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A NOTE ON COMMUTATIVITY PRESERVING MAPS ON $M_n(\mathbb{R})$

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Abstract. Let $M_n(\mathbb{F})$ be the set of all $n\times n$ matrices over a field $\mathbb{F}$. Surjective maps which preserve the commutativity relation on $M_n(\mathbb{F})$ only in one direction have been recently classified for the case when $\mathbb{F}$ is an algebraically closed field. We show that the same result holds also when $\mathbb{F}=\mathbb{R}$ is the field of real numbers and $n\geqslant 7$ is odd.

1. Introduction and statement of the result

Let $M_n(\mathbb{F})$ be the set of all $n\times n$ matrices over a field $\mathbb{F}$. A map $\Phi: M_n(\mathbb{F})\to M_n(\mathbb{F})$ preserves commutativity if $\Phi(A)\Phi(B)=\Phi(B)\Phi(A)$ whenever $AB=BA$, $A,B\in M_n(\mathbb{F})$. If Φ is bijective and both Φ and Φ^{-1} preserve commutativity, then we say that Φ preserves commutativity in both directions. In recent decades commutativity preserving linear maps were extensively studied (see for example [3] and references therein). Motivated by applications in quantum mechanic some authors found interest to study a more difficult problem of characterizing non-linear commutativity preserving maps. In [12], Šemrl characterized continuous, bijective maps on the set of all $n\times n$ complex matrices $M_n(\mathbb{C})$, where $n\geqslant 3$, which preserve commutativity in both directions. He also studied such maps without the continuity assumption and showed that essentially the same result as for linear bijections is true on the set of rank-one complex matrices and that outside this set a map can be very nonlinear. The same author considered in [13] injective, continuous maps on $M_n(\mathbb{C})$, $n>3$, that preserve commutativity. Fošner [7] proved, using the real Jordan canonical form, that an analogous result holds true for the set of real matrices $M_n(\mathbb{R})$, $n>3$.

Dolinar and Kuzma further relaxed in [4] the assumptions on a map. Namely, they assumed Φ preserves commutativity only in one direction, is surjective, and leaves invariant the set of non-central elements. They showed that the last assumption is indispensable. Their result, which was proved by techniques that combine graph theory, linear algebra, and projective geometry, follows.

THEOREM 1. *Let $\mathbb{F}$ be an algebraically closed field and $n\geqslant 5$. Assume a surjective map $\Phi: M_n(\mathbb{F})\to M_n(\mathbb{F})$ preserves commutativity and suppose further that $\Phi(X)\in\mathbb{F}I$, where I is the identity matrix, implies $X\in\mathbb{F}I$. Then there exist functions $\alpha,\gamma\colon M_n(\mathbb{F})\to\mathbb{F}\setminus\{0\}$, a field isomorphism $\sigma:\mathbb{F}\to\mathbb{F}$, and an invertible matrix $S\in M_n(\mathbb{F})$ such that one of the following holds true:*

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(i) $\Phi(X) = \alpha(X)SX^{\sigma}S^{-1} + \gamma(X)I, \quad \mathrm{rk}X = 1;$

(ii) $\Phi(X) = \alpha(X)S(X^{\sigma})^{t}S^{-1} + \gamma(X)I, \quad \mathrm{rk}X = 1.$

Here $\mathrm{rk}X$ denotes the rank of the matrix $X \in M_n(\mathbb{F})$. The question is whether the same result holds also when $\mathbb{F} = \mathbb{R}$ is the field of real numbers (recall that the only field isomorphism of real numbers is the identity, so in this case σ is automatically identity). We will show that at least when n is an odd number greater then 5, the answer is positive. It turned out that some partial results from [4] can be directly applied in the real case however some techniques which were developed in [4] for the case of complex matrices using the Jordan canonical form had to be altered. Thus we will give at the beginning of the next section a brief description of the real Jordan canonical form and then the proof of our main result will follow. Let us state our main result.

THEOREM 2. *Let $n \geqslant 7$ be an odd number. Assume a surjective map $\Phi : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ preserves commutativity and suppose further that $\Phi(X) \in \mathbb{R}I$, where I is the identity matrix, implies $X \in \mathbb{R}I$. Then there exist functions α, γ: $M_n(\mathbb{R}) \to \mathbb{R} \setminus \{0\}$ and an invertible matrix $S \in M_n(\mathbb{R})$ such that one of the following holds true:*

(i) $\Phi(X) = \alpha(X)SXS^{-1} + \gamma(X)I, \quad \mathrm{rk}X = 1;$

(ii) $\Phi(X) = \alpha(X)SX^{t}S^{-1} + \gamma(X)I, \quad \mathrm{rk}X = 1.$

2. Proofs

Let us start with briefly recalling some properties of the real Jordan canonical form, see also [9]. Suppose $A \in M_n(\mathbb{R})$ and let $\lambda \in \mathbb{C}$ be a non-real eigenvalue of A. The structure of Jordan blocks in the Jordan canonical form corresponding to λ is the same as the structure of Jordan blocks corresponding to the conjugate eigenvalue $\overline{\lambda}$. Thus, all Jordan blocks of all sizes corresponding to non-real eigenvalues occur in conjugate pairs of equal sizes. Suppose $J_2(\lambda)$, the 2×2 Jordan block corresponding to λ, appears in the Jordan canonical form of A. Then $J_2(\overline{\lambda})$ also appears in the Jordan canonical form of A with the same multiplicity as $J_2(\lambda)$. The block matrix

$$\begin{bmatrix} J_2(\lambda) & 0 \\ 0 & J_2(\overline{\lambda}) \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \overline{\lambda} & 1 \\ 0 & 0 & 0 & \overline{\lambda} \end{bmatrix} \tag{1}$$

is permutation-similar to the matrix

$$\begin{bmatrix} D(\lambda) & I \\ 0 & D(\lambda) \end{bmatrix}$$

where $D(\lambda) \equiv \begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix} \in M_2(\mathbb{C})$ and I is the 2×2 identity matrix. Suppose now $\lambda = a+ib$, where $a,b \in \mathbb{R}$, and let $S = \begin{bmatrix} -i & -i \\ 1 & -1 \end{bmatrix}$. Then

$$SD(\lambda)S^{-1} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

Let $C(a,b) \equiv \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Then (1) is similar via $\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}$ to a real 4×4 block

$$\begin{bmatrix} a & b & 1 & 0 \\ -b & a & 0 & 1 \\ 0 & 0 & a & b \\ 0 & 0 & -b & a \end{bmatrix} = \begin{bmatrix} C(a,b) & I \\ 0 & C(a,b) \end{bmatrix}.$$

It turns out (see [9]) that in general, each matrix $\begin{bmatrix} J_k(\lambda) & 0 \\ 0 & J_k(\overline{\lambda}) \end{bmatrix} \in M_{2k}(\mathbb{C})$, where $J_k(\lambda)$ is a $k\times k$ Jordan block corresponding to a non-real eigenvalue λ, is similar to a $2k\times 2k$ real block

$$C_k(a,b) \equiv \begin{bmatrix} C(a,b) & I & 0 & \dots & 0 \\ 0 & C(a,b) & I & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & I \\ 0 & 0 & \dots & 0 & C(a,b) \end{bmatrix}. \tag{2}$$

This observation leads us to the real Jordan canonical form.

PROPOSITION 1. (see [9]) *Each matrix $A \in M_n(\mathbb{R})$ is similar to a block diagonal real matrix of the form*

$$C_{n_1}(a_1,b_1) \oplus C_{n_2}(a_2,b_2) \oplus \dots \oplus C_{n_p}(a_p,b_p) \oplus J_{n_q}(\lambda_q) \oplus \dots \oplus J_{n_r}(\lambda_r) \tag{3}$$

where $\lambda_k = a_k + ib_k$, $a_k,b_k \in \mathbb{R}$, $1 \leqslant k \leqslant p$, are non-real eigenvalues of A, and $\lambda_q,\dots,\lambda_r$ are real eigenvalues of A. Each real block triangular matrix $C_{n_k}(a_k,b_k) \in M_{2n_k}$ is of the form (2) and corresponds to a pair of Jordan blocks $J_{n_k}(\lambda_k)$, $J_{n_k}(\overline{\lambda_k}) \in M_{n_k}$ with non-real λ_k from the Jordan canonical form of A. The real Jordan blocks $J_{n_q}(\lambda_q),\dots,J_{n_r}(\lambda_r)$ in (3) are exactly the Jordan blocks from the Jordan canonical form of A corresponding to real eigenvalues of A.

REMARK 1. It is not evident from the above approach that the similarity matrix that transforms A into (3) can be chosen to be real. It turns out (see [9]) that if $A \in M_n(\mathbb{R})$, there is always a real nonsingular matrix S such that $S^{-1}AS$ is in the real Jordan canonical form (3).

We continue by proving the main result. Throughout the proof we will assume that $n \geqslant 7$. Let us remark that some lemmas, which we will present and prove in the continuation, are true also for smaller n. Let $\mathbb{F}$ be a field. Given a subset $\Delta \subseteq M_n(\mathbb{F})$, its centralizer is the vector space

$$\mathscr{C}_{\mathbb{F}}(\Delta) = \{X \in M_n(\mathbb{F}) : AX = XA \text{ for all } A \in \Delta\}.$$

If $\Delta = \{A\}$ is a singleton, we will shortly write $\mathscr{C}_{\mathbb{F}}(A) = \mathscr{C}_{\mathbb{F}}(\{A\})$. If $\mathscr{C}_{\mathbb{F}}(A) = \mathscr{C}_{\mathbb{F}}(B)$ for some $A, B \in M_n(\mathbb{F})$, then we say that A and B are $\mathscr{C}$-equivalent.

We say that $A \in M_n(\mathbb{F})$ is a predecessor of $B \in M_n(\mathbb{F})$ or B is a successor of A if $\mathscr{C}_{\mathbb{F}}(A) \subseteq \mathscr{C}_{\mathbb{F}}(B)$. If for every $X \in M_n(\mathbb{F})$, $\mathscr{C}_{\mathbb{F}}(A) \subseteq \mathscr{C}_{\mathbb{F}}(X) \subseteq \mathscr{C}_{\mathbb{F}}(B)$ implies $\mathscr{C}_{\mathbb{F}}(X) \in \{\mathscr{C}_{\mathbb{F}}(A), \mathscr{C}_{\mathbb{F}}(B)\}$, then we say that A is an immediate predecessor of B.

A non-scalar $M \in M_n(\mathbb{F})$ is maximal if for every non-scalar $X \in M_n(\mathbb{F})$ with $\mathscr{C}_{\mathbb{F}}(M) \subseteq \mathscr{C}_{\mathbb{F}}(X)$, it follows $\mathscr{C}_{\mathbb{F}}(M) = \mathscr{C}_{\mathbb{F}}(X)$. We will denote by $\mathbb{F}[A]$ the unital $\mathbb{F}$-subalgebra of $M_n(\mathbb{F})$ generated by the matrix $A \in M_n(\mathbb{F})$. Let $M \in M_n(\mathbb{F})$ be a maximal matrix. Then, by [5, Theorem 3.2] M belongs to one of the following classes: (1) M is $\mathscr{C}$-equivalent to an idempotent, or (2) M is $\mathscr{C}$-equivalent to a square-zero matrix, or (3) M is similar to $C \oplus C \oplus \ldots \oplus C$ where C is a companion matrix of an irreducible polynomial such that there is no proper intermediate field between $\mathbb{F}$ and $\mathbb{F}[C]$.

Since we will study matrices as elements of $M_n(\mathbb{R})$, we will from now on shortly write $\mathscr{C}(\Delta) = \mathscr{C}_{\mathbb{R}}(\Delta)$. We know (see [10, Corrolary 1, page 113]) that

$$\mathscr{C}(\mathscr{C}(A)) = \mathbb{R}[A] \tag{4}$$

for any $A \in M_n(\mathbb{R})$. Also, by Theorem 2.8 in [5], $A \in M_n(\mathbb{R})$, $n \geqslant 2$, is non-derogatory (its minimal polynomial equals its characteristic polynomial) if and only if

$$\mathscr{C}(A) = \mathbb{R}[A]. \tag{5}$$

Authors divided in [4] the proof of Theorem 1 into two parts. In the first part, authors characterized rank-one matrices in terms of commutativity of a certain $n-2$ tuple of matrices. In the second part, authors first used this key lemma (Lemma 2.8) to prove some new lemmas. For example, they showed that $\Phi(A)$ is $\mathscr{C}$-equivalent to a matrix of rank-one whenever A is a rank-one matrix. By such auxiliary results and with tools of projective geometry, authors then concluded the proof of Theorem 1. It turns out that the proof of Theorem 2 may be structured in a similar way. Moreover, if every matrix $A \in M_n(\mathbb{F})$ has an eigenvalue $\lambda_A \in \mathbb{F}$, the second part of the proof of Theorem 1 works for any (not necessarily algebraically closed) field $\mathbb{F}$ with sufficiently many elements. Note that when n is odd, every matrix $A \in M_n(\mathbb{R})$ has a real eigenvalue. So, in order to prove Theorem 2, we will prove a result similar to Lemma 2.8 from [4].

We begin with an auxiliary result.

LEMMA 1. *Let a non-scalar matrix $A \in M_n(\mathbb{R})$ be non-maximal. Then there exist a maximal matrix M and a matrix B, which is an immediate predecessor of M, such that*

$$\mathscr{C}(A) \subseteq \mathscr{C}(B) \subsetneq \mathscr{C}(M).$$

Proof. Since A is not maximal, there exists a non-scalar matrix X_1 with $\mathscr{C}(A) \subsetneq \mathscr{C}(X_1) \subseteq M_n(\mathbb{R})$. If X_1 is also not maximal, we can continue the chain with $\mathscr{C}(X_2)$ for some non-scalar matrix X_2. By comparing the dimensions of these vector spaces, such chain must terminate in finite steps, and clearly ends with a maximal matrix. By comparing the dimensions one more time, we can also assume that in this chain X_{i-1} is always an immediate predecessor of X_i with $\mathscr{C}(X_{i-1}) \subsetneq \mathscr{C}(X_i)$. $\square$

Note that the size of any block $C_{n_i}(a_i, b_i)$ in the real Jordan canonical form (3) of $A \in M_n(\mathbb{R})$ that corresponds to a conjugate pair of (non-real) eigenvalues of A must be even. We define the signature of a matrix $A \in M_n(\mathbb{R})$ to be a tuple, consisting of the sizes of its blocks from the real Jordan canonical form (3) of A, ordered non-increasingly. Thus, the diagonalizable matrix has the signature $(1,1,\dots,1)$ which we shortly write (1^n) while the nilpotent matrix with maximal nilindex has the signature (n). If the signature of A is for example $(4,3^3,2^2,1^{n-17})$, then the real Jordan canonical form of A consists of one block (a Jordan block that corresponds to a real eigenvalue or a block that corresponds to a pair of non-real eigenvalues of A) of size 4, three Jordan blocks of size 3, two blocks (each of them may be a Jordan block that corresponds to a real eigenvalue or a block that corresponds to a pair of non-real eigenvalues of A) of size 2, and other $n-17$ Jordan blocks are of size 1.

LEMMA 2. *Let $A \in M_n(\mathbb{R})$ be non-maximal and an immediate predecessor of a maximal matrix. The following two statements are equivalent.*

(i) Each maximal matrix M with $\mathscr{C}(M) \supseteq \mathscr{C}(A)$ is $\mathscr{C}$-equivalent to a rank-one matrix.

(ii) The signature of A is either

(a) $(3,2^s,1^t)$ for some $s,t \geqslant 0$ with $3+2s+t=n$ and all eigenvalues of A must be the same.

(b) $(2,1^{n-2})$ and A is similar to $\left[\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right] \oplus a \oplus \cdots \oplus a \oplus c$, for some $a,c \in \mathbb{R}$ with $a \neq c$.

(c) $(2^{(n-1)/2},1)$, where n is odd, and A is similar to $\left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] \oplus \cdots \oplus \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] \oplus c$, for some $a,b,c \in \mathbb{R}$ with $b \neq 0$.

Proof. Let $A \in M_n(\mathbb{R})$ be non-maximal and an immediate predecessor of a maximal matrix. Suppose first that statement (ii) holds. If (a) holds, then all eigenvalues of A are real and hence the real Jordan canonical form of A is the same as the Jordan canonical form of A. Similarly if (b) holds. So, by applying a similarity and by subtracting a suitable scalar matrix we may conclude that in case (a), $A = J_3(0) \oplus (\bigoplus_{i=1}^{s} J_2(0)) \oplus 0_t$ and in case (b), $A = J_2(0) \oplus 0_{n-3} \oplus (c-a)$. Recall (see (4)) that every matrix M with $\mathscr{C}(A) \subseteq \mathscr{C}(M)$ is a polynomial in A. So, every maximal matrix M with $\mathscr{C}(A) \subseteq \mathscr{C}(M)$ is $\mathscr{C}$-equivalent in the former case to $J_3(0)^2 \oplus 0_{2s+t}$ which is of rank-one, and in the latter case to $J_2(0) \oplus 0_{n-2}$ or to $0_{n-1} \oplus 1$ which are again both of rank-one.

If (c) holds, then up to similarity and by [2, Proposition 4.1],

$$\mathbb{R}[A] = \mathbb{R}\left[\left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right] \oplus \cdots \oplus \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right]\right] \oplus \mathbb{R}.$$

It is easy to see that every maximal matrix in $\mathbb{R}[A]$ is $\mathscr{C}$-equivalent to E_{nn} where E_{ij} is the matrix with all entries equal to zero except the (i,j)-entry which is equal to one.

Suppose now that statement (ii) does not hold. We will prove that then also statement (i) does not hold. Without loss of generality we may assume that A is already in the real Jordan canonical form and let r be the size of a maximal block. Let us write

$$A = A_1 \oplus A_2, \tag{6}$$

where A_1 contains at least one block of size r and the eigenvalues of A_1 coincide with the eigenvalues of this block, while the intersection of spectrums, $\mathrm{Sp}(A_1) \cap \mathrm{Sp}(A_2)$, is empty. By [2, Proposition 4.1] we have

$$\mathbb{R}[A] = \mathbb{R}[A_1] \oplus \mathbb{R}[A_2].$$

Note that if A_1 corresponds to a real eigenvalue, then we may (and will) without loss of generality assume that $A_1 = \bigoplus_{i=1}^{k} J_{n_i}(0)$ with $n_1 = r$.

Suppose first that $r \geqslant 4$. If A_1 corresponds to a real eigenvalue, then $M = A_1^{r-2} \oplus 0 \in \mathbb{R}[A]$ is a square-zero matrix, hence maximal, and of rank at least two. By (4), $\mathscr{C}(A) \subseteq \mathscr{C}(M)$ so (i) does not hold. If $\mathrm{Sp}(A_1) = \{a \pm ib\}$ for some $a,b \in \mathbb{R}$, $b \neq 0$, then there exists an integer $t \geqslant 1$ such that

$$M = ((A_1 - aI)^2 + b^2 I)^t \oplus 0 \in \mathbb{R}[A]$$

is a square-zero matrix of rank greater then 1 and is thus not $\mathscr{C}$-equivalent to a matrix of rank-one, so again (i) does not hold.

Let $r = 3$. Then the size of the largest block is odd and therefore $\mathrm{Sp}(A_1) \subseteq \mathbb{R}$. By subtracting a suitable scalar matrix we can assume that $\mathrm{Sp}(A_1) = \{0\}$. Suppose first that the signature of A is not $(3, 2^s, 1^t)$. Then there are at least two Jordan blocks of size 3. It follows that A has only one eigenvalue for otherwise $B = A_1 \oplus 0$ is a non-maximal matrix with $\mathscr{C}(A) \subsetneq \mathscr{C}(B)$. So, $A = A_1 = \bigoplus_{i=1}^{k} J_{r_i}(0)$ is a nilpotent. Since there is more then one Jordan block of size 3 in this direct sum, we may conclude that $M = A_1^2$ is a square-zero matrix of rank greater then 1 and (i) does not hold.

Now suppose the signature of A is $(3, 2^s, 1^t)$ and A has at least two eigenvalues. Then $M = I_{m_1} \oplus 0$, where $m_1 \geqslant 3$ is the size of A_1, is a maximal matrix with $\mathscr{C}(A) \subseteq \mathscr{C}(M)$. Since $\mathscr{C}(M) = \mathscr{C}(I - M) = \mathscr{C}(0 \oplus 1_{n-m_1})$, such M is not $\mathscr{C}$-equivalent to a matrix of rank-one unless $n - m_1 = 1$. This is possible only if A_2 is of size 1. Then, however, A has exactly two eigenvalues and one of them is simple. In this case A is $\mathscr{C}$-equivalent to $J_3(0) \oplus (\bigoplus_{i=1}^{s} J_2(0)) \oplus 1$. All the immediate successors B of A are polynomials in A and (assuming $\mathscr{C}(A) \neq \mathscr{C}(B)$) it is easily seen to be $\mathscr{C}$-equivalent to $J_3(0) \oplus (\bigoplus_{i=1}^{s} J_2(0)) \oplus 0$, or to $J_3(0)^2 \oplus (\bigoplus_{i=1}^{s} J_2(0)^2) \oplus 1$. But neither of them is maximal, a contradiction to the hypothesis that A is an immediate predecessor of a maximal matrix.

Let now $r=2$. The signature of A is then $(2^s,1^{n-2s})$, $s\geqslant 1$. If A has only one eigenvalue, then A is $\mathscr{C}$-equivalent to a square-zero matrix and hence maximal, a contradiction. Therefore A has at least two eigenvalues.

Suppose first $s=1$. Then in decomposition (6), the block A_1 contains a cell of size 2. If A has at least three eigenvalues, then clearly A_2 contains at least two cells of size 1. If A has exactly two eigenvalues, then the block A_2 contains at least two cells of size 1 because we have assumed (ii) does not hold and therefore neither of the two eigenvalues of A is of algebraic multiplicity one. Hence, there exists a maximal matrix $M=0_{m_1}\oplus I_{n-m_1}\in\mathbb{R}[A_1]\oplus\mathbb{R}[A_2]=\mathbb{R}[A]$ where $m_1\geqslant 2$ is the size of A_1 and $n-m_1\geqslant 2$ is the size of A_2. Such M is not $\mathscr{C}$-equivalent to a matrix of rank-one, so (i) does not hold.

Let $s\geqslant 2$. First, assume that that a 2×2 block in A_1 corresponds to a pair of conjugate eigenvalues of A, $\lambda=a+ib$ and $\overline{\lambda}=a-ib$, $b\neq 0$. Let m_1 be the size of A_1. Then

$$M=(A_1-aI_{m_1})^2\oplus 0_{n-m_1}=-b^2I_{m_1}\oplus 0_{n-m_1}\in\mathbb{R}[A]$$

is a maximal matrix which is $\mathscr{C}$-equivalent to a matrix of rank-one only if $n-m_1=1$. Since we assumed that (ii) does not hold, this last option is not possible. So again, (i) does not hold. Still assuming $s\geqslant 2$, suppose now that every 2×2 block is induced by a real eigenvalue, i.e. all the eigenvalues of A are real. Then there exists a polynomial p where $N=p(A)$ is square-zero matrix which is of rank at least 2 and thus not $\mathscr{C}$-equivalent to a matrix of rank-one.

Finally, let $r=1$. Every block in the real Jordan canonical form of A is of the size 1 and thus corresponds to a real eigenvalue of A. Also, A has at least three (real) eigenvalues for otherwise A is scalar or maximal. Therefore there exists a polynomial p such that $p(A)=0_{k_1}\oplus I_{k_2}\oplus(\alpha I_{k_3})$ where $\alpha\in\mathbb{R}$, $k_i\geqslant 1$ for every $i\in\{1,2,3\}$, and $k_1+k_2+k_3=n$. If $k_2=k_3=1$, then $M=0_{k_1}\oplus 1\oplus 1\in\mathbb{R}[A]$ is a maximal matrix that is not $\mathscr{C}$-equivalent to a matrix of rank-one. If $k_2\geqslant 2$ or $k_3\geqslant 2$, then the same holds for $M=0_{k_1}\oplus I_{k_2}\oplus 0_{k_3}$ or $M=0_{k_1+k_2}\oplus I_{k_3}$, respectively. $\square$

Recall that Hilbert matrix $H\in M_n(\mathbb{F})$ is of the following form $H=\left(\frac{1}{i+j-1}\right)_{ij}$. Note that all the minors of H are nonzero. By $e_1,e_2,\ldots,e_n$ we will denote the standard basis vectors. For a field $\mathbb{F}$, the graph $\Gamma=\Gamma(M_n(\mathbb{F}))$ with the vertex set $V(\Gamma)=M_n(\mathbb{F})\setminus\mathbb{F}I$ and the edge set

$$E(\Gamma)=\{(X,Y)\in V(\Gamma)\times V(\Gamma): XY=YX \text{ and } X\neq Y\}$$

is called a commuting graph. It is known (see [8, 11]) that for $\mathbb{F}=\mathbb{R}$ and $n\geqslant 3$, Γ is connected with diameter four. We will denote by $d(A,B)$ the distance between vertices $A,B\in\Gamma$, i.e. the minimal number k for which there exist $(k+1)$-tuple of non-scalar matrices $A=X_0,\ X_1,\ \ldots,\ X_k=B$ such that X_i commutes with X_{i+1}. Such tuple forms a path which we will denote by $A=X_0-X_1-\ldots-X_k=B$.

The first lemma below may be proved in the same way as the corresponding lemma in [4] by noting that Hilbert matrix is a special case of Cauchy matrix $\left(\frac{1}{x_i-y_j}\right)_{ij}$, where $x_i,y_j\in\mathbb{R}$ with $x_i\neq y_j$ for all i and j. We omit the proof.

LEMMA 3. *Define matrices B_i by $B_ie_k = e_{k+1}$ for $k = 1,2,\ldots,n-2$, $B_ie_{n-1} = h_i$ and $B_ih_i = 0$, where h_i is the i-th column of the Hilbert matrix H. Then $d(B_i,B_j) = 4$ for $i \neq j$.*

The next lemma is new.

LEMMA 4. *Let h_i be the i-th column of the Hilbert matrix and let $B_1, B_2, \ldots, B_{n-2}$ be nilpotent matrices defined in Lemma 3. If $X_i \in \mathscr{C}(B_i)$ are non-scalar matrices and $A \in M_n(\mathbb{R})$ is an arbitrary matrix, then for every $T \in \mathscr{C}(\{A,X_1,\ldots,X_{n-2}\})$ there exists $\lambda \in \mathbb{R}$ such that $Th_i = \lambda h_i$ for every $i = 1,2,\ldots,(n-2)$.*

Proof. It suffices to prove the lemma for X_i maximal since we can find, by Lemma 1, maximal matrices $\hat{X}_i$ with $B_i \in \mathscr{C}(X_i) \subseteq \mathscr{C}(\hat{X}_i)$ for which clearly

$$\mathscr{C}(\{A,X_1,\ldots,X_{n-2}\}) \subseteq \mathscr{C}(\{A,\hat{X}_1,\ldots,\hat{X}_{n-2}\}).$$

So, assume X_i are maximal. Observe first that B_i, $i = 1,2,\ldots,(n-2)$, are non-derogatory matrices. So, since $X_i \in \mathscr{C}(B_i)$, we may conclude by (5) that X_i is a polynomial in B_i. Also, X_i is $\mathscr{C}$-equivalent to a square-zero matrix since it is maximal and since B_i is nilpotent. Then, after subtracting a suitable scalar and multiplying it with a suitable scalar we may assume that for every $i = 1,2,\ldots,(n-2)$,

$$X_i = B_i^{k_i} + \sum_{k=k_i+1}^{n-1} \lambda_{i,k} B_i^k \qquad \text{for some } k_i \geqslant \tfrac{n}{2} \text{ and some } \lambda_{i,k} \in \mathbb{R}.$$

As T commutes with X_i, the space $\operatorname{Ker} X_i = \operatorname{Lin}\{e_{n-k_i+1},\ldots,e_{n-1},h_i\}$ is invariant for T, i.e. $T(\operatorname{Ker} X_i) \subseteq \operatorname{Ker} X_i$. Recall that h_i are columns of the Hilbert matrix and the 2×2 minor at positions $(1,i),(1,j),(n,i),(n,j)$ of the Hilbert matrix is nonzero. Hence, the projections of h_i and h_j, where $i,j \in \{1,2,\ldots,n-2\}$, onto their 1-st and n-th components are linearly independent. From here we deduce easily that

$$W = \operatorname{Ker} X_1 \cap \operatorname{Ker} X_2 \cap \ldots \cap \operatorname{Ker} X_{n-2} = \operatorname{Lin}\{e_p,\ldots,e_{n-1}\}$$

for a suitable p, with $2 \leqslant p \leqslant \frac{n}{2}+1$. Clearly, W is invariant for T. Since $n \geqslant 7$, we have that $\dim W \geqslant 3$. Now,

$$\begin{aligned} Th_i = TX_ie_{n-k_i} = X_iTe_{n-k_i} &\in \operatorname{Im} X_i = \operatorname{Lin}\{e_{k_i+1},\ldots,e_{n-1},h_i\} \\ &\subseteq \operatorname{Lin}\{e_p,\ldots,e_{n-1},h_i\} = W + \mathbb{R}h_i. \end{aligned}$$

Hence, considering the quotient space $\mathbb{R}^n/W$, there exist $\lambda_i \in \mathbb{R}$, where $i = 1,2,\ldots,(n-2)$, such that

$$Th_i = \lambda_i h_i \pmod{W}.$$

Denote $\dot{h}_i = h_i + W \in \mathbb{R}^n/W$, the projection of h_i in the quotient space. Since $r = \dim(\mathbb{R}^n/W) = n - \dim W \leqslant n-3$ the $n-2$ vectors $\dot{h}_1,\ldots,\dot{h}_{n-2}$ must be linearly dependent. Recall that every $r\times r$ minor of the Hilbert matrix is nonzero. In particular, the vectors $\dot{h}_1,\ldots,\dot{h}_r$ form a basis for $\mathbb{R}^n/W$. Also, $\dot{h}_{r+1} = \sum_{i=1}^r \alpha_i \dot{h}_i$ for some $\alpha_i \in \mathbb{R}$, which are all nonzero because otherwise the appropriate $r\times r$ minor of the Hilbert matrix would be zero.

Hence also $h_{r+1}=\sum_{i=1}^{r}\alpha_i h_i \pmod{W}$ and after applying T, for which W is invariant, we have $\lambda_{r+1}h_{r+1}=Th_{r+1}=\sum_{i=1}^{r}\alpha_i\lambda_i h_i \pmod{W}$. Since $\alpha_i\neq 0$, we deduce that $\lambda_{r+1}=\lambda_1=\lambda_2=\ldots=\lambda_r$. Likewise we see that $\lambda_{r+k}=\lambda_1$ for $k=2,3,\ldots,n-2-r$. Therefore,

$$(T-\lambda_1 I)h_i\in W, \qquad i=1,2,\ldots,n-2.$$

Observe that $\{h_1,\ldots,h_{n-2},e_{n-2},e_{n-1}\}$ is a basis for $\mathbb{R}^n$ and that $e_{n-2},e_{n-1}\in W$. We deduce that

$$\operatorname{Im}(T-\lambda_1 I)\subseteq W.$$

Hence, $(T-\lambda_1 I)h_i=(T-\lambda_1 I)X_i e_{n-k_i}=X_i(T-\lambda_1 I)e_{n-k_i}\in X_i(\operatorname{Im}(T-\lambda_1 I))\subseteq X_i(W)=\{0\}$ for $i=1,2,\ldots,n-2$. $\square$

With the next five lemmas we will show that when $A\in M_n(\mathbb{R})$ is of a certain form, we may find $n-2$ nilpotent matrices $B_1,B_2,\ldots,B_{n-2}$ pairwise at distance 4 such that whatever the choice of non-scalar $X_i\in\mathscr{C}(B_i)$, the centralizer of the set $\{A,X_1,\ldots,X_{n-2}\}$ is trivial, i.e. $\mathscr{C}(\{A,X_1,\ldots,X_{n-2}\})=\mathbb{R}I$.

LEMMA 5. *Let $A=0\oplus\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]\oplus\cdots\oplus\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]$ and let $B_1,\ B_2,\ \ldots,\ B_{n-2}$ be nilpotent matrices defined in Lemma 3. If $X_i\in\mathscr{C}(B_i)$ are non-scalar matrices, then we have $\mathscr{C}(\{A,X_1,\ldots,X_{n-2}\})=\mathbb{R}I$.*

Proof. Observe first that n is an odd number. Suppose that there exists a non-scalar $T\in\mathscr{C}(\{A,X_1,\ldots,X_{n-2}\})$. By Lemma 4, we may subtract from T a suitable scalar matrix to achieve that $Th_i=0$ for $i=1,2,\ldots,n-2$, where, yet again, h_i is the i-th column of the Hilbert matrix. This already implies that $\operatorname{rk}T\leqslant 2$, and since $\{h_1,\ldots,h_{n-2},e_{n-1},e_n\}$ is a basis for $\mathbb{R}^n$, we only need to show that $Te_{n-1}=Te_n=0$.

Since $T=(t_{ij})_{ij}\in\mathscr{C}(A)$, it follows that $T=t_{11}\oplus\dot{T}$ where $t_{11}\in\mathbb{R}$ and

$$\dot{T}=\left(\begin{bmatrix}\alpha_{ij}&\beta_{ij}\\-\beta_{ij}&\alpha_{ij}\end{bmatrix}\right)_{1\leqslant i,j\leqslant\frac{n-1}{2}}\in M_{n-1}(\mathbb{R}).$$

Observe that the rank of $\dot{T}$ is even. From $\operatorname{rk}T\leqslant 2$, either $\dot{T}=0$ or $t_{11}=0$. In the latter case, we may conclude that the rank of T is even. Also, $Te_1=0$ and therefore $\operatorname{Ker}T$ contains the $n-1$ dimensional space $\{e_1,h_1,\ldots,h_{n-2}\}$. So $\operatorname{rk}T\leqslant 1$ and thus $T=0$. In the former case, $T=t_{11}\oplus 0$ and therefore $Te_{n-1}=0=Te_n$. $\square$

The next four lemmas may be proved in the same way as corresponding lemmas in [4]. Nevertheless, for the sake of completeness and since Lemma 4 holds for an arbitrary matrix $A\in M_n(\mathbb{R})$, we will present new and shorter proofs.

LEMMA 6. *Let $A=I_r\oplus 0_{n-r}\in M_n(\mathbb{R})$ be an idempotent with $2\leqslant r=\operatorname{rk}A\leqslant\frac{n}{2}$ and let $B_1,\ B_2,\ \ldots,\ B_{n-2}$ be nilpotent matrices defined in Lemma 3. If $X_i\in\mathscr{C}(B_i)$ are non-scalar matrices, then we have $\mathscr{C}(\{A,X_1,\ldots,X_{n-2}\})=\mathbb{R}I$.*

Proof. Suppose that there exists a non-scalar $T \in \mathscr{C}(\{A, X_1, \ldots, X_{n-2}\})$. Without loss of generality we may by Lemma 4 assume that $Th_i = 0$ for $i = 1, 2, \ldots, n-2$, where h_i is the i-th column of the Hilbert matrix. Also, since $T = (t_{ij})_{ij} \in \mathscr{C}(A)$, it follows that $T = \dot{T} \oplus \ddot{T}$ where $\dot{T} \in M_r(\mathbb{R})$ and $\ddot{T} \in M_{n-r}(\mathbb{R})$. We have $Th_i = (\dot{T}\dot{h}_i) \oplus (\ddot{T}\ddot{h}_i)$ and hence $\dot{T}\dot{h}_i = 0$ and $\ddot{T}\ddot{h}_i = 0$ for every $i = 1, 2, \ldots, n-2$. Recall that $n \geqslant 7$. So, $n - 2 > \frac{n}{2}$. Since $2 \leqslant r \leqslant \frac{n}{2}$ and since each $r \times r$ and $(n-r) \times (n-r)$ minor of the Hilbert matrix is nonzero, the $n-2$ vectors $\dot{h}_1, \ldots, \dot{h}_{n-2}$ span the whole space $\mathbb{R}^r$, and the $n-2$ vectors $\ddot{h}_1, \ldots, \ddot{h}_{n-2}$ span the whole space $\mathbb{R}^{n-r}$. Thus, $\dot{T} = 0$ and $\ddot{T} = 0$ and therefore $T = 0$, a contradiction. □

LEMMA 7. *Let*

$$A = \begin{bmatrix} 0_r & 0 & 0 \\ 0 & 0_{n-2r} & 0 \\ I_r & 0 & 0_r \end{bmatrix} \in M_n(\mathbb{R})$$

be a square-zero matrix with $2 \leqslant r = \operatorname{rk} A \leqslant \frac{n}{2}$ and let $B_1, B_2, \ldots, B_{n-2}$ be nilpotent matrices defined in Lemma 3. If $X_i \in \mathscr{C}(B_i)$ are non-scalar matrices, then we have $\mathscr{C}(\{A, X_1, \ldots, X_{n-2}\}) = \mathbb{R}I$.

Proof. Suppose that there exists a non-scalar $T \in \mathscr{C}(\{A, X_1, \ldots, X_{n-2}\})$. By Lemma 4 we may assume that $Th_i = 0$ for $i = 1, 2, \ldots, n-2$. Since $T = (t_{ij})_{ij} \in \mathscr{C}(A)$, it follows that

$$T = \begin{pmatrix} \dot{T} & 0 & 0 \\ * & * & 0 \\ * & * & \dot{T} \end{pmatrix}$$

where $\dot{T} \in M_r(\mathbb{R})$. We have $\dot{T}\dot{h}_i = 0$ for $i = 1, 2, \ldots, n-2$ which yields $\dot{T} = 0$ since $n - 2 > \frac{n}{2} \geqslant r$. By $r \geqslant 2$, we have $Te_n = 0 = Te_{n-1}$. Since $\{h_1, \ldots, h_{n-2}, e_{n-1}, e_n\}$ is a basis for the space $\mathbb{R}^n$, we may conclude that $T = 0$, a contradiction. □

LEMMA 8. *Let $s \geqslant 0$, let $A = J_3(0) \oplus \bigoplus_{i=1}^{s} J_2(0) \oplus 0_{n-3-2s}$ where the middle term is omitted if $s = 0$, and let $B_1, B_2, \ldots, B_{n-2}$ be nilpotent matrices defined in Lemma 3. If $X_i \in \mathscr{C}(B_i)$ are non-scalar matrices, then we have $\mathscr{C}(\{A, X_1, \ldots, X_{n-2}\}) = \mathbb{R}I$.*

Proof. Suppose that there exists a non-scalar $T \in \mathscr{C}(\{A, X_1, \ldots, X_{n-2}\})$. By Lemma 4, we may assume that $Th_i = 0$ for $i = 1, 2, \ldots, n-2$ which yields $\operatorname{rk} T \leqslant 2$. Since $T = (t_{ij})_{ij} \in \mathscr{C}(A)$, we have $t_{11} = t_{22} = t_{33}$ and $t_{12} = t_{23}$. Also, the third row of T can have a nonzero entry only at position $(3, 3)$ and the first column can have a nonzero entry only at position $(1, 1)$. Further,

$$t_{(2i)2} = t_{(2i+1)3}, \quad t_{(2i+1)2} = 0, \quad t_{j2} = 0 \quad \text{for } i = 2, \ldots, s+1, \; j \geqslant 2s+4. \tag{7}$$

It follows that $t_{11} = t_{22} = t_{33} = 0$ for otherwise $\operatorname{rk} T \geqslant 3$. Thus, $Te_1 = 0$. Therefore, $T(\operatorname{Lin}\{h_1, \ldots, h_{n-2}, e_1\}) = \{0\}$ and thus $\operatorname{rk} T \leqslant 1$. Note that $\{e_1, e_3, h_1, \ldots, h_{n-2}\}$ forms a basis for the space $\mathbb{R}^n$. So, if $Te_3 = 0$, then $T = 0$, a contradiction. If $Te_2 \neq 0$, then there exists a nonzero $\alpha \in \mathbb{R}$, such that $Te_2 = \alpha Te_3$. Since $t_{22} = 0$, $t_{12} = t_{23}$, and since (7) holds, this is possible only if $Te_2 = 0 = Te_3$, a contradiction. So, $Te_2 = 0$. Since $\{e_1, e_2, h_1, \dots, h_{n-2}\}$ is a basis for the space $\mathbb{R}^n$, we may again conclude that $T = 0$, a contradiction. $\square$

LEMMA 9. *Let $A = 1 \oplus J_2(0) \oplus 0_{n-3}$ and let B_1, B_2, $\dots$, B_{n-2} be nilpotent matrices defined in Lemma 3. If $X_i \in \mathscr{C}(B_i)$ are non-scalar matrices, then we have $\mathscr{C}(\{A, X_1, \dots, X_{n-2}\}) = \mathbb{R}I$.*

Proof. Suppose that there exists a non-scalar $T \in \mathscr{C}(\{A, X_1, \dots, X_{n-2}\})$. By Lemma 4 we may assume that $Th_i = 0$ for $i = 1, 2, \dots, n-2$. So, $\operatorname{rk} T \leqslant 2$. Since $T = (t_{ij})_{ij} \in \mathscr{C}(A)$, it follows that $T = t_{11} \oplus \dot{T}$ where the second column and the third row of T can have a nonzero entry $t_{22} = t_{33}$ only at positions $(2,2)$ and $(3,3)$, respectively. Suppose first $t_{11} = 0$. Then $Te_1 = 0$ and thus $\operatorname{rk} T \leqslant 1$. Let $Te_2 \neq 0$. Then there exists a nonzero $\alpha \in \mathbb{R}$ such that $Te_2 = \alpha Te_3$. This implies, $t_{33} = 0$. Since $t_{22} = t_{33}$, we may conclude that the second column of T vanishes, i.e. $Te_2 = 0$, a contradiction. It follows that $T(\operatorname{Lin}\{e_1, e_2, h_1, \dots, h_{n-2}\}) = \{0\}$ which yields $T = 0$, a contradiction.

Suppose now $t_{11} \neq 0$. Since $\operatorname{rk} T \leqslant 2$, it follows that $\operatorname{rk} \dot{T} \leqslant 1$. So, $\dot{T}\dot{e}_2 = \alpha \dot{T}\dot{e}_3$ for some $\alpha \in \mathbb{R}$ and thus also $Te_2 = \alpha Te_3$. Suppose $Te_2 \neq 0$. So, $\alpha \neq 0$ and therefore $t_{33} = 0$ which yields $Te_2 = 0$, a contradiction. So, $Te_2 = 0$ and thus $\operatorname{rk} T \leqslant 1$. If $Te_1 \neq 0$, there exits $\beta \in \mathbb{R}$, $\beta \neq 0$, such that $Te_1 = \beta Te_3$. But then $t_{11} = 0$, since $t_{13} = 0$, and thus $Te_1 = 0$, a contradiction. Again, it follows that $T(\operatorname{Lin}\{e_1, e_2, h_1, \dots, h_{n-2}\}) = \{0\}$ and therefore $T = 0$, a contradiction. $\square$

The following key lemma corresponds to Lemma 2.8 in [4].

LEMMA 10. *Let n be an odd number. The following two statements are equivalent for a non-scalar $A \in M_n(\mathbb{R})$.*

(i) A is $\mathscr{C}$-equivalent to a rank-one matrix.

(ii) For every $n-2$ tuple of matrices B_1, $B_2, \dots,$ B_{n-2} which are pairwise at distance 4, there exist a non-scalar matrix $Y \in \mathscr{C}(A)$ and paths $B_i - X_{ij} - Y - Z_{ij} - B_j$ of length 4 connecting B_i to B_j, $i \neq j$, with Y in the middle.

Proof. Observe first that every real $n \times n$ matrix has a real eigenvalue since we assumed that n is odd. Hence, we may prove that (i) implies (ii) in the same way as in the proof of Lemma 2.8 in [4].

Conversely, let us assume that (i) does not hold. We will distinguish different cases. Let us first list all of them.

Suppose first $A = M$ is a maximal matrix.

(a) By assumption $A = M$ is not $\mathscr{C}$-equivalent to a rank-one matrix.

Suppose now A is not maximal. Then by Lemma 1 there exists $A_1 \in M_n(\mathbb{R})$, an immediate predecessor of some maximal matrix M, with $\mathscr{C}(A) \subseteq \mathscr{C}(A_1)$. This gives us four more cases.

(b) There exists a maximal matrix M, which is not $\mathscr{C}$-equivalent to a matrix of rank-one, with $\mathscr{C}(A_1) \subseteq \mathscr{C}(M)$.

If (b) is not true, that is each maximal matrix M with $\mathscr{C}(A_1) \subseteq \mathscr{C}(M)$ is $\mathscr{C}$-equivalent to a matrix of rank-one, then by Lemma 2 we obtain the last three cases.

(c) The signature of A_1 equals $(3, 2^s, 1^t)$ with $s, t \geqslant 0$ and $3 + 2s + t = n$ and all the eigenvalues of A_1 are the same.

(d) The signature of A_1 equals $(2, 1^{n-2})$ and A_1 has two distinct eigenvalues, one with algebraic multiplicity 1.

(e) The signature of A_1 equals $(2^{(n-1)/2}, 1)$, where A_1 has two conjugate complex eigenvalues and a real eigenvalue with algebraic multiplicity 1.

Let us now show that for all cases (a)–(e), statement (ii) does not hold. In cases (a) and (b), M belongs by [5, Theorem 3.2] to one of the following classes:

(1) M is $\mathscr{C}$-equivalent to an idempotent, or (2) M is $\mathscr{C}$-equivalent to a square-zero matrix, or (3) M is similar to $C \oplus C \oplus \ldots \oplus C$ where C is a companion matrix of an irreducible polynomial such that there is no proper intermediate field between $\mathbb{R}$ and $\mathbb{R}[C]$. In case (1) we may assume that M is an idempotent with rank between 2 and $\frac{n}{2}$, and in case (2) we may assume that M is a square-zero matrix with rank between 2 and $\frac{n}{2}$. In case (3) we may without loss of generality assume that matrix M is of the following form $M = C \oplus C \oplus \ldots \oplus C$ where $C = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. So, in the last case, the dimension of M is even and therefore, by assumption, this case can not occur. By Lemmas 6 and 7, we may conclude that in cases (1) and (2) there exist $n-2$ nilpotent matrices $B_1, B_2, \ldots, B_{n-2}$ pairwise at distance 4 such that whatever the choice of non-scalar $X_i \in \mathscr{C}(B_i)$, we always have $\mathscr{C}(\{M, X_1, \ldots, X_{n-2}\}) = \mathbb{R}I$. Since $\mathscr{C}(A) \subseteq \mathscr{C}(M)$, we also have $\mathscr{C}(\{A, X_1, \ldots, X_{n-2}\}) = \mathbb{R}I$. Hence, no matter the choice of $X_{ij} \in \mathscr{C}(B_i) \setminus \mathbb{R}I$ there does not exist a non-scalar Y which would commute with A and with all X_{ij}, $i = 1, 2, \ldots, n-2$. To summarize, for cases (1) and (2), (ii) does not hold, and case (3) can not occur. We may conclude that for cases (a) and (b), statement (ii) does not hold.

If (c), or (d), or (e) holds, then we use Lemmas 8, 9, 5, respectively, and repeat the arguments above to see that (ii) does not hold. $\square$

Let us now show that Φ preserves the set of rank-one matrices modulo $\mathscr{C}$-equivalence. To do this we will apply Lemma 10 while taking into account that map Φ preserves scalar matrices in both directions (see Lemma 3.1 in [4]).

LEMMA 11. *Let n be an odd number. If $\operatorname{rk} A = 1$, $A \in M_n(\mathbb{R})$, then $\Phi(A)$ is $\mathscr{C}$-equivalent to a matrix of rank-one.*

Proof. Let $A \in M_n(\mathbb{R})$ be of rank one and let $B = \Phi(A)$. Choose any tuple of $n-2$ matrices $B_i \in M_n(\mathbb{R})$, $i = 1, 2, \ldots, n-2$, which are pairwise at distance four (recall that such matrices exist by Lemma 3). By surjectivity of Φ, there exist $n-2$ matrices $A_i \in M_n(\mathbb{R})$, with $\Phi(A_i) = B_i$ for each i. Recall that if $n \geqslant 3$, $\Gamma = \Gamma(M_n(\mathbb{R}))$ is a connected graph with diameter four (see [8, 11]). So, since Φ does not increase the distance, we have $d(A_i, A_j) = 4$ for $i \neq j$. Since $\operatorname{rk} A = 1$, Lemma 10 implies that there exist a non-scalar matrix $Y \in \mathscr{C}(A)$ and paths $A_i - X_{ij} - Y - Z_{ij} - A_j$. It follows that $B_i = \Phi(A_i) - \Phi(X_{ij}) - \Phi(Y) - \Phi(Z_{ij}) - \Phi(A_j) = B_j$ are paths of length at most four and hence of length four since $d(B_i,B_j)=4$. These paths that connect B_i and B_j have the same matrix $\Phi(Y)$ in the middle, and this matrix $\Phi(Y)$ also commutes with $\Phi(A)=B$. By applying again Lemma 10, we may conclude that B is $\mathscr{C}$-equivalent to a matrix of rank-one. □

Since we assumed that n is odd, every real $n\times n$ matrix has a real eigenvalue and thus a real eigenvector. So, all lemmas from the third section of [4] can be used for the set of real $n\times n$ matrices with n odd. For example, in the proof of [4, Lemma 3.3] it is required that every matrix C commutes with a rank-one matrix and this is equivalent to the fact that C has an eigenvector. Therefore, from now on we will only briefly sketch the lengthy arguments from the third section of [4].

Following [4], let us now modify Φ to $\hat{\Phi}$, which maps rank-one matrices to rank-one matrices and annihilates the zero matrix as follows. Let $m(\lambda_X)$ be the algebraic multiplicity of an eigenvalue $\lambda_X\in \mathrm{Sp}(\Phi(X))$. We then define the map $\hat{\Phi}: M_n(\mathbb{R})\to M_n(\mathbb{R})$ with

$$\hat{\Phi}(X)=\begin{cases}\Phi(X)-\lambda_X I, & \operatorname{rk}X=1,\ \lambda_X\in \mathrm{Sp}(\Phi(X)) \text{ with } m(\lambda_X)>1\\ 0, & X=0\\ \Phi(X), & \text{otherwise}\end{cases}.$$

Observe that $\mathscr{C}(\hat{\Phi}(X))=\mathscr{C}(\Phi(X))$ for every $X\in M_n(\mathbb{R})$. So, $\hat{\Phi}$ also preserves commutativity but it is surjective only modulo $\mathscr{C}$-equivalence, i.e. for every matrix $Y\in M_n(\mathbb{R})$ there exists a matrix $X\in M_n(\mathbb{R})$ such that $\mathscr{C}(\hat{\Phi}(X))=\mathscr{C}(Y)$.

If $R\in M_n(\mathbb{R})$ is a rank-one nilpotent, then we may show as in the proof of Lemma 3.7 in [4] that $\hat{\Phi}(R)$ is also a rank-one nilpotent. Recall that $A\in M_n(\mathbb{R})$ is of rank-one if and only if $A=xf^t$ for some nonzero vectors $x,f\in\mathbb{R}^n$ where f^t denotes the transpose. As in [4] we may replace, if necessary, $\hat{\Phi}$ with the map $X\mapsto \hat{\Phi}(X)^t$ and then prove the following lemma in the same way as [4, Lemma 3.12].

LEMMA 12. *There exist maps $\phi,\psi:\mathbb{R}^n\to\mathbb{R}^n$ with the following properties:*

(i) $\phi(\mathbb{R}x)\subseteq\mathbb{R}\phi(x)$ *and* $\psi(\mathbb{R}f)\subseteq\mathbb{R}\psi(f)$ *for every* $x,f\in\mathbb{R}\setminus\{0\}$;

(ii) Both ϕ *and* ψ *annihilate only the zero vector;*

(iii) $\mathbb{R}\phi(\mathbb{R}^n)=\mathbb{R}\psi(\mathbb{R}^n)=\mathbb{R}^n$;

(iv) For every rank-one nilpotent xf^t *we have*

$$\hat{\Phi}(xf^t)\in\mathbb{R}\phi(x)\psi(f)^t \quad \textit{and} \quad \psi(f)^t\phi(x)=0.$$

By Lemma 12 it follows that maps ϕ and ψ induce a well-defined surjections on the projective space $\mathbb{PR}^n=\{[x]=\mathbb{R}x : x\in\mathbb{R}^n\setminus\{0\}\}$ which we again denote by ϕ and ψ, respectively. As in the proof of Lemma 3.15 in [4] we may prove that $\phi,\psi:\mathbb{PR}^n\to\mathbb{PR}^n$ are also injective maps.

Recall that a map $\chi:\mathbb{PR}^n\to\mathbb{PR}^n$ is a projective morphism if for every $[x],[y],[z]\in\mathbb{PR}^n$ where $[x]\in[y]+[z]$, we have $\chi([x])\in\chi([y])+\chi([z])$. As in the proof of Lemma 3.16 in [4], we observe that maps ϕ and ψ are projective morphisms.

We are now in position to conclude the proof of Theorem 2.

Proof of Theorem 2. Since the map ϕ is a projective morphism, it follows by the fundamental theorem of projective geometry (see for example [6]) that $\phi([x]) = [A(x^{\sigma})]$ for some field homomorphism $\sigma\colon \mathbb{R} \to \mathbb{R}$ and some linear map $A\colon \mathbb{R}^n \to \mathbb{R}^n$. Recall that the map ϕ is bijective, and therefore A is invertible. Also, $\sigma = \mathrm{id}_{\mathbb{R}}$ because the identity is the only nonzero homomorphism of $\mathbb{R}$ (see [1]). Likewise $\psi([f]) = [Bf]$ for some invertible linear B.

We replace $\hat{\Phi}$ by $X \mapsto A^{-1}\hat{\Phi}(X)A$ and denote the new map again by $\hat{\Phi}$. The new map satisfies $\hat{\Phi}(xf^t) \in x(\tilde{B}f)^t$ where $\tilde{B} = A^tB$. Note that $\tilde{B}$ is invertible. Since $\hat{\Phi}$ maps rank-one nilpotents to rank-one nilpotents, we have that $f^tx = 0$ implies $(\tilde{B}f)^tx = 0$ for every vector $x \in \mathbb{R}^n$. Therefore $\tilde{B}f \in \mathbb{R}f$ for every vector $f \in \mathbb{R}^n$. If we take f equal to e_i, $i = 1,2,\dots,n$, and then equal to $e_i + e_{i+1}$, $i = 1,2,\dots,n-1$, we deduce that $\tilde{B}$ is a scalar matrix. Thus

$$\hat{\Phi}(xf^t) \in \mathbb{R}xf^t \setminus \{0\} \tag{8}$$

for every rank-one nilpotent xf^t.

It is easy to see that the only rank-one matrices which commute with each of the rank-one nilpotents E_{2i}, E_{i2}, $i = 3,4,\dots,n$, are in the set $\mathbb{R}E_{11}$. Hence by (8), $\hat{\Phi}(\mathbb{R}E_{11} \setminus \{0\}) \subseteq \mathbb{R}E_{11} \setminus \{0\}$. Since every rank-one idempotent P is similar to E_{11}, we likewise deduce that $\hat{\Phi}(\mathbb{R}P \setminus \{0\}) \subseteq \mathbb{R}P \setminus \{0\}$. Hence, for every rank-one matrix X, we have $\hat{\Phi}(X) = \alpha(X)X$ where $\alpha : M_n(\mathbb{R}) \to \mathbb{R} \setminus \{0\}$ is a suitable function. From here the main result follows easily. $\square$

REFERENCES

[1] J. ACZÉL, J. DHOMBRES, *Functional equations in several variables*, Encyclopedia Math. Appl. **31**, Cambridge University Press, Cambridge, 1989.

[2] J. BRAČIČ AND B. KUZMA, *Reflexivity defect of spaces of linear operators*, Linear Algebra Appl. **430** (2009), no. 1, 344–359.

[3] M. D. CHOI, A. A. JAFARIAN, AND H. RADJAVI, *Linear maps preserving commutativity*, Linear Algebra Appl. **87** (1987), 227–241.

[4] G. DOLINAR AND B. KUZMA, *Homomorphisms of commutativity relation*, Linear Multilinear Algebra **64** (2016), no. 5, 897–922.

[5] G. DOLINAR, A. E. GUTERMAN, B. KUZMA, AND P. OBLAK, *Extremal maximal centralizers*, Linear Algebra Appl. **438** (2013), no. 7, 2904–2910.

[6] C.-A. FAURE, *An elementary proof of the fundamental theorem of projective geometry*, Geom. Dedicata **90** (2002), no. 1, 145–151.

[7] A. FOŠNER, *Commutativity preserving-maps on $M_n(\mathbb{R})$*, Glas. Mat. **44** (2009), no. 1, 127–140.

[8] J. M. GRAU, A. OLLER-MARCÉN, AND C. TASIS, *On the diameter of the commuting graph of the full matrix ring over the real numbers*, Bull. Iranian Math. Soc. **43** (2017), no. 1, 217–221.

[9] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.

[10] N. JACOBSON, *Lectures in algebra*, volume II: linear algebra. Vol. 31, Graduate texts in mathematics, Springer-Verlag, New York, 1953.

[11] C. MIGUEL, *A note on a conjecture about commuting graphs*, Linear Algebra Appl. **438** (2013), no. 12, 4750–4756.

[12] P. ŠEMRL, *Non-linear commutativity preserving maps*, Acta. Sci. Math. (Szeged) **71** (2005), 781–819.
[13] P. ŠEMRL, *Commutativity preserving maps*, Linear Algebra Appl. **429** (2008), 1051–1070.

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