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ESTIMATES ON SINGULAR VALUES OF FUNCTIONS OF PERTURBED OPERATORS

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Abstract. This is a conitunation of [1] and [2]. We prove that if function f belongs to the class $\Lambda_\omega \stackrel{\text{def}}{=} \{f : \omega_f(\delta) \leqslant \text{const}\ \omega(\delta)\}$ for an arbitrary modulus of continuity ω, then $s_j(f(A)-f(B)) \leqslant c\cdot\omega_*\left((1+j)^{-\frac{1}{p}}\|A-B\|_{S_p^l}\right)\cdot\|f\|_{\Lambda_\omega}$ for arbitrary self-adjoint operators A, B and all $1 \leqslant j \leqslant l$, where $\omega_*(x) \stackrel{\text{def}}{=} x\int_x^\infty \frac{\omega(t)}{t^2}dt$ $(x>0)$. The result is then generalized for contractions, maximal dissipative operators, normal operators and n-tuples of commuting self-adjoint operators.

1. Introduction

In this note we study the behavior of functions of operators under perturbations. We are going to find estimates for the singular values $s_n(f(A)-f(B))$, where both A and B are arbitrary self-adjoint or unitary operators. These results are based on the methods developed in [1] and [3] for estimates of operator norms $\|f(A)-f(B)\|$, in these papers the authors proved if f belongs to the Hölder class $\Lambda_\alpha(\mathbb{R})$ with $0<\alpha<1$, then $\|f(A)-f(B)\| \leqslant \text{const}\ \|f\|_{\Lambda_\alpha}\|A-B\|^\alpha$ for all pairs of self-adjoint or unitary operators A and B. The authors also generalized their results to the class Λ_ω, and obtained estimate $\|f(A)-f(B)\| \leqslant \text{const}\ \|f\|_{\Lambda_\omega}\omega_*\|A-B\|$.

In [2], it was shown that for functions f in the Hölder class $\Lambda_\alpha(\mathbb{R})$ with $0<\alpha<1$ and if $1<p<\infty$, the operator $f(A)-f(B)$ belongs to $\mathbf{S}_{p/\alpha}$, whenever A and B are arbitrary self-adjoint operators such that $A-B\in\mathbf{S}_p$. In particular, it was proved that if $0<\alpha<1$, then there exists a constant $c>0$ such that for every $l\geqslant 0$, $p\in[1,\infty)$, $f\in\Lambda_\alpha(\mathbb{R})$, and for arbitrary self-adjoint operators A and B on Hilbert space with bounded $A-B$, the following inequality holds for every $j\leqslant l$:

$$s_j(f(A)-f(B)) \leqslant c \quad \|f\|_{\Lambda_\alpha(\mathbb{R})}(1+j)^{-\frac{\alpha}{p}}\|A-B\|^\alpha_{S_p^l} \quad \text{(see (3.1) for definition).}$$

In section §3, we generalize this estimate to the class Λ_ω and also obtain some lower-bound estimates for rank one perturbations which also extend the results in [2]. In section §4, similar estimates are given without proofs in case of contractions, maximal dissipative operators, normal operators and n-tuples of commuting self-adjoint operators.

Necessary information on Space Λ_ω is given in section §2. We refer the reader to [1] for more detailed information.

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2. Space Λ_ω

Let ω be a modulus of continuity, i.e., ω is a nondecreasing continuous function on $[0,\infty)$ such that $\omega(0)=0$, $\omega(x)>0$ for $x>0$, and

$$\omega(x+y)\leqslant\omega(x)+\omega(y),\quad x,y\in[0,\infty).$$

We denote by $\Lambda_\omega(\mathbb{R})$ the space of functions on $\mathbb{R}$ such that

$$\|f\|_{\Lambda_\omega(\mathbb{R})}\stackrel{\text{def}}{=}\sup_{x\neq y}\frac{|f(x)-f(y)|}{\omega(|x-y|)}.$$

The space $\Lambda_\omega(\mathbb{T})$ on the unit circle can be defined in a similar way.

We continue with the class Λ_ω of functions on $\mathbb{T}$ first. Let w be an infinitely differentiable function on $\mathbb{R}$ such that

$$w\geqslant 0,\quad \operatorname{supp} w\subset\left[\frac{1}{2},2\right],\ \text{ and }\ w(x)=1-w\left(\frac{x}{2}\right)\ \text{ for }\ x\in[1,2]. \tag{2.1}$$

Define a C^∞ function v on $\mathbb{R}$ by

$$v(x)=1\ \text{ for }\ x\in[-1,1]\ \text{ and }\ v(x)=w(|x|)\ \text{ if }\ |x|\geqslant 1. \tag{2.2}$$

Define trigonometric polynomials W_n, $W_n^\sharp$ and V_n by

$$W_n(z)=\sum_{k\in\mathbb{Z}}w\left(\frac{k}{2^n}\right)z^k,\quad n\geqslant 1,\quad W_0(z)=\bar{z}+1+z,\ \text{ and }\ W_n^\sharp(z)=\overline{W_n(z)},\quad n\geqslant 0$$

and

$$V_n(z)=\sum_{k\in\mathbb{Z}}v\left(\frac{k}{2^n}\right)z^k,\quad n\geqslant 1.$$

V_n is called de la Vallée Poussin type kernel.

If f is a distribution on $\mathbb{T}$, we define f_n, $n\geqslant 0$ by

$$f_n=f*W_n+f*W_n^\sharp,\quad n\geqslant 1,\ \text{ and }\ f_0=f*W_0,$$

Then $f=\sum_{n\geqslant 0}f_n$ and $f-f*V_n=\sum_{k=n+1}^\infty f_n$.

Now we proceed to the real line case. We use the same functions w, v as in (2.1), (2.2), and define functions W_n, $W_n^\sharp$ and V_n on $\mathbb{R}$ by

$$\mathscr{F}W_n(x)=w\left(\frac{x}{2^n}\right),\quad \mathscr{F}W_n^\sharp(x)=\mathscr{F}W_n(-x),\quad n\in\mathbb{Z}$$

and

$$\mathscr{F}V_n(x)=v\left(\frac{x}{2^n}\right),\quad n\in\mathbb{Z},$$

where $\mathscr{F}$ is the *Fourier transform*:

$$(\mathscr{F}f)(t)=\int_{\mathbb{R}}f(x)e^{-ixt}\,dx,\quad f\in L^1.$$

V_n is also called de la Vallée Poussin type kernel.

If f is a tempered distribution on $\mathbb{R}$, we define f_n by

$$f_n = f * W_n + f * W_n^\sharp, \quad n \in \mathbb{Z}.$$

We will use the same notation Λ_ω, W_n, $W_n^\sharp$ and V_n on $\mathbb{R}$ and on $\mathbb{T}$ in the following discussion.

In [1], it is proved that there exists a constant c such that for an arbitrary modulus of continuity ω and for an arbitrary function f in Λ_ω, the following inequalities hold for all $n \in \mathbb{Z}$, in $\mathbb{R}$ case, or for all $n \geqslant 0$, in $\mathbb{T}$ case:

$$\|f - f * V_n\|_{L^\infty} \leqslant c\omega(2^{-n})\|f\|_{\Lambda_\omega} \tag{2.3}$$

$$\|f * W_n\|_{L^\infty} \leqslant c\omega(2^{-n})\|f\|_{\Lambda_\omega}, \quad \|f * W_n^\sharp\|_{L^\infty} \leqslant c\omega(2^{-n})\|f\|_{\Lambda_\omega} \tag{2.4}$$

Let $\mathscr{S}'(\mathbb{R})$ be the space of all tempered distributions on $\mathbb{R}$. Denote by $\mathscr{S}'_+(\mathbb{R})$ the set of all $f \in \mathscr{S}'(\mathbb{R})$ such that $\operatorname{supp} \mathscr{F} f \subset [0, \infty)$. Put $\big(\Lambda_\omega(\mathbb{R})\big)_+ \stackrel{\text{def}}{=} \Lambda_\omega(\mathbb{R}) \cap \mathscr{S}'_+(\mathbb{R})$ and $\mathbb{C}_+ \stackrel{\text{def}}{=} \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Then a function in $\Lambda_\omega(\mathbb{R})$ belongs to the space $\big(\Lambda_\omega(\mathbb{R})\big)_+$ if and only if it has a (unique) continuous extension to the closed upper half-plance $\operatorname{clos} \mathbb{C}_+$ that is analytic in the open upper half-plane $\mathbb{C}_+$ with at most a polynomial growth rate at infinity.

3. Estimates on singular values of functions of perturbed self-adjoint and unitary operators

Recall that if T is a bounded linear operator on Hilbert space, then the singular values $s_j(T)$, $j \geqslant 0$, are defined by

$$s_j(T) = \inf\{\|T - R\| : \operatorname{rank} R \leqslant j\}.$$

For $l \geqslant 0$ and $p \geqslant 1$, we consider the norm S_p^l (see [9]) defined by

$$\|T\|_{S_p^l} \stackrel{\text{def}}{=} \Big(\sum_{j=0}^{l} (s_j(T))^p \Big)^{\frac{1}{p}}. \tag{3.1}$$

It is shown in [13] and [2] that if f is an entire function of exponential type at most σ that is bounded on $\mathbb{R}$, and A, B are self-adjoint operators with bounded $A - B$, then

$$\|f(A) - f(B)\| \leqslant \text{const}\, \sigma \|f\|_{L^\infty} \|A - B\|, \tag{3.2}$$

and

$$\|f(A) - f(B)\|_{S_p^l} \leqslant \text{const}\, \sigma \|f\|_{L^\infty} \|A - B\|_{S_p^l}. \tag{3.3}$$

For the proof and more details, see [1], [2], [8], [10], [11] and [13].

Given a modulus of continuity ω, define functions ω_* and $\omega_\sharp$ by

$$\omega_*(x) = x \int_x^\infty \frac{\omega(t)}{t^2} dt, \quad x > 0$$

and

$$\omega_\sharp(x) = x\int_x^\infty \frac{\omega(t)}{t^2}dt + \int_0^x \frac{\omega(t)}{t}dt, \quad x > 0.$$

In this paper, we assume that $\omega_\sharp$ is finite valued whenever it is used.

For example, if we define ω by

$$\omega(x) = x^\alpha, \quad x > 0, \quad 0 < \alpha < 1,$$

then $\omega_\sharp(x) \leqslant \text{const}\ \omega(x)$.

It is well known (see [6], Ch.3, Theorem 13.30) that if ω is a modulus of continuity, then the Hilbert transform maps Λ_ω into itself if and only if $\omega_\sharp(x) \leqslant \text{const}\ \omega(x)$.

THEOREM 3.1. *There exists a constant $c > 0$ such that for every modulus of continuity ω, every f in $\Lambda_\omega(\mathbb{R})$ and for arbitrary self-adjoint operators A and B, the following inequality holds for all l and for all j, $1 \leqslant j \leqslant l$:*

$$s_j(f(A) - f(B)) \leqslant c\cdot\omega_*\big((1+j)^{-\frac{1}{p}}\|A-B\|_{S_p^l}\big)\cdot\|f\|_{\Lambda_\omega}. \tag{3.4}$$

Proof. A and B can be taken as bounded operators (see [3], Lemma 4.4), then we may further assume f is bounded. Let $R_N = \sum_{n=-\infty}^N (f_n(A) - f_n(B))$, $Q_N = (f - f*V_N)(A) - (f - f*V_N)(B)$. Here f_n and the de la Vallée Poussín type kernel V_N are defined as in §2. Then $f(A) - f(B) = R_N + Q_N$, with convergence in the uniform operator topology as shown in [1]. Note that for any integer $m \in \mathbb{Z}$, functions f_m and $f - f*V_m$ are entire functions of exponential type at most 2^{m+1}. Thus it follows from (3.2), (3.3), (2.3), and (2.4) that

$$\|Q_N\| \leqslant c\cdot\omega(2^{-N})\cdot\|f\|_{\Lambda_\omega},$$

and

$$\begin{aligned}\|R_N\|_{S_p^l} &\leqslant \sum_{n=-\infty}^N \|f_n(A) - f_n(B)\|_{S_p^l}\\ &\leqslant c\cdot\sum_{n=-\infty}^N \big(2^n\cdot\|f_n\|_{L^\infty}\big)\cdot\|A-B\|_{S_p^l}\\ &\leqslant c\cdot 2^N\cdot\omega_*(2^{-N})\cdot\|A-B\|_{S_p^l}\cdot\|f\|_{\Lambda_\omega} \quad \text{(see [1])}\end{aligned}$$

Then

$$\begin{aligned}s_j(f(A)-f(B)) &\leqslant s_j(R_N) + \|Q_N\| \leqslant (1+j)^{-1/p}\cdot\|R_N\|_{S_p^l} + \|Q_N\|\\ &\leqslant c\cdot\big[(1+j)^{-\frac{1}{p}}\cdot 2^N\cdot\omega_*(2^{-N})\|A-B\|_{S_p^l} + \omega(2^{-N})\big]\cdot\|f\|_{\Lambda_\omega}\end{aligned}$$

Take N such that $1 \leqslant (1+j)^{-\frac{1}{p}}\cdot 2^N\cdot\|A-B\|_{S_p^l} \leqslant 2$ and use the fact that $\omega(t) \leqslant \omega_*(t)$ for any $t > 0$, we get (3.4). $\square$

THEOREM 3.2. *There exists a constant $c>0$ such that for every modulus of continuity ω, every f in $\Lambda_\omega(\mathbb{T})$ and for arbitrary unitary operators U and V, the following inequality holds for all l and for all j, $1\leqslant j\leqslant l$:*

$$s_j(f(U)-f(V))\leqslant c\cdot\omega_*\big((1+j)^{-\frac{1}{p}}\|U-V\|_{S_p^l}\big)\cdot\|f\|_{\Lambda_\omega}. \tag{3.5}$$

Proof. If $(1+j)^{-\frac{1}{p}}\cdot\|U-V\|_{S_p^l}\leqslant 2$, the proof is similar to Theorem 3.1 with $R_N=\sum_{n=0}^N(f_n(U)-f_n(U))$; if $(1+j)^{-\frac{1}{p}}\cdot\|U-V\|_{S_p^l}>2$, then

$$s_j(f(U)-f(V))\leqslant\|f(U)-f(V)\|\leqslant c\cdot\omega_*(\|U-V\|)\cdot\|f\|_{\Lambda_\omega}\leqslant c\cdot\omega_*(2)\cdot\|f\|_{\Lambda_\omega}. \quad\square$$

COROLLARY 3.3. *Let ω be a modulus of continuity such that*

$$\omega_*(x)\leqslant\text{const}\,\omega(x),\quad x\geqslant 0.$$

Then for an arbitrary function $f\in\Lambda_\omega(\mathbb{R})$ and for arbitrary self-adjoint operators A and B, the following inequality holds for all l and for all j, $1\leqslant j\leqslant l$:

$$s_j(f(A)-f(B))\leqslant\text{const}\,\omega\big((1+j)^{-\frac{1}{p}}\|A-B\|_{S_p^l}\big)\|f\|_{\Lambda_\omega}.$$

Let H, $\mathscr{H}$ be the Hankel operators defined in [2].

THEOREM 3.4. *Let ω be a modulus of continuity on $\mathbb{T}$. There exist unitary operators U, V and a real function h in $\Lambda_{\omega_\sharp}((T))$ such that*

$$\operatorname{rank}(U-V)=1\qquad\textit{and}\qquad s_m(h(U)-h(V))\geqslant\omega\big((1+m)^{-1}\big).$$

Proof. Consider the operators U and V on space $L_2(\mathbb{T})$ with respect to the normalized Lebesgue measure on $\mathbb{T}$ defined by (see [2])

$$Uf=\bar{z}f\ \text{ and }\ Vf=\bar{z}f-2(f,\,1)\bar{z},\quad f\in L^2.$$

For $f\in C(\mathbb{T})$, we have

$$\big((f(U)-f(V))z^j,z^k\big)=-2\begin{cases}\hat{f}(j-k), & \text{if } j\geqslant 0,\ k<0;\\ \hat{f}(j-k), & \text{if } j<0,\ k\geqslant 0;\\ 0, & \text{otherwise.}\end{cases}$$

Define function g by

$$g(\zeta)=\sum_{n=1}^{\infty}\omega(4^{-n})(\zeta^{4^n}+\bar{\zeta}^{4^n}),\quad \zeta\in\mathbb{T}.$$

Then we have

$$\|g*W_n\|_{L^\infty}\leqslant\text{const}\,\omega(2^{-n}),\quad \|g*W_n^\sharp\|_{L^\infty}\leqslant\text{const}\,\omega(2^{-n}),\quad n\geqslant 0.$$

Let ξ, η be two arbitrarily different fixed points on $\mathbb{T}$, choose $N \geqslant 0$ such that $\frac{1}{2} \leqslant \frac{2^{-N}}{|\xi-\eta|} \leqslant 1$, then

$$\begin{aligned}|g(\xi)-g(\eta)| &\leqslant \sum_{n=0}^{N}|g_n(\xi)-g_n(\eta)|+|(g-g*V_N)(\xi)-(g-g*V_N)(\eta)| \\ &\leqslant \sum_{n=0}^{N}|g_n(\xi)-g_n(\eta)|+2\sum_{n=N+1}^{\infty}\|g_n\|_{L^\infty} \\ &\leqslant \text{const}\sum_{n=0}^{N}2^n\|g_n\|_{L^\infty}|\xi-\eta|+2\sum_{n=N+1}^{\infty}\|g_n\|_{L^\infty} \\ &\leqslant \text{const}\sum_{n=0}^{N}2^n\omega(2^{-n})|\xi-\eta|+\text{const}\sum_{n=N+1}^{\infty}\omega(2^{-n}) \\ &\leqslant \text{const}\ \omega_*(|\xi-\eta|)+\text{const}\int_0^{2^{-N}}\frac{\omega(t)}{t}dt \\ &\leqslant \text{const}\ \omega_\sharp(|\xi-\eta|).\end{aligned}$$

Consider the matrix $\Gamma_g = \{\hat{g}(-j-k)\}_{j\geqslant 1,k\geqslant 0} = \{\hat{g}(j+k)\}_{j\geqslant 1,k\geqslant 0}$.

Let $n \geqslant 1$. Define matrix $T_n = \{\hat{g}(j+k+4^{n-1}+1)\}_{0\leqslant j,k\leqslant 3\cdot 4^{n-1}}$, then

$$T_n = \begin{bmatrix} & & & \omega(4^{-n}) \\ & & \omega(4^{-n}) & \\ & \cdot^{\cdot^{\cdot}} & & \\ \omega(4^{-n}) & & & \end{bmatrix}.$$

If R is any matrix with the same size of T_n such that $\operatorname{rank}(R) < 3\cdot 4^{n-1}$, then $\|T_n - R\| \geqslant \omega(4^{-n})$. It follows that $s_j(T_n) \geqslant \omega(4^{-n})$ for $j < 3\cdot 4^{n-1}$. For each T_n, there is some orthogonal projection P_n such that $T_n = P_n\Gamma_g P_n$, hence $s_j(\Gamma_g) \geqslant s_j(T_n) \geqslant \omega(4^{-n})$ for all n and for all j, $j < 3\cdot 4^{n-1}$. Thus for all $j \geqslant 0$, we have

$$s_j(\Gamma_g) \geqslant \omega\Big(\frac{3}{16}\cdot(j+1)^{-1}\Big) \geqslant \frac{3}{32}\cdot\omega\big((j+1)^{-1}\big).$$

To complete the proof, it suffices to take $h = \frac{32}{3}g$. $\square$

COROLLARY 3.5. *Let ω be a modulus of continuity such that*

$$\omega_\sharp(x) \leqslant \text{const}\ \omega(x), \quad 0 \leqslant x \leqslant 2.$$

There exist unitary operators U, V and a real function h in $\Lambda_\omega(T)$ such that

$$\operatorname{rank}(U-V) = 1 \qquad \text{and} \qquad s_m(h(U)-h(V)) \geqslant \omega\big((1+m)^{-1}\big).$$

THEOREM 3.6. *Let ω be a modulus of continuity on $\mathbb{T}$ and f be a continuous function on $\mathbb{T}$. If for all unitary operators U and V, we have*

$$s_n(f(U)-f(V)) \leqslant \text{const}\ \omega\big((1+n)^{-\frac{1}{p}}\|U-V\|_{S_p}\big), \quad \textit{for all} \quad n \geqslant 0,$$

then $f \in \Lambda_\omega(\mathbb{T})$.

Proof. Let $\zeta,\eta\in\mathbb{T}$, we can select commuting unitary operators U and V such that $s_0(U-V)=s_1(U-V)=\ldots=s_n(U-V)=|\zeta-\eta|$ and $s_k(U-V)=0$, $k\geqslant n+1$. Then $s_n(f(U)-f(V))=|f(\zeta)-f(\eta)|$, $\|U-V\|_{S_p}=(1+n)^{\frac{1}{p}}\cdot|\zeta-\eta|$. □

THEOREM 3.7. *Let ω be a modulus of continuity on $\mathbb{R}$ and f be a continuous function on $\mathbb{R}$. If for all self-adjoint operators A and B, we have*

$$s_n(f(A)-f(B))\leqslant \text{const}\,\omega\big((1+n)^{-\frac{1}{p}}\|A-B\|_{S_p}\big),\quad \textit{for all}\quad n\geqslant 0,$$

then $f\in\Lambda_\omega(\mathbb{R})$.

Proof. Similar to Theorem 3.6. □

THEOREM 3.8. *Let ω be a modulus of continuity over $\mathbb{R}$. There exist self-adjoint operators A, B, and a real function f in $\Lambda_{\omega_\sharp}(\mathbb{R})$ such that*

$$\operatorname{rank}(A-B)=1\ \ \textit{and}\ \ s_m(f(A)-f(B))\geqslant\omega\big((1+m)^{-1}\big),\ \ \textit{for all}\ \ m\geqslant 0.$$

Proof. WLOG, we assume $\omega(t)=\omega(2)$, for all $t\geqslant 2$, that is, ω can be regarded as a modulus of continuity on $\mathbb{T}$.

We then choose a function (see [2], Lemma 9.6) $\rho\in C^\infty(\mathbb{T})$ such that $\rho(\zeta)+\rho(i\zeta)=1$, $\rho(\zeta)=\rho(\bar{\zeta})$ for all $\zeta\in\mathbb{T}$, and ρ vanishes in a neighborhood of the set $\{-1,1\}$. Note that $\rho\in\Lambda_\omega(\mathbb{T})$, since $\omega(st)\geqslant\frac{s}{2}\omega(t)$, for all $t\geqslant 0$ and s, $0<s<1$.

Define function g_1 by

$$g_1(\zeta)=\sum_{n=1}^{\infty}\omega(4^{-n})(\zeta^{4^n}+\bar{\zeta}^{4^n}),\qquad \zeta\in\mathbb{T}.$$

Then $g_1\in\Lambda_{\omega_\sharp}(\mathbb{T})$. If $g_0\overset{\text{def}}{=}C\rho g_1$ for a sufficient large number C, then $g_0\in\Lambda_{\omega_\sharp}(\mathbb{T})$, vanishes in a neighborhood of the set $\{-1,1\}$ and $g_0(\zeta)=g_0(\bar{\zeta})$ for all $\zeta\in\mathbb{T}$, and $s_m(H_{g_0})\geqslant\omega\big((1+m)^{-1}\big)$ for all $m\geqslant 0$.

Define $\varphi(x)=(x^2+1)^{-1}$ (as in [2], Theorem 9.9), then there exists a compactly supported real bounded function f such that $f(\varphi(x))=g_0(\frac{x-i}{x+i})$ and a simple calculation shows that f belongs to $\Lambda_{\omega_\sharp}(\mathbb{R})$. Denote $L^2_e(\mathbb{R})$ the subspace of even functions in $L^2(\mathbb{R})$. Consider operators A and B on $L^2_e(\mathbb{R})$ defined by $A(g)=\mathbf{H}^{-1}M_\varphi\mathbf{H}(g)$ and $B(g)=\varphi g$, here $\mathbf{H}$ is the Hilbert transform defined on $L^2(\mathbb{R})$ (see [2]) and M_φ is the multiplication by φ. Then $\operatorname{rank}(A-B)=1$, and we have

$$s_m(f(B)-f(A))\geqslant\sqrt{2}s_m(\mathscr{H}_{f\circ\varphi})=\sqrt{2}s_m(H_{g_0})\geqslant\sqrt{2}\omega\big((1+m)^{-1}\big).\quad\square$$

4. Estimates for other types of operators

The following estimates are given without proofs in case of contractions, maximal dissipative operators, normal operators and n-tuples of commuting self-adjoint operators.

THEOREM 4.1. *There exists a constant $c>0$ such that for every modulus of continuity ω, every f in $\big(\Lambda_\omega(\mathbb{R})\big)_+$ and for arbitrary contractions T and R on Hilbert space, the following inequality holds for all l and for all j, $1\leqslant j\leqslant l$:*

$$s_j(f(T)-f(R))\leqslant c\,\omega_*\big((1+j)^{-\frac{1}{p}}\|T-R\|_{S_p^l}\big)\|f\|_{\Lambda_\omega}.$$

To prove this result, the following result is important (see [1], [2] and [12]):

There exists a constant c such that for arbitrary trigonometric polynomial f of degree n and for arbitrary contractions T and R on Hilbert space,

$$\|(f(T)-f(R)\|_{S_p}\leqslant c\,n\|f\|_{L^\infty}\|T-R\|_{S_p}.$$

Denote $\mathscr{F}$ the *Fourier transform* on $L_1(\mathbb{R}^n)$, $n\geqslant 1$ by:

$$(\mathscr{F}f)(t)=\int_{\mathbb{R}^\ltimes} f(x)e^{-i(x,t)}dx,$$

where

$$x=(x_1,\dots,x_n),\quad t=(t_1,\dots,t_n),\quad (x,t)\overset{\text{def}}{=}x_1t_1+\dots+x_nt_n.$$

THEOREM 4.2. *There exists a constant $c>0$ such that for every modulus of continuity ω, every f in $\big(\Lambda_\omega(\mathbb{R})\big)_+$ and for arbitrary maximal dissipative operators L and M with bounded difference, the following inequality holds for all l and for all j, $1\leqslant j\leqslant l$:*

$$s_j(f(L)-f(M))\leqslant c\,\omega_*\big((1+j)^{-\frac{1}{p}}\|L-M\|_{S_p^l}\big)\|f\|_{\Lambda_\omega}.$$

To prove this result, the following result is important (see [4]):

There exists a constant $c>0$ such that for every function f in $H^\infty(\mathbb{C}_+)$ with $\operatorname{supp}\mathscr{F}f\subset[0,\sigma]$, $\sigma>0$, and for arbitrary maximal dissipative operators L and M with bounded difference,

$$\|f(L)-f(M)\|_{S_p}\leqslant c\quad\sigma\|f\|_{L^\infty}\|L-M\|_{S_p}.$$

THEOREM 4.3. *There exists a constant $c>0$ such that for every modulus of continuity ω, every f in $\Lambda_\omega(\mathbb{R}^2)$ and for arbitrary normal operators N_1 and N_2, the following inequality holds for all l and for all j, $1\leqslant j\leqslant l$:*

$$s_j(f(N_1)-f(N_2))\leqslant c\,\omega_*\big((1+j)^{-\frac{1}{p}}\|N_1-N_2\|_{S_p^l}\big)\|f\|_{\Lambda_\omega}.$$

To prove this result, the following result is important (see [5]):

There exists a constant $c > 0$ such that for every bounded continuous function f on $\mathbb{R}^2$ with

$$\operatorname{supp} \mathscr{F} f \subset \{\zeta \in \mathbb{C} : |\zeta| \leqslant \sigma\}, \quad \sigma > 0,$$

and for arbitrary normal operators N_1 and N_2,

$$\|(f(N_1) - f(N_2)\|_{S_p} \leqslant c \quad \sigma \|f\|_{L^\infty} \|N_1 - N_2\|_{S_p}.$$

THEOREM 4.4. *Let n be a positive integer and $p \geqslant 1$. There exists a positive number c_n such that for every modulus of continuity ω, every f in $\Lambda_\omega(\mathbb{R}^n)$ and for arbitrary n-tuples of commuting self-adjoint operators $(A_1,\dots,A_n)$ and $(B_1,\dots,B_n)$, the following inequality holds for all l and for all j, $1 \leqslant j \leqslant l$:*

$$s_j(f(A_1,\dots,A_n) - f(B_1,\dots,B_n)) \leqslant c_n \max_{1 \leqslant j \leqslant n} \omega_* \left((1+j)^{-\frac{1}{p}} \|A_j - B_j\|_{S_p^l}\right) \|f\|_{\Lambda_\omega}.$$

To prove this result, the following result is important (see [7]):

There exists a constant $c_n > 0$ such that for every bounded continuous function f on $\mathbb{R}^n$ with

$$\operatorname{supp} \mathscr{F} f \subset \{\xi \in \mathbb{R}^n : |\xi| \leqslant \sigma\}, \quad \sigma > 0,$$

and for arbitrary n-tuples of commuting self-adjoint operators $(A_1,\dots,A_n)$ and $(B_1,\dots, B_n)$,

$$\|f(A_1,\dots,A_n) - f(B_1,\dots,B_n)\|_{S_p} \leqslant c_n \quad \sigma \|f\|_{L^\infty} \max_{1 \leqslant j \leqslant n} \|A_j - B_j\|_{S_p}.$$

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