-1}\|^{-1}$. We want that $\|T-(T+A)\| = \|T^{-1}\|^{-1}$. Suppose not and $\|A\| < \|T^{-1}\|^{-1}$. Then

$$\|A-T+T\| = \|T-(T+A)\| < \|T^{-1}\|^{-1},$$

which implies (T+A) is invertible, a contradiction. Hence $\|A\| = \|T^{-1}\|^{-1}$. □

REMARK 4. The converse of Theorem 6 may not be true. Let H denote the Hilbert space $(\ell^2, \|.\|_2)$ and $\{e_n\}_{n\in\mathbb{N}}$ be the standard orthonormal basis for $(\ell^2, \|.\|_2)$. Consider $T \in B(H)$ defined as $T(e_n) = \left(1 + \frac{1}{(n+1)}\right) e_n$ for all $n \geqslant 1$. Then T is invertible and satisfies condition (B) as H is a Hilbert space, but T^{-1} is not norm attaining.

REMARK 5. If X is finite dimensional, then any $T \in B(X)$ attains its norm, and hence from Theorem 6, $B(X)$ satisfies condition (B).

Recall that if $1 \leqslant p < \infty$ and $\{X_\alpha\}_{\alpha\in\Lambda}$ is a family of Banach spaces, then their ℓ_p-direct sum is the space

$$X = \left\{ x \in \prod_{\alpha\in\Lambda} X_\alpha : \sum_{\alpha\in\Lambda} \|x_\alpha\|^p < \infty \right\}$$

endowed with the norm

$$\|x\| = \left(\sum_{\alpha\in\Lambda} \|x_\alpha\|^p \right)^{\frac{1}{p}}$$

THEOREM 7. *Let X be the ℓ_p direct sum of the family $\{X_\alpha : \alpha \in \lambda\}$ of finite dimensional Banach spaces, $1 < p < \infty$. Then $B(X)$ satisfies condition (B).*

Proof. Let $T \in G(B(X))$, then

$$\frac{1}{\|T^{-1}\|} = \inf\{\|Tx\| : \|x\| = 1\}.$$

From ([10], Lemma 3.4) we have that there exists a $S \in B(X)$ such that $\|S\| \leqslant \frac{1}{\|T^{-1}\|}$ and $\inf\{\|(T+S)x\| : \|x\| = 1\} = 0$ which gives us that $T+S$ is singular, and hence acts as the required singular element on the boundary. □

3. End remarks

REMARK 6. The (B) condition has been dealt with clearly for commutative Banach algebras, C^*-Algebra and some cases of B(X). Further it is required to understand and clearly characterize all Banach algebras that satisfy condition (B) and which do not.

REMARK 7. After observing examples 1, 2, 5 and Theorem 3, it is natural to ask if we can extend Corollary 1 to the following: Let A be a Banach algebra satifying condition (B), then A is commutative iff $\|a^2\| = \|a\|^2$ for every $a \in A$.

REMARK 8. Finally, can we simplify the proof of Theorem 5 in [8], if we add in the hypothesis that the complex commutative Banach algebra satisfies condition (B)?

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