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HYPERCYCLICITY OF WEIGHTED TRANSLATIONS ON ORLICZ SPACES

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Abstract. In this paper, we study the hypercyclicity of the weighted translation $C_{u,g}$ defined on Orlicz space $L^{\Phi}(G)$ where G is a locally compact group, $g \in G$ and u is a weight function on G. It is shown that when $g \in G$ is a torsion element, then $C_{u,g}$ cannot be hypercyclic. However, for an aperiodic element $g \in G$, necessary and sufficient conditions for $C_{u,g}$ and its adjoint are given to be hypercyclic.

1. Introduction and preliminaries

A bounded linear operator T on a Fréchet space X is called *hypercyclic* if there is a vector $x \in X$ whose orbit $\{T^n x : n = 0, 1, 2, \ldots\}$ is dense in X, where T^n stands for the n-th iterate of T and T^0 is the identity map. Such a vector is called a hypercyclic vector for the operator T. We recall the well-known equivalence between hypercyclicity and *topological transitivity*. An operator T acting on a Fréchet space X is hypercyclic if and only if for each pair of no-empty open sets (U,V) in X, there exists an $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. Further, an operator T satisfies the Hypercyclic Criterion if and only if the operator $T \oplus T$ is hypercyclic on $X \oplus X$. An operator T on a Fréchet space X is *weakly mixing* if and only if $T \oplus T$ is hypercyclic on $X \oplus X$. It is readily seen that weakly mixing maps are topologically transitive but in the topological setting, the converse is not true. For example, any irrational rotation of the circle $\mathbb{T}$ is topologically transitive but it is not weakly mixing. An operator T is *topologically mixing* whenever for each pair of no-empty open sets (U,V) in X, there exists an $N \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geqslant N$. The operators of the form "identity plus a backward shift" are the example of topologically mixing operators which are also hypercyclic. The books [2] and [5] are the best interesting references in the dynamics of linear operators.

Let G be a locally compact group with the identity e and a right Haar measure μ. A continuous, even and convex function $\Phi : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ is called a *Young's function* whenever $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$. Usually for each Young's function Φ, another Young's function $\Psi : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ defined by

$$\Psi(y) := \sup\{x|y| - \Phi(x) : \ x \geqslant 0\}$$

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is associated which is called *complementary Young's function* of Φ.

Let $L^{\Phi}(G)$ denote the set of all Borel measurable functions f on G such that $\int_G \Phi(k|f|)d\mu < \infty$ for some constant $k > 0$. It is plain that $L^{\Phi}(G)$ is a vector space and equipped with the norm

$$N_{\Phi}(f) = \inf\left\{k > 0 : \int_G \Phi\Big(\frac{|f|}{k}\Big)d\mu \leqslant 1\right\}$$

which is a Banach space and called an *Orlicz space*. A Young's function Φ is said to satisfy condition Δ_2-regular if there is a constant $k > 0$ such that $\Phi(2t) \leqslant k\Phi(t)$ for large values of t when $\mu(G) < \infty$. In case $\mu(G) = \infty$, $\Phi(2t) \leqslant k\Phi(t)$ for each $t > 0$. For further information the interested reader is referred to [7]. It is well known that the hypercyclic phenomenon is occurred only on infinite-dimensional and separable spaces([2, 5]). Hence we assume that G is second countable and Young's function Φ is Δ_2-regular([7]). A bounded continuous function $u : G \to (0,\infty)$ is called a *weight*. For $g \in G$ let ν_g be the unit point mass at g. Given a weight u on G and $g \in G$, a weighted translation $C_{u,g} : L^{\Phi}(G) \to L^{\Phi}(G)$ is defined by

$$C_{u,g}(f) := u \cdot f * \nu_g \qquad f \in L^{\Phi}(G)$$

where $f * \nu_g$ is the following convolution

$$f * \nu_g(t) := \int_G f(tx^{-1})d\nu_g(x) = f(tg^{-1}) \qquad t \in G.$$

Indeed it is the right translation of f by g^{-1}. Further, it is easy to see that $f * \nu_g \in L^{\Phi}(G)$ whenever $f \in L^{\Phi}(G)$. For if, consider

$$\int_G \Phi(k|f * \nu_g(t)|)d\mu(t) = \int_G \Phi(k|f(tg^{-1})|)d\mu(t) = \int_G \Phi(k|f(y)|)d\mu(y) < \infty$$

where $tg^{-1} = y$ and $d\mu(t) = d\mu(yg) = d\mu(y)$.

Since the spectrum of hypercyclic operators meets the unit circle ([2] or [5]), then a weighted translation $C_{u,g}$ cannot be hypercyclic when $\|u\|_\infty \leqslant 1$. Another case which $C_{u,g}$ cannot be hypercyclic, appears whenever g is a torsion element. Recall that an element $g \in G$ is called a *torsion element* if it is of finite order. An element $g \in G$ is called *periodic* if the closed subgroup $G(g)$ generated by g is compact. Further, an element in G is *aperiodic* if it is not periodic. The hypercyclicity of the weighted translations on $L^p(G)$ for $1 \leqslant p < \infty$ has been widely studied in [3] and [4]. In this paper, we study the hypercyclicity of the weighted translation $C_{u,g}$ on Orlicz space $L^{\Phi}(G)$. For an aperiodic element $g \in G$, we give a necessary and sufficient condition for $C_{u,g}$ to be hypercyclic. Moreover, it is shown that when $g \in G$ is a torsion element then $C_{u,g}$ cannot be hypercyclic.

2. Hypercyclicity of weighted translations On $L^{\Phi}(G)$

One of the hypercyclicity criteria is the following which is known as Kitai's hypercyclicity criterion.

DEFINITION 2.1. ([6]) Let X be a topological vector space and $T : X \to X$ be a bounded linear operator. We say that T satisfies the *hypercyclicity criterion* if there exist an increasing sequence of integers (n_k), two dense sets $D_1, D_2 \subset X$ and a sequence of maps $S_{n_k} : D_2 \to X$ (not necessarily linear or continuous) such that

- $T^{n_k}(x) \to 0$ for any $x \in D_1$;
- $S_{n_k}(y) \to 0$ for any $y \in D_2$;
- $T^{n_k} S_{n_k}(y) \to 0$ for any $y \in D_2$.

For the possible setting, $n_k = k$ and $D_1 = D_2$, it is called *Kitai's hypercyclicity criterion.*

In this section, we characterize the hypercyclicity of the weighted translation $C_{u,g}$ when $g \in G$ is torsion and aperiodic. For an aperiodic, a given necessary and sufficient condition is proved by Kitai's hypercyclicity criterion.

LEMMA 2.2. *Let $g \in G$ be a torsion element. Then a weighted translation $C_{u,g}$: $L^{\Phi}(G) \to L^{\Phi}(G)$ is not hypercyclic.*

Proof. The method of proof is similar to the one used in [3]. Let $m \in \mathbb{N}$ be the order of the element g i.e., $g^m = e$. For each $t \in G$, let $u_{m,g}(t) := \prod_{i=0}^{m-1} u(tg^{-i})$ where $g^0 = e$. We shall proceed the proof with the two cases $\|u_{m,g}\|_\infty \leqslant 1$ and $\|u_{m,g}\|_\infty > 1$. The first case proceeds along the same lines as the proof of Lemma 1.1 in [3]. The orbit of $C_{u,g}$ at $L^{\Phi}(G)$ may appear like

$$\begin{gathered}\{f, C_{u,g}(f), C_{u,g}^2(f), \ldots, C_{u,g}^{m-1}(f),\\ u_{m,g}f, u_{m,g}C_{u,g}(f), u_{m,g}C_{u,g}^2(f), \ldots, u_{m,g}C_{u,g}^{m-1}(f),\\ u_{m,g}^2 f, u_{m,g}^2 C_{u,g}(f), u_{m,g}^2 C_{u,g}^2(f), \ldots, u_{m,g}^2 C_{u,g}^{m-1}(f),\\ \vdots\\ \}.\end{gathered}$$

Indeed, because of $\|u_{m,g}\|_\infty \leqslant 1$, it is clear that the orbit of the weighted translation $C_{u,g}$ is bounded and hence it cannot be dense in $L^{\Phi}(G)$.

For the case $\|u_{m,g}\|_\infty > 1$, suppose on contrary that $C_{u,g}$ is hypercyclic. Then one may readily find a compact subset $K \subseteq G$ and an $\varepsilon > 0$ such that $\mu(K) > \frac{2}{\Phi(\frac{1}{\varepsilon})}$. Moreover we may assume that $u(x) > 1$ for all $x \in K$, since u is continuous. The hypercyclicity of $C_{u,g}$ guaranties the hypercyclicity of the its m-th iterate, say $C_{u,g}^m$. To see this well-known fact consult, [1]. Let χ_K be the characteristic function of K.

Clearly $\chi_K \in L^\Phi(G)$, since $N_\Phi(\chi_K) = \frac{1}{\Phi^{-1}(\frac{1}{\mu(K)})}$ and μ is a regular measure. Recall that for a Young's function Φ, $\Phi^{-1} : [0,+\infty) \to [0,+\infty]$ is defined by $\Phi^{-1}(y) := \inf\{x \geqslant 0 : \Phi(x) > y\}$ with $\inf(\emptyset) = +\infty$.

That a Young's function Φ is assumed to be Δ_2-regular, ensures that the set of all simple functions and the set of all continuous functions with the compact supports are dense in Orlicz space $L^\Phi(G)$ (c.f., [7]). Hence, one may find $f \in L^\Phi(G)$ and $n \in \mathbb{N}$, sufficiently large such that

$$N_\Phi(f - 2\chi_K) < \varepsilon \quad \text{and} \quad N_\Phi((C^m_{u,g})^n f) < \varepsilon.$$

Set $S = \{t \in K : |f(t)| < 1\}$. Then

$$\begin{aligned} \varepsilon > N_\Phi(f - 2\chi_K) &\geqslant N_\Phi(\chi_S(f - 2\chi_K)) \\ &\geqslant N_\Phi(\chi_S) \\ &= \frac{1}{\Phi^{-1}(\frac{1}{\mu(S)})}. \end{aligned}$$

Therefore, we have $\mu(S) < \frac{1}{\Phi(\frac{1}{\varepsilon})}$. On the other hand,

$$\begin{aligned} \varepsilon > N_\Phi((C^m_{u,g})^n f) &\geqslant N_\Phi(\chi_{K-S}(C^m_{u,g})^n f) \\ &\geqslant N_\Phi((u^n_{m,g} f)\chi_{K-S}) \\ &\geqslant N_\Phi(\chi_{K-S}) \\ &= \frac{1}{\Phi^{-1}(\frac{1}{\mu(K-S)})}. \end{aligned}$$

Similarly, we obtain that $\mu(K-S) < \frac{1}{\Phi(\frac{1}{\varepsilon})}$. But we know that $\mu(K) = \mu(S) + \mu(K - S) < \frac{2}{\Phi(\frac{1}{\varepsilon})}$ which is a contradiction. $\square$

THEOREM 2.3. *Let $g \in G$ be an aperiodic element and let $C_{u,g}$ be a weighted translation on $L^\Phi(G)$. Then the following conditions are equivalent:*

(i) $C_{u,g}$ is hypercyclic.

(ii) For each compact subset $K \subseteq G$ with $\mu(K) > 0$, there is a sequence of Borel sets $\{V_k\} \subseteq K$ such that $\mu(V_k) \to \mu(K)$ as $k \to \infty$ and both sequences

$$u_{n,g} := (\prod_{i=0}^{n-1} u * \nu^i_g)^{-1} \quad \text{and} \quad u_{n,g^{-1}} := \prod_{i=1}^{n} u * \nu^i_{g^{-1}}$$

possess respectively subsequences $\{u_{n_k,g}\}_{k=1}^\infty$ and $\{u_{n_k,g^{-1}}\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} \|u_{n_k,g}|_{V_k}\|_\infty = \lim_{k\to\infty} \|u_{n_k,g^{-1}}|_{V_k}\|_\infty = 0.$$

Proof. We take the same approach as in [3]. However, the novelty of our approach lies on the structure of Orlicz spaces and the hypercyclicity criterion (Definition 2.1). Suppose that $C_{u,g}$ is hypercyclic. Let $K \subseteq G$ be a compact set with $\mu(K) > 0$ and let $\varepsilon > 0$. Since $g \in G$ is an aperiodic element, then by Lemma 2.1 in [3], there exists an $N \in \mathbb{N}$ such that $K \cap Kg^{-n} = \emptyset$ for $n > N$. We know that the set of all hypercyclic vectors for $C_{u,g}$ and the set of all simple functions form dense subsets in $L^{\Phi}(G)$. Of course, both these facts depend on Young's function Φ which is assumed to be Δ_2-regular. Hence there exist a hypercyclic vector $f \in L^{\Phi}(G)$ and $n_0 \in \mathbb{N}$, $n_0 > N$, such that

$$N_\Phi(f - \chi_K) < \varepsilon_1^2 \quad \text{and} \quad N_\Phi(C_{u,g}^{n_0} f - \chi_K) < \varepsilon_1^2$$

where ε_1 is chosen in such a way that $0 < \varepsilon_1 < \frac{\varepsilon}{1+\varepsilon}$. Put $P_{\varepsilon_1} = \{t \in K : |f(t) - 1| \geqslant \varepsilon_1\}$. Now note that

$$\begin{aligned}
\varepsilon_1^2 &> N_\Phi(f - \chi_K) \\
&\geqslant N_\Phi(\chi_K(f-1)) \\
&\geqslant N_\Phi(\chi_{P_{\varepsilon_1}}(f-1)) \\
&\geqslant N_\Phi(\chi_{P_{\varepsilon_1}} \varepsilon_1) \\
&= \frac{\varepsilon_1}{\Phi^{-1}(\frac{1}{\mu(P_{\varepsilon_1})})}.
\end{aligned}$$

Then $\Phi^{-1}(\frac{1}{\mu(P_{\varepsilon_1})}) > \frac{1}{\varepsilon_1}$ and so $\frac{1}{\mu(P_{\varepsilon_1})} > \Phi(\frac{1}{\varepsilon_1})$ which yields that $\mu(P_{\varepsilon_1}) < \frac{1}{\Phi(\frac{1}{\varepsilon_1})}$. Let $R_{\varepsilon_1} = \{t \in G - K : |f(t)| \geqslant \varepsilon_1\}$. Then $\mu(R_{\varepsilon_1}) < \frac{1}{\Phi(\frac{1}{\varepsilon_1})}$ since

$$\begin{aligned}
\varepsilon_1^2 &> N_\Phi(f - \chi_K) \\
&\geqslant N_\Phi(\chi_{G-K} f) \\
&\geqslant N_\Phi(\chi_{R_{\varepsilon_1}} f) \\
&\geqslant N_\Phi(\chi_{R_{\varepsilon_1}} \varepsilon_1) \\
&= \frac{\varepsilon_1}{\Phi^{-1}(\frac{1}{\mu(R_{\varepsilon_1})})}.
\end{aligned}$$

Let $S_{n_0,\varepsilon_1} = \{t \in K : |u_{n_0,g}(t)^{-1} f(tg^{-n_0}) - 1| \geqslant \varepsilon_1\}$. Then, consider the following

$$\begin{aligned}
\varepsilon_1^2 &> N_\Phi(C_{u,g}^{n_0} f - \chi_K) \\
&\geqslant N_\Phi(\chi_{S_{n_0,\varepsilon_1}}(C_{u,g}^{n_0} f - \chi_K)) \\
&= \inf\Big\{k > 0 : \int_{S_{n_0,\varepsilon_1}} \Phi\Big(\frac{1}{k}|u_{n_0,g}(t)^{-1} f(tg^{-n_0}) - \chi_K(t)|\Big) d\mu(t) \leqslant 1\Big\} \\
&\geqslant N_\Phi(\varepsilon_1 \chi_{S_{n_0,\varepsilon_1}}) \\
&= \varepsilon_1 \frac{1}{\Phi^{-1}(\frac{1}{\mu(S_{n_0,\varepsilon_1})})}
\end{aligned}$$

to deduce that $\mu(S_{n_0,\varepsilon_1}) < \frac{1}{\Phi(\frac{1}{\varepsilon_1})}$. But for each $t \in K - (S_{n_0,\varepsilon_1} \cup R_{\varepsilon_1} g^{n_0})$, we have

$$u_{n_0,g}(t) < \frac{|f(tg^{-n_0})|}{1-\varepsilon_1} < \frac{\varepsilon_1}{1-\varepsilon_1} < \varepsilon,$$

since $K \cap Kg^{n_0} = \emptyset$. Let $U_{n_0,\varepsilon_1} = \{t \in K : |u_{n_0,g^{-1}}(t)f(t)| \geqslant \varepsilon_1\}$. Again, by the assumption $K \cap Kg^{n_0} = \emptyset$ and the fact that μ is a right invariant Haar measure, we have

$$\begin{aligned}
\varepsilon_1^2 &> N_\Phi(C_{u,g}^{n_0} f - \chi_K) \\
&= \inf\Big\{k > 0 : \int_G \Phi\Big(\frac{1}{k}|u_{n_0,g}(t)^{-1} f(tg^{-n_0}) - \chi_K(t)|\Big) d\mu(t) \leqslant 1\Big\} \\
&= \inf\Big\{k > 0 : \int_G \Phi\Big(\frac{1}{k}|u_{n_0,g^{-1}}(t) f(t) - \chi_K(tg^{n_0})|\Big) d\mu(t) \leqslant 1\Big\} \\
&\geqslant \inf\Big\{k > 0 : \int_{U_{n_0,\varepsilon_1}} \Phi\Big(\frac{1}{k}|u_{n_0,g^{-1}}(t) f(t) - \chi_K(tg^{n_0})|\Big) d\mu(t) \leqslant 1\Big\} \\
&= \inf\Big\{k > 0 : \int_{U_{n_0,\varepsilon_1}} \Phi\Big(\frac{1}{k}|u_{n_0,g^{-1}}(t) f(t)|\Big) d\mu(t) \leqslant 1\Big\} \\
&= N_\Phi(\chi_{U_{n_0,\varepsilon_1}} u_{n_0,g^{-1}} f) \\
&\geqslant \varepsilon_1 N_\Phi(\chi_{U_{n_0,\varepsilon_1}}) \\
&= \varepsilon_1 \frac{1}{\Phi^{-1}(\frac{1}{\mu(U_{n_0,\varepsilon_1})})},
\end{aligned}$$

which implies in turn that $\mu(U_{n_0,\varepsilon_1}) < \frac{1}{\Phi(\frac{1}{\varepsilon_1})}$. Note that for each $t \in K - (U_{n_0,\varepsilon_1} \cup P_{\varepsilon_1})$, we have

$$u_{n_0,g^{-1}}(t) < \frac{\varepsilon_1}{|f(t)|} < \frac{\varepsilon_1}{1-\varepsilon_1} < \varepsilon.$$

Eventually, define $V_{n_0,\varepsilon_1} := K - (P_{\varepsilon_1} \cup R_{n_0,\varepsilon_1} \cup S_{n_0,\varepsilon_1} \cup U_{n_0,\varepsilon_1})$. It is evident that, $\mu(K - V_{n_0,\varepsilon_1}) < \frac{4}{\Phi(\frac{1}{\varepsilon_1})}$, $\|u_{n_0,g^{-1}}|_{V_{n_0,\varepsilon_1}}\|_\infty < \varepsilon$ and $\|u_{n_0,g}|_{V_{n_0,\varepsilon_1}}\|_\infty < \varepsilon$.

Proceeding inductively, for each $k \in \mathbb{N}$ there is a Borel set $V_k \subseteq K$ and $n_1 < n_2 < \ldots < n_k < \ldots$ such that $\mu(K - V_k) < \frac{4}{\Phi(\frac{1}{k})}$, $\|u_{n_k,g^{-1}}|_{V_k}\|_\infty < \frac{1}{k}$ and $\|u_{n_k,g}|_{V_k}\|_\infty < \frac{1}{k}$.

For the reverse implication, we use Kitai's hypercyclicity criterion (Definition 2.1) essentially. Let $\{V_k\} \subseteq K$, $\{u_{n_k,g}\}$ and $\{u_{n_k,g^{-1}}\}$ be items satisfying condition (ii). We use the fact that the set of all continuous functions with compact supports say $C_c(G)$, is dense in $L^\Phi(G)$, since Young's function Φ is assumed to be Δ_2-regular. For more details see [7]. We mean the support of a function f by the set $\overline{\{t \in G : f(t) \neq 0\}}$ which is denoted by $\sigma(f)$, for simplicity. Take $D_1 = D_2 = C_c(G)$ and define the maps $S_{n_k,g} : C_c(G) \to L^\Phi(G)$ by

$$S_{n_k,g}(f) := u_{n_k,g} f * \nu_{g^{-1}}.$$

In this circumstance, we have $C_{u,g}^{n_k}(S_{n_k,g}(f)) = f$. It remains to show that $N_\Phi(C_{u,g}^{n_k} f) \to 0$ and $N_\Phi(S_{n_k,g}(f)) \to 0$ as $k \to \infty$. Let $\varepsilon > 0$ and let $\{u_{n_k,g^{-1}}\}$ be bounded on $\sigma(f)$ by M. By the hypothesis, there exists an $N \in \mathbb{N}$ such that $\mu(\sigma(f) - V_N) < \frac{\varepsilon}{MN_\Phi(f)}$. Now, by Egoroff's theorem, there is a Borel set $W_N \subseteq \sigma(f)$ such that $\mu(W_N - \sigma(f)) < \frac{\varepsilon}{MN_\Phi(f)}$ and $\{u_{n_k,g^{-1}}\}$ converges to 0 uniformly on W_N. Hence, there exists an $\acute{N} \in \mathbb{N}$ such that for each $n_k > \acute{N}$, $u_{n_k,g^{-1}} < \frac{\varepsilon}{N_\Phi(f)}$ on W_N. Now, by the change of variable formula, for $n_k > \acute{N}$ we have

$$
\begin{aligned}
&N_\Phi(C_{u,g}^{n_k} f) = N_\Phi(C_{u,g}^{n_k} f \chi_{\sigma(f)}) \\
&= \inf\Big\{k > 0 : \int_{\sigma(f)g^{n_k}} \Phi\Big(\frac{1}{k}|u(t)u(tg^{-1})\dots u(tg^{-(n_k-1)})f(tg^{-n_k})|\Big)\,d\mu(t) \leqslant 1\Big\} \\
&= \inf\Big\{k > 0 : \int_{\sigma(f)} \Phi\Big(\frac{1}{k}|u(tg^{n_k})u(tg^{(n_k-1)})\dots u(tg)f(t)|\Big)\,d\mu(t) \leqslant 1\Big\} \\
&\leqslant N_\Phi(u_{n_k,g^{-1}} f \chi_{W_N}) + N_\Phi(u_{n_k,g^{-1}} f \chi_{\sigma(f)-W_N}) \\
&< \frac{\varepsilon}{N_\Phi(f)} N_\Phi(f) + \frac{2\varepsilon}{MN_\Phi(f)} MN_\Phi(f) \\
&= 3\varepsilon.
\end{aligned}
$$

By repeating the similar method for the sequence $\{u_{n_k,g}\}$, one may obtain that $N_\Phi(S_{n_k,g}(f)) < 3\varepsilon$ and the proof is completed. $\square$

PROPOSITION 2.4. *Let $g \in G$ be an aperiodic element and let $C_{u,g}$ be a weighted translation on $L^\Phi(G)$. Then the following conditions are equivalent:*

(i) *$C_{u,g}$ satisfies the Hypercyclic Criterion.*

(ii) *$C_{u,g}$ is hypercyclic.*

(iii) *$C_{u,g} \oplus C_{u,g}$ is hypercyclic.*

(iv) *$C_{u,g}$ is weakly mixing.*

Proof. We only prove the implication $(ii) \Rightarrow (iii)$. In fact, the condition (ii) in Theorem 2.3 implies that $C_{u,g} \oplus C_{u,g}$ is topologically transitive. For if, consider two pairs of non-empty open sets (U_1, V_1) and (U_2, V_2) in $L^\Phi(G)$. Choose the functions $f_i, h_i \in C_c(G)$ with $f_i \in U_i$ and $h_i \in V_i$ (i=1,2). Let $K = \sigma(f_1) \cup \sigma(f_2) \cup \sigma(g_1) \cup \sigma(g_2)$ be a compact set in G. Let $\{V_k\} \subseteq K$, $\{u_{n_k,g}\}_{k=1}^\infty$ and $\{u_{n_k,g^{-1}}\}_{k=1}^\infty$ be satisfied the condition (ii) in Theorem 2.3. There exists an $N_1 \in \mathbb{N}$, such that for all $n > N_1$, $K \cap Kg^{\pm n} = \emptyset$ since g is aperiodic. Moreover, for each $\varepsilon > 0$ there exists $N_2 \in \mathbb{N}$ such that for each $k > N_2$ and $n_k > N_1$, $u_{n_k,g^{-1}} < \frac{\varepsilon}{N_\Phi(f_i)}$ on V_k. Hence, for $k > N_2$, by the change of variable formula we have

$$\begin{aligned}
&N_\Phi(C^{n_k}_{u,g} f_i \chi_{V_k})\\
&= \inf\Big\{k>0 : \int_{V_k g^{n_k}} \Phi\Big(\frac{1}{k}|u(t)u(tg^{-1})\ldots u(tg^{-(n_k-1)}) f_i(tg^{-n_k})|\Big)d\mu(t) \leqslant 1\Big\}\\
&= \inf\Big\{k>0 : \int_{V_k} \Phi\Big(\frac{1}{k}|u(tg^{n_k})u(tg^{-(n_k-1)})\ldots u(tg) f_i(t)|\Big)d\mu(t) \leqslant 1\Big\}\\
&= N_\Phi(u_{n_k,g^{-1}} f_i \chi_{V_k})\\
&< \varepsilon.
\end{aligned}$$

Now define a map $D_{u,g}$ on the subspace $L^\Phi_c(G)$ of functions in $L^\Phi(G)$ with compact support by $D_{u,g}(f) := \frac{f}{u} * \nu_{g^{-1}}$. Then for each $f \in L^\Phi_c(G)$, $C_{u,g}D_{u,g}(f) = f$. Again, there exists $N_3 \in \mathbb{N}$ such that for each $k > N_3$ and $n_k > N_1$ such that $u_{n_k,g} < \frac{\varepsilon}{N_\Phi(h_i)}$ on V_k. For $k > N_3$ note that

$$\begin{aligned}
&N_\Phi(D^{n_k}_{u,g} h_i \chi_{V_k})\\
&= \inf\Big\{k>0 : \int_{V_k g^{-n_k}} \Phi\Big(\frac{1}{k|u(tg)\ldots u(tg^{n_k})|}|h_i(tg^{n_k})|\Big)d\mu(t) \leqslant 1\Big\}\\
&= \inf\Big\{k>0 : \int_{V_k} \Phi\Big(\frac{1}{k|u(tg^{-(n_k-1)})u(tg^{-(n_k-2)})\ldots u(t)|}|h_i(t)|\Big)d\mu(t) \leqslant 1\Big\}\\
&= N_\Phi(u_{n_k,g} h_i \chi_{V_k})\\
&< \varepsilon.
\end{aligned}$$

For each $k \in \mathbb{N}$, let

$$\rho_{i,k} = f_i \chi_{V_k} + D^{n_k}_{u,g} h_i \chi_{V_k}.$$

Clearly $\rho_{i,k} \in L^\Phi(G)$,

$$N_\Phi(\rho_{i,k} - f) \leqslant N_\Phi(f_i)\mu(K - V_k) + N_\Phi(D^{n_k}_{u,g} h_i \chi_{V_k})$$

and

$$N_\Phi(C^{n_k}_{u,g}\rho_{i,k} - h_i) \leqslant N_\Phi(h_i)\mu(K - V_k) + N_\Phi(C^{n_k}_{u,g} f_i \chi_{V_k}).$$

Hence $\lim_{k\to\infty} \rho_{i,k} = f_i$, $\lim_{k\to\infty} C^{n_k}_{u,g}\rho_{i,k} = h_i$ and $C^{n_k}_{u,g}(U_i) \cap V_i \neq \emptyset$ for some $k \in \mathbb{N}$. □

COROLLARY 2.5. *Let $g \in G$ be an aperiodic element and let $C_{u,g}$ be a weighted translation on $L^\Phi(G)$. Then the following conditions are equivalent:*

(i) $C_{u,g}$ is topologically mixing.

(ii) For each compact subset $K \subseteq G$ with $\mu(K) > 0$, there is a sequence of Borel sets $\{V_n\} \subseteq K$ such that $\mu(V_n) \to \mu(K)$ as $n \to \infty$ and both sequences

$$u_{n,g} = \Big(\prod_{i=0}^{n-1} u * \nu^i_g\Big)^{-1} \quad \textit{and} \quad u_{n,g^{-1}} = \prod_{i=1}^{n} u * \nu^i_{g^{-1}}$$

satisfy

$$\lim_{n\to\infty} \|u_{n,g}|_{V_n}\|_\infty = \lim_{n\to\infty} \|u_{n,g^{-1}}|_{V_n}\|_\infty = 0.$$

Proof. Using the full sequences $\{u_{n,g}\}$ and $\{u_{n,g^{-1}}\}$ instead of subsequences, the implication $(ii) \Rightarrow (i)$ holds by Theorem 2.3. Indeed, we have used the fact that an operator on a separable F-space satisfying the hypercyclicity criterion with respect to the full sequence (n), is in turn topologically mixing [2]. For the reverse implication, let $K \subseteq G$ be compact with $\mu(K) > 0$, $\varepsilon > 0$ and $\chi_K \in L^\Phi(G)$ be the characteristic function. Take $U = \{f \in L^\Phi(G) : N_\Phi(f - \chi_K) < \varepsilon\}$ which is a non-empty open subset. Since $C_{u,g}$ is assumed to be topologically mixing and g is an aperiodic element, one may find $N \in \mathbb{N}$ such that for all $n > N$, $C^n_{u,g}(U) \cap U \neq \emptyset$ and $K \cap Kg^n = \emptyset$ (c.f. Lemma 2.1 in [3]) hold simultaneously. Hence for each $n > N$, we can choose a function $f_n \in U$ meanwhile $C^n_{u,g} f_n \in U$. Then $N_\Phi(f - \chi_K) < \varepsilon$ and $N_\Phi(C^n_{u,g} f_n - \chi_K) < \varepsilon$. Now, the rest of proof can be proceed by the similar arguments used in the proof of Theorem 2.3. □

PROPOSITION 2.6. *Let $g \in G$ be an aperiodic element and let $C^*_{u,g} : L^\Psi(G) \to L^\Psi(G)$ be the adjoint of a weighted translation $C_{u,g}$ on $L^\Phi(G)$ provided that Ψ is assumed to be Δ_2-regular. Then $C^*_{u,g}$ is hypercyclic if and only if for each compact subset $K \subseteq G$ with $\mu(K) > 0$, there is a sequence of Borel sets $\{V_k\} \subseteq K$ such that $\mu(V_k) \to \mu(K)$ as $k \to \infty$ and both sequences*

$$d_{n,g^{-1}} := \Big(\prod_{i=1}^{n} u * \nu^i_{g^{-1}}\Big)^{-1} \quad \text{and} \quad d_{n,g} := \prod_{i=0}^{n-1} u * \nu^i_g$$

possess respectively subsequences $\{d_{n_k,g^{-1}}\}_{k=1}^\infty$ and $\{d_{n_k,g}\}_{k=1}^\infty$ such that

$$\lim_{k\to\infty} \|d_{n_k,g^{-1}}|_{V_k}\|_\infty = \lim_{k\to\infty} \|d_{n_k,g}|_{V_k}\|_\infty = 0.$$

Proof. Let $\langle \cdot, \cdot \rangle : L^\Phi(G) \times L^\Psi(G) \to \mathbb{C}$ be the duality defined by $\langle h, f \rangle = \int_G hf\, d\mu$, for any $h \in L^\Phi(G)$ and $f \in L^\Psi(G)$ ([7, Corollary 4.1.9]). Now, consider the following computations

$$\begin{aligned}
\langle h, C^*_{u,g} f \rangle &= \langle C_{u,g} h, f \rangle = \langle uh * \nu_g, f \rangle \\
&= \int_G u(t) h(tg^{-1}) f(t)\, d\mu(t) \\
&= \int_G u(tg) h(t) f(tg)\, d\mu(tg) \\
&= \int_G h(t) u * \nu_{g^{-1}}(t) f * \nu_{g^{-1}}(t)\, d\mu(t) \\
&= \langle h, u * \nu_{g^{-1}} \cdot f * \nu_{g^{-1}} \rangle.
\end{aligned}$$

Therefore, the adjoint of $C_{u,g}$ is obtained by

$$C^*_{u,g} f = u * \nu_{g^{-1}} \cdot f * \nu_{g^{-1}},$$

which is again a weighted translation. Moreover, one may easily check that

$$C_{u,g}^{*^n} f = [\prod_{i=1}^{n} u * v_{g^{-1}}^{i}] f * v_{g^{-1}}^{n}.$$

Hence, by scrutinizing the proof of Theorem 2.3, it is inferred that $C_{u,g}^{*}$ is hypercyclic if the sequences $(\prod_{i=1}^{n} u * v_{g^{-1}}^{i})^{-1}$ and $\prod_{i=0}^{n-1} u * v_{g}^{i}$ satisfy condition (ii) of Theorem 2.3. $\square$

REMARK 2.7. In fact, $C_{u,g}^{*}$ is hypercyclic if the weight function $u * v_{g^{-1}}$ satisfies that condition for g^{-1} while $C_{u,g}$ is hypercyclic whenever the weight function u satisfies so for g. However, the hypercyclicity of $C_{u,g}^{*}$ and $C_{u,g}$ can be coincided in some senses. As a specific example, one may consider the bilateral weighted shift on $\mathbb{Z}$, the group of all integer numbers which is due to H. N. Salas [8].

EXAMPLE 2.8. *Consider the following Young's functions*

$$\begin{aligned} \Phi_1(t) &= (e+|t|)\ln(e+|t|) - e, \\ \Phi_2(t) &= |t|^{\alpha}(1+|\log|t||) \quad \alpha > 1, \\ \Phi_3(t) &= |t|^{\alpha}\ln^{\beta}(|t|+e) \quad \alpha > 1,\ \beta \geqslant 1, \end{aligned}$$

where e is Napier's constant. It is not so hard to check that all three mentioned functions are Δ_2-regular. Especially Ψ_2 and Ψ_3, the complementary of Φ_2 and Φ_3 respectively, are also Δ_2-regular. Define the weight function u on $G = \mathbb{R}$ by

$$u(t) = \begin{cases} \frac{1}{2}, & 1 \leqslant t, \\ -\frac{t}{2}+1, & -1 \leqslant t \leqslant 1, \\ \frac{3}{2}, & t \leqslant -1. \end{cases}$$

Let $K = [a,b]$. Take $V_k = [a, b-\frac{1}{k})$. For $g > 0$, choose $k_0 \in \mathbb{N}$ such that $a + n_0 g > 1$. Then for each $k \geqslant k_0$ and $t \in V_k$ we have

$$\begin{aligned} 0 < u_{k,g^{-1}}(t) &= u(t+g)u(t+2g)\cdots u(t+kg) \\ &\leqslant u(a+g)u(a+2g)\cdots u(a+k_0 g) \\ &\leqslant M, \end{aligned}$$

where M is a constant independent of k. Moreover, note that for each $t \in V_k$ and $q \geqslant k_0 g$, we have $u(t+q) = \frac{1}{2}$. Hence for $k \geqslant k_0$,

$$\begin{aligned} u_{k,g^{-1}}(t) &= u(a+g)u(a+2g)\cdots u(a+k_0 g)u(a+(k_0+1)g)\cdots u(a+kg) \\ &\leqslant M(\frac{1}{2})^{k-k_0} \to 0 \ \text{ as } \ k \to \infty. \end{aligned}$$

The same argument can be applied to the sequence $\{u_{k,g}\}_{k=1}^{\infty}$ convincing that the condition (ii) of Theorem 2.3 is established and hence $C_{u,g}$ is hypercyclic.

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REFERENCES

[1] S. I. ANSARI, *Hypercyclic and cyclic vectors*, J. Funct. Anal. **128** (1995), no. 2, 374–383.
[2] F. BAYART, É. MATHERON, *Dynamics of linear operators*, Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2009.
[3] C. CHEN, C.-H. CHU, *Hypercyclic weighted translations on groups*, Proc. Amer. Math. Soc. **139** (2011), no. 8, 2839–2846.
[4] C. CHEN, C.-H. CHU, *Hypercyclicity of weighted convolution operators on homogeneous spaces*, Proc. Amer. Math. Soc. **137** (2009), no. 8, 2709–2718.
[5] K.-G. GROSSE-ERDMANN, A. P. MANGUILLOT, *Linear chaos*, Universitext, Springer, London, 2011.
[6] C. KITAI, *Invariant closed sets for linear opeartors*, Thesis (Ph. D.), University of Toronto (Canada), 1982.
[7] M. M. RAO, Z. D. REN, *Theory of Orlicz spaces*, Monographs and Textbooks in Pure and Applied Mathematics, **146**, Marcel Dekker, Inc., New York, 1991.
[8] H. N. SALAS, *Hypercyclic weighted shifts*, Trans. Amer. Math. Soc., **347** (3) (1995) 993–1004.

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