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CHAOTIC OPERATORS ON HYPERGROUPS

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Abstract. In this paper, we initiate a study of chaos, in the sense of Devaney, and topological transitivity on the L^p space of hypergroups, and give some sufficient and necessary conditions for weighted translation operators on hypergroups to be chaotic and transitive in terms of the Haar measure, weight functions and center elements of hypergroups. A characterization of topologically mixing weighted translations on hypergroups is also given.

1. Introduction

Recently, chaotic, topologically transitive and mixing weighted translation operators on locally compact groups are characterized in [4, 5], which subsumes some previous works on the discrete group $\mathbb{Z}$ in [6, 17]. We note that locally compact groups are a special case of hypergroups which were introduced in [8, 12, 18]. Roughly speaking, a hypergroup is a locally compact Hausdorff space with a convolution and involution such that the corresponding space of regular Borel measures is an associative Banach algebra. For instance, the double coset space $G//H = \{HgH : g \in G\}$, in which H is a non-normal compact subgroup of the locally compact group G, does not inherit a group structure from G. However, the space of regular Borel measures on $G//H$ has an algebra structure induced by that of G. Classical examples of hypergroups include locally compact groups, the double coset spaces, the dual object of a compact group, the polynomial hypergroups (see [3]). Hence, naturally we intend to consider linear chaos and topological dynamics on a wider setting of hypergroups.

Let X be a separable Banach space. An operator T on X is called *topologically transitive* if for each non-empty open sets $U,V \subseteq X$, there exists some $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. If $T^n(U) \cap V \neq \emptyset$ holds from some n onwards, then T is called *topologically mixing*. Following Devaney [7], we call T chaotic if it is topologically transitive and the set of periodic elements of T, denoted by $\mathscr{P}(T) := \{x \in X : \exists\, n \in \mathbb{N} \text{ s.t. } T^n x = x\}$, is dense in X. In this setting, topological transitivity coincides with hypercyclicity. An operator T on X is called *hypercyclic* if there exists a vector $x \in X$ such that the set $\{x, Tx, T^2x, \cdots, T^nx, \cdots\}$ is dense in X. One of the criterions for

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topological transitivity (hypercyclicity) of T is the so called *blow up/collapse property* [10, 11], that is, for any non-empty open sets U, V and W in X with $0 \in W$, there exists some $n \in \mathbb{N}$ such that $T^n(U) \cap W \neq \emptyset$ and $T^n(W) \cap V \neq \emptyset$. In this paper, we show that for the operators on L^p spaces related to hypergroups, the blow up/collapse property and topological transitivity are equivalent.

In the investigation on linear dynamics, the weighted shifts on $\ell^p(\mathbb{N}_0)$ or $\ell^p(\mathbb{Z})$ are concrete examples to demonstrate the theory of transitivity and linear chaos. For unilateral shifts T on $\ell^p(\mathbb{N}_0)$, S. Rolewicz [15] showed that αT is topologically transitive whenever $|\alpha| > 1$. H. Salas characterized transitive bilateral weighted shifts on $\ell^p(\mathbb{Z})$ in [17]. Also, K. Costakis and M. Sambarino in [6] gave a sufficient and necessary condition for bilateral weighted shifts on $\ell^p(\mathbb{Z})$ to be mixing. Linear chaos and topological dynamics have been studied intensively in the past three decades. For more details, refer to [2, 11, 13].

In this paper, we investigate topologically transitive, mixing and chaotic weighted translation operators on the L^p space of hypergroups, extending the results on $\mathbb{Z}$ and locally compact groups. In Section 2, we give some preliminaries of hypergroups, and introduce the center elements of hypergroups, and the corresponding operators. In Section 3, we will give some sufficient and necessary conditions for weighted translation operators on hypergroups to be chaotic, mixing and transitive in terms of weight functions and center elements of hypergroups.

2. Center of hypergroups

In this section, we recall the definition of hypergroups and some related topics. We refer to the classical papers and book [3, 8, 12, 18] for more details about hypergroups (see also [14] and [20]). Let K be a locally compact Hausdorff space, and $M(K)$ be the Banach space of regular complex Borel measures on K. The predual of $M(K)$ is the Banach space $C_0(K)$ of complex-valued continuous functions on K vanishing at infinity. The support of a measure $\mu \in M(K)$ and the Dirac measure at $x \in K$ are denoted by $\operatorname{supp}(\mu)$ and δ_x, respectively.

DEFINITION 2.1. Suppose that K is a locally compact Hausdorff space, $(\mu, \nu) \mapsto \mu * \nu$ is a bilinear positive-continuous mapping from $M(K) \times M(K)$ into $M(K)$ (called *convolution*), and $x \mapsto x^-$ is an involutive homeomorphism on K (called *involution*) such that:

1. $(M(K), +, *)$ is a complex associative algebra;
2. for all $x, y \in K$, $\delta_x * \delta_y$ is a probability measure with compact support;
3. there exists a (necessarily unique) element $e \in K$ (called *identity*) such that for all $x \in K$, $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$;
4. for all $x, y \in K$, $e \in \operatorname{supp}(\delta_x * \delta_y)$ if and only if $x = y^-$;
5. for all $x, y \in K$, $(\delta_x * \delta_y)(\check{\psi}) = (\delta_{y^-} * \delta_{x^-})(\psi)$, where $\psi \in C_0(K)$ and $\check{\psi}(t) := \psi(t^-) \quad (t \in K)$;

6. the mapping $(x,y) \mapsto \text{supp}(\delta_x * \delta_y)$ from $K \times K$ into $\mathbf{C}(K)$ is continuous, where $\mathbf{C}(X)$ is the space of all non-empty compact subsets of K equipped with the Michael topology.

Then $(K, *, ^-, e)$ is called a *hypergroup*.

Throughout this paper, K is a hypergroup with convolution $*$, involution $^-$ and identity e. For $\mu, \sigma \in M(K)$, the convolution $\mu * \sigma$ is given by

$$\int_K f d(\mu * \sigma) = \int_K \int_K \int_K f d(\delta_x * \delta_y) d\mu(x) d\sigma(y) \qquad (f \in C_0(K)).$$

If K is a locally compact group, then it is a hypergroup with the convolution $\delta_x * \delta_y = \delta_{xy}$ and the inverse mapping $x \mapsto x^{-1}$ as involution.

Let $f : K \to \mathbb{C}$ be a Borel measurable function. For each $x_1, \ldots, x_n \in K$, we put

$$f(x_1 * \ldots * x_n) := \int_K f d(\delta_{x_1} * \cdots * \delta_{x_n}),$$

if the integral exists. So by [12, 3.1F], we have

$$f_{x_1}(x_2 * \ldots * x_n) = f(x_1 * \ldots * x_n) = f^{x_n}(x_1 * \ldots * x_{n-1}),$$

where for any $x, y \in K$, $f^y(x) = f_x(y) := f(x * y)$. Given a measure $\mu \in M(K)$ and a Borel function f on K, we define the convolution $f * \mu$ by

$$f * \mu(x) = \int_K f(x * y^-) d\mu(y) \qquad (x, y \in K)$$

if the integral exists. In particular, $f * \delta_y(x) = f(x * y^-) = f^{y^-}(x)$ is viewed as the right translation of f by y^-.

In the sequel, we will study weighted translation operators on the L^p space of K with respect to a right Haar measure. A non-zero non-negative regular Borel measure λ of K is called a *(right) Haar measure* if for each $x \in K$, $\lambda * \delta_x = \lambda$. It is not known whether every hypergroup has a Haar measure. However, it is known that compact hypergroups, commutative hypergroups, discrete hypergroups, double coset hypergroups and nilpotent hypergroups admit a Haar measure (see [1, 12, 19]). In what follows, we assume that K is a hypergroup with a right Haar measure λ. For all $1 \leqslant p < \infty$, we denote by $L^p(K)$ the L^p space with respect to the Haar measure λ, where as usual, for each $f \in L^p(K)$, $\|f\|_p := (\int_K |f|^p d\lambda)^{\frac{1}{p}}$. We recall the definition of the center of K.

DEFINITION 2.2. The center of a hypergroup K is defined by

$$\text{Ma}(K) := \{x \in K : \delta_x * \delta_{x^-} = \delta_e = \delta_{x^-} * \delta_x\}.$$

The study on the center of hypergroups was initiated by Dunkl [8, 1.6], and called the *maximum subgroup* in an equivalent definition by Jewett [12, 10.4]. It should be noted that if K is a locally compact group, then $\text{Ma}(K) = K$.

EXAMPLE 2.3. Let G be a central group i.e. $G/Z \equiv \mathrm{Inn}(G)$ is compact, where $Z := \{x \in G : \text{for each } y \in G, xy = yx\}$ and $\mathrm{Inn}(G)$ is the inner automorphisms group of G with the normalized right Haar measure σ. Then $(x,s) \mapsto s(x)$ from $G \times \mathrm{Inn}(G)$ into G is a continuous action of the compact group $\mathrm{Inn}(G)$ on G. For each $x \in G$ and $s \in \mathrm{Inn}(G)$, we put $[x] := \{s(x) : s \in \mathrm{Inn}(G)\}$, and $G_I := \{[x] : x \in G\}$. Then G_I with the operation

$$(\delta_{[x]} * \delta_{[y]})(\phi) := \int_{\mathrm{Inn}(G)} \phi([s(x)y])d\sigma(s), \qquad (\phi \in C_0(G_I))$$

as convolution and $[x]^- := [x^{-1}]$ as involution, is a hypergroup and admits a (right) Haar measure (for more details, see [12, 8.3]). We have $\mathrm{Ma}(G_I) = \{[z] : z \in Z\}$ [16]. In particular, if $G = SU(2)$, then $G_I = [0, 2\pi]$ and $\mathrm{Ma}(G_I) = \{0, 2\pi\}$. Also if $G = SO(2)$, then $G_I = [0, \pi]$ and $\mathrm{Ma}(G_I) = \{0\}$.

EXAMPLE 2.4. Let $\mathbb{Z}_+^\infty$ be the one-point compactification of $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$. Fix a prime number p. For all $m, n \in \mathbb{Z}_+$, we define

$$\delta_m * \delta_n := \begin{cases} \delta_{\min\{m,n\}}, & m \neq n \\ \frac{p-2}{p-1}\delta_n + \sum_{k=n+1}^{\infty} \frac{1}{p^{k-n}}\delta_k, & m = n \end{cases}$$

and $\delta_m * \delta_\infty = \delta_\infty * \delta_m := \delta_m$. Then $\mathbb{Z}_+^\infty$ is a Hermitian hypergroup with the identity ∞. This important class of compact countable hypergroups was introduced by Dunkl and Ramirez in [9]. The dual of $\mathbb{Z}_+^\infty$ equals to $\{\chi_n : n = 0, 1, 2, ..\}$, where the function χ_n is defined on $\mathbb{Z}_+^\infty$ by

$$\chi_n(m) := \begin{cases} 1, & m \geqslant n \text{ or } m = \infty \\ \frac{-1}{p-1}, & m = n-1 \\ 0, & m \leqslant n-2. \end{cases}$$

Then we have the following convolution on $\widehat{\mathbb{Z}_+^\infty} \cong \mathbb{Z}_+$:

$$\delta_{\chi_n} * \delta_{\chi_m} := \begin{cases} \delta_{\chi_{\max\{m,n\}}}, & n \neq m \\ \frac{1}{p^{n-1}(p-1)}\delta_{\chi_0} + \sum_{k=1}^{n-1} p^{k-n}\delta_{\chi_k} + \frac{p-2}{p-1}\delta_{\chi_n}, & n = m \end{cases}$$

with χ_0 as the identity. If we identify χ_n and n, then $\mathbb{Z}_+$ is a Hermitian hypergroup (see [9]). We have $\mathrm{Ma}(\mathbb{Z}_+) = \{0\}$ and $\mathrm{Ma}(\mathbb{Z}_+^\infty) = \{\infty\}$.

The center elements of a hypergroup have many nice properties. For instance, for each $a \in \mathrm{Ma}(K)$ and $y \in K$, the sets $\mathrm{supp}(\delta_a * \delta_y)$ and $\mathrm{supp}(\delta_y * \delta_a)$ are singletons by [12, 10.4B]. However, in general, $\mathrm{supp}(\delta_x * \delta_y)$ does not need to be a singleton for $x, y \in K$. Hence if $a \in \mathrm{Ma}(K)$ and $n \in \mathbb{N}$, we denote the unique element of the singleton $\mathrm{supp}(\overbrace{\delta_a * \ldots * \delta_a}^{n-\text{times}})$ by a^n. Also, if we write

$$f * \delta_a^n = f * (\overbrace{\delta_a * \cdots * \delta_a}^{n-\text{times}}),$$

then $f*\delta_a^n = f*\delta_{a^n}$. Also we have the right invariance of the norm of a Borel function after translation by $a \in \mathrm{Ma}(K)$. In general, for $x \in K$ and $f \in L^p(K)$, one only has $\|f^x\|_p \leqslant \|f\|_p$.

LEMMA 2.5. *Let $a \in Ma(K)$ and $f \in L^p(K)$. Then $\|f^a\|_p = \|f\|_p$.*

Proof. Assume $a \in \mathrm{Ma}(K)$ and $f \in L^p(K)$. Then for $g(x) := |f^a(x)|^p$, $\mathrm{supp}(\delta_x * \delta_{a^-}) := \{t_x\}$,

$$\begin{aligned}
\|f^a\|_p^p &= \int_K |f^a(x)|^p d\lambda(x) \\
&= \int_K g(x) d\lambda(x) \\
&= \int_K g^{a^-}(x) d\lambda(x) \qquad \text{(apply [12, 3.3F] for right Haar measure)} \\
&= \int_K g(x*a^-) d\lambda(x) \\
&= \int_K g(t_x) d\lambda(x) \\
&= \int_K |f^a(t_x)|^p d\lambda(x) \\
&= \int_K |f(t_x*a)|^p d\lambda(x) \\
&= \int_K |f(x)|^p d\lambda(x) = \|f\|_p^p
\end{aligned}$$

since

$$f(t_x*a) = \int_K f d(\delta_{t_x} * \delta_a) = \int_K f d(\delta_x * \delta_{a^-} * \delta_a) = f(x). \quad \square$$

In this paper, we investigate linear dynamics of weighted translation operators induced by weight functions and center elements of hypergroups. Every bounded continuous function $w : K \to (0,\infty)$ is called a *weight* on K. Let $a \in \mathrm{Ma}(K)$ and w be a weight on K. Then a *weighted translation operator* $T_{a,w} : L^p(K) \longrightarrow L^p(K)$ is defined by

$$T_{a,w}(f) := w \cdot (f * \delta_a) \qquad (f \in L^p(K)).$$

In fact,

$$(f*\delta_a)(x) = \int_K f(x*y^-) d\delta_a(y) = f(x*a^-) = f^{a^-}(x) \qquad (x \in K).$$

For $f \in L^p(K)$, one has $T_{a,w}(f) \in L^p(K)$ by [12, 3.3B]. It is clear that if $w \equiv 1$, then by Lemma 2.5 we have $\|T_{a,1}\| = 1$ and so $T_{a,1}$ can not be hypercyclic.

Note that in general, for any $x \in K$, there is no relation between $(fg)^x$ and $f^x g^x$. However, we have the following lemma for center elements.

LEMMA 2.6. *Let $a \in Ma(K)$ and $f, g : K \to \mathbb{C}$ be Borel measurable functions. Then $(fg)^a = f^a g^a$.*

Proof. Let $a \in \mathrm{Ma}(K)$ and $x \in K$. Let $\{b\} := \mathrm{supp}(\delta_x * \delta_a)$. Since $\delta_x * \delta_a$ is a probability measure, we have $\delta_x * \delta_a = \delta_b$. Hence

$$\begin{aligned}(fg)^a(x) = (fg)(x*a) &= \int_K f(t)g(t)\,d(\delta_x * \delta_a)(t) \\ &= f(b)g(b) \\ &= \int_K f\,d(\delta_x * \delta_a) \int_K g\,d(\delta_x * \delta_a) \\ &= f^a(x)g^a(x). \quad \square\end{aligned}$$

REMARK 2.7. By Lemma 2.6,

$$T_{a,w}^2 f = T_{a,w}(T_{a,w}f) = T_{a,w}(wf^{a^-}) = w(wf^{a^-})^{a^-} = ww^{a^-} f^{(a^-)^2}$$

thus

$$T_{a,w}^m f = ww^{a^-} \dots w^{(a^-)^{m-1}} f^{(a^-)^m}$$

for all $m \in \mathbb{N}$.

Throughout this paper, we assume that $w, w^{-1} \in L^\infty(K)$. Under this assumption, the operator $T_{a,w}$ has an inverse as follows.

LEMMA 2.8. *Let $w^{-1} \in L^\infty(K)$ and $a \in Ma(K)$. Then the operator $S_{a,w} : L^p(K) \longrightarrow L^p(K)$ defined by*

$$S_{a,w}(f) = \frac{f}{w} * \delta_{a^-} \qquad (f \in L^p(K)),$$

is the inverse of $T_{a,w}$.

Proof. By Lemma 2.6, for each $x \in K$ and $a \in \mathrm{Ma}(K)$

$$S_{a,w}f(x) = \left(\frac{f}{w} * \delta_{a^-}\right)(x) = \left(\frac{f}{w}\right)(x*a) = \frac{1}{w(x*a)} f(x*a).$$

Hence

$$\begin{aligned}T_{a,w}(S_{a,w}f)(x) = w(x)(S_{a,w}f)(x*a^-) &= w(x)\int_K S_{a,w}f(t)d(\delta_x * \delta_{a^-})(t) \\ &= w(x)\int_K \left(\frac{f}{w}\right)(t*a)d(\delta_x * \delta_{a^-})(t) \\ &= w(x)\int_K \left(\frac{f}{w}\right)^a(t)d(\delta_x * \delta_{a^-})(t) \\ &= w(x)\left(\frac{f}{w}\right)^a(x*a^-)\end{aligned}$$

$$= w(x)\left(\frac{f}{w}\right)_x (a^- * a) \text{ (by [12, 3.1F])}$$
$$= w(x)\left(\frac{f}{w}\right)(x) = f(x).$$

Similarly, one can show that $S_{a,w}(T_{a,w}f) = f$. □

3. Chaotic operators on hypergroups

In this section, we will first give sufficient and necessary conditions for weighted translations on hypergroups to be topologically transitive and mixing. By applying the characterization of transitivity, the description of chaos follows. Let us define the convolution of two subsets A and B of a hypergroup K by

$$A * B := \bigcup \{\mathrm{supp}(\delta_x * \delta_y) : x \in A, y \in B\}.$$

For each $n \in \mathbb{N}$ and $x \in K$, we put

$$A * \{x\}^n = (\cdots(A * \overbrace{\{x\}) * \cdots) * \{x\}}^{n-\text{times}}.$$

For $a \in \mathrm{Ma}(K)$, $x \in K$ and $n \in \mathbb{N}$, we define

$$\varphi_n(x) := w(x * a)w(x * a^2) \cdots w(x * a^n),$$

and

$$\widetilde{\varphi}_n(x) := \frac{1}{w(x)w(x * a^-) \cdots w(x * (a^-)^{n-1})}.$$

Also, we put

$$\nu_n(E) := \int_E \varphi_n^p(x) d\lambda(x) \qquad \text{and} \qquad \widetilde{\nu}_n(E) := \int_E \widetilde{\varphi}_n^p(x) d\lambda(x),$$

where $1 \leqslant p < \infty$ and E is a Borel subset of K.

Now we are ready to give a characterization for transitivity of $T_{a,w}$ using a different approach from that in [5] where the property of aperiodic elements (cf. Definition 3.3) was applied in the proof of [5, Theorem 2.3]. Here we do not use aperiodicity to obtain Theorem 3.1. In particular, if K is a locally compact group, then $M(K) = K$. Hence Theorem 3.1 can be regarded as an extension of [5, Theorem 2.3] from aperiodic elements of locally compact groups to center elements of hypergroups.

THEOREM 3.1. *Let K be a hypergroup and $a \in Ma(K)$. Let $1 \leqslant p < \infty$ and $w, w^{-1} \in L^\infty(K)$. If $T_{a,w}$ is a weighted translation operator on $L^p(K)$, then the following are equivalent.*

(i) *$T_{a,w}$ is topologically transitive.*

(ii) *$T_{a,w}$ satisfies the blow up/collapse property.*

(iii) *For each compact subset* $C \subseteq K$ *with* $\lambda(C) > 0$*, there are a sequence of Borel sets* (E_k) *in* C*, and a sequence* (n_k) *of positive numbers such that* $\lambda(C) = \lim_{k\to\infty} \lambda(E_k)$ *and* $\lim_{k\to\infty} \nu_{n_k}(E_k) = \lim_{k\to\infty} \widetilde{\nu}_{n_k}(E_k) = 0$.

Proof. (iii) $\Rightarrow$ (ii). Let U, V and W be non-empty open subsets of $L^p(K)$ with $0 \in W$. Since the space $C_c(K)$ of continuous functions on G with compact support is dense in $L^p(K)$, we can pick $f, g \in C_c(K)$ with $f \in U$ and $g \in V$. Let C be the union of the compact supports of f and g. Assume that $E_k \subseteq C$, $\nu_{n_k}(E_k)$ and $\widetilde{\nu}_{n_k}(E_k)$ satisfy condition (iii).

Choose $\varepsilon > 0$ such that $B(0,\varepsilon) := \{h \in L^p(K) : \|h - 0\|_p < \varepsilon\} \subseteq W$, $B(f,\varepsilon) \subseteq U$, and $B(g,\varepsilon) \subseteq V$. By condition (iii), there exists $N \in \mathbb{N}$ such that $\nu_{n_k}(E_k)\|f\|_\infty^p < \varepsilon^p$ and $\|f\|_\infty^p \lambda(C \setminus E_k) < \varepsilon^p$ for all $k > N$. Using Lemma 2.5 and the right invariance of the Haar measure λ, we have the following estimate:

$$\begin{aligned}
&\|T_{a,w}^{n_k}(f\chi_{E_k})\|_p^p \\
&= \int_K |w(x)w(x*a^-)\cdots w(x*(a^-)^{n_k-1})|^p |f(x*(a^-)^{n_k})|^p |\chi_{E_k}(x*(a^-)^{n_k})|^p d\lambda(x) \\
&= \int_K |w(x*a^{n_k})w(x*a^{n_k-1})\cdots w(x*a)|^p |f(x)|^p |\chi_{E_k}(x)|^p d\lambda(x) \\
&= \int_{E_k} |w(x*a^{n_k})w(x*a^{n_k-1})\cdots w(x*a)|^p |f(x)|^p d\lambda(x) \\
&= \int_{E_k} \varphi_{n_k}^p(x)|f(x)|^p d\lambda(x) \leqslant \nu_{n_k}(E_k)\|f\|_\infty^p < \varepsilon^p
\end{aligned}$$

implying $T_{a,w}^{n_k}(f\chi_{E_k}) \in W$. Moreover, $f\chi_{E_k} \in B(f,\varepsilon) \subseteq U$ by

$$\begin{aligned}
\|f - f\chi_{E_k}\|_p^p &= \int_G |f(x)\chi_C(x) - f(x)\chi_{E_k}(x)|^p d\lambda(x) \\
&\leqslant \|f\|_\infty^p \lambda(C \setminus E_k) < \varepsilon^p.
\end{aligned}$$

Hence, we attain

$$T_{a,w}^{n_k}(f\chi_{E_k}) \in T_{a,w}^{n_k}(U) \cap W.$$

By applying a similar argument to $S_{a,w}$ with $\widetilde{\nu}_{n_k}(E_k)$,

$$\lim_{k\to\infty} \|g - g\chi_{E_k}\|_p^p = 0 \qquad \text{and} \qquad \lim_{k\to\infty} \|S_{a,w}^{n_k}(g\chi_{E_k})\|_p^p = 0.$$

Therefore, $g\chi_{E_k} \in V$ and $S_{a,w}^{n_k}(g\chi_{E_k}) \in W$, for some k. Hence,

$$T_{a,w}^{n_k} S_{a,w}^{n_k}(g\chi_{E_k}) = g\chi_{E_k} \in T_{a,w}^{n_k}(W) \cap V.$$

Hence the operator $T_{a,w}$ satisfies the blow up/collapse property, and $T_{a,w}$ is topologically transitive.

(i) $\Rightarrow$ (iii). Let $T_{a,w}$ be transitive, and let $C \subseteq K$ be a compact set with $\lambda(C) > 0$. We denote by $\chi_C \in L^p(K)$ the characteristic function on C. Given $\varepsilon \in (0,1)$, by the assumption of topological transitivity of $T_{a,w}$, there exist a vector $f \in L^p(K)$ and some $m \in \mathbb{N}$ such that

$$\|f - \chi_C\|_p < \varepsilon^2 \quad \text{and} \quad \|T^m_{a,w} f + \chi_C\|_p < \varepsilon^2.$$

Without loss of generality, we may assume that f is real-valued by the continuity of the mapping $h \in L^p(K,\mathbb{C}) \mapsto Re\ h \in L^p(K,\mathbb{R})$ and the fact that $T_{a,w}$ commutes with it. Also, the mapping $h \in L^p(K,\mathbb{R}) \mapsto h^+ \in L^p(K,\mathbb{R})$ commutes with $T_{a,w}$ where $h^+ = \max\{0,h\}$. Therefore, for a Borel set $F \subseteq K$, we have

$$\begin{aligned} \|(T^m_{a,w} f^+)\chi_F\|_p &\leqslant \|(T^m_{a,w} f)^+\|_p = \|(T^m_{a,w} f - (-\chi_C) + (-\chi_C))^+\|_p \\ &\leqslant \|(T^m_{a,w} f - (-\chi_C))^+\|_p + \|(-\chi_C)^+\|_p \\ &= \|(T^m_{a,w} f - (-\chi_C))^+\|_p \leqslant \|T^m_{a,w} f + \chi_C\|_p < \varepsilon^2, \end{aligned}$$

and

$$\begin{aligned} \|f^- \chi_F\|_p &\leqslant \|f^-\|_p = \|(f - \chi_C + \chi_C)^-\|_p \\ &\leqslant \|(f - \chi_C)^-\|_p + \|\chi_C^-\|_p \\ &= \|f - \chi_C\|_p < \varepsilon^2, \end{aligned}$$

where $f^- = \max\{0,-f\}$. Let $A := \{x \in C : |f(x) - 1| \geqslant \varepsilon\}$. Then

$$\varepsilon^{2p} > \|f - \chi_C\|_p^p \geqslant \int_A |f(x) - 1|^p d\lambda(x) \geqslant \varepsilon^p \lambda(A).$$

Similarly, for $B := \{x \in C : |T^m_{a,w} f(x) + 1| \geqslant \varepsilon\}$,

$$\varepsilon^{2p} > \|T^m_{a,w} f + \chi_C\|_p^p \geqslant \int_B |T^m_{a,w} f(x) + 1|^p d\lambda(x) \geqslant \varepsilon^p \lambda(B).$$

Setting $E = C \setminus (A \cup B)$, it follows that $\lambda(C \setminus E) < 2\varepsilon^p$,

$$f(x) > 1 - \varepsilon > 0 \qquad \text{and} \qquad T^m_{a,w} f(x) < \varepsilon - 1 < 0 \qquad (x \in E).$$

Hence, by the right invariance of the Haar measure λ,

$$\begin{aligned} \varepsilon^{2p} &> \|(T^m_{a,w} f^+)\chi_{E*\{a\}^m}\|_p^p \\ &= \int_K |\chi_{E*\{a\}^m}(x)|^p |T^m_{a,w} f^+(x)|^p d\lambda(x) \\ &= \int_K |\chi_{E*\{a\}^m}(x)|^p |w(x)w(x*a^-)\cdots w(x*(a^-)^{m-1})|^p |f^+(x*(a^-)^m)|^p d\lambda(x) \\ &\geqslant \int_K |\chi_{E*\{a\}^m}(x*a^m)|^p |w(x*a^m)w(x*a^{m-1})\cdots w(x*a)|^p |f^+(x)|^p d\lambda(x) \\ &= \int_E |w(x*a^m)w(x*a^{m-1})\cdots w(x*a)|^p |f^+(x)|^p d\lambda(x) \\ &> (1-\varepsilon)^p \int_E \varphi_m^p(x) d\lambda(x) = (1-\varepsilon)^p \nu_m(E), \end{aligned}$$

and

$$\begin{aligned}
\varepsilon^{2p} &> \|f^{-}\chi_{E*\{a^{-}\}^m}\|_p^p = \left\|\left(S_{a,w}^m T_{a,w}^m f^{-}\right)\chi_{E*\{a^{-}\}^m}\right\|_p^p \\
&= \int_K |\chi_{E*\{a^{-}\}^m}(x)|^p \left|S_{a,w}^m (T_{a,w}^m f^{-})(x)\right|^p d\lambda(x) \\
&= \int_K |\chi_{E*\{a^{-}\}^m}(x)|^p \frac{1}{|w(x*a)w(x*a^2)\cdots w(x*a^m)|^p} |T_{a,w}^m f^{-}(x*a^m)|^p d\lambda(x) \\
&\geqslant \int_K |\chi_{E*\{a^{-}\}^m}(x*(a^{-})^m)|^p \frac{1}{|w(x*(a^{-})^{m-1})w(x*(a^{-})^{m-2})\cdots w(x)|^p} \\
&\quad \times |T_{a,w}^m f^{-}(x)|^p d\lambda(x) \\
&= \int_E \frac{1}{|w(x*(a^{-})^{m-1})w(x*(a^{-})^{m-2})\cdots w(x)|^p} |(T_{a,w}^m f)^{-}(x)|^p d\lambda(x) \\
&> (1-\varepsilon)^p \int_E \widetilde{\varphi}_m^p(x) d\lambda(x) = (1-\varepsilon)^p \widetilde{\nu}_m(E),
\end{aligned}$$

which gives condition (iii). □

By strengthening the condition (iii) in Theorem 3.1, we give a sufficient and necessary condition for weighted translation operators on hypergroups to be topologically mixing.

COROLLARY 3.2. *Let K be a hypergroup and $a \in Ma(K)$. Let $1 \leqslant p < \infty$ and $w, w^{-1} \in L^\infty(K)$. If $T_{a,w}$ is a weighted translation operator on $L^p(K)$, then the following are equivalent.*

(i) *$T_{a,w}$ is topologically mixing.*

(ii) *For each compact subset $C \subseteq K$ with $\lambda(C) > 0$, there is a sequence of Borel sets (E_n) in C such that $\lambda(C) = \lim_{n\to\infty} \lambda(E_n)$ and $\lim_{n\to\infty} \nu_n(E_n) = \lim_{n\to\infty} \widetilde{\nu}_n(E) = 0$.*

Proof. (ii) ⇒ (i). Let U and V be nonempty open subsets of $L^p(K)$. We pick $f, g \in C_c(K)$ with $f \in U$ and $g \in V$. Let C be the union of the compact supports of f and g. Assume $E_n \subseteq C$, $\nu_n(E_n)$ and $\widetilde{\nu}_n(E_n)$ satisfy condition (ii). As in the proof of Theorem 3.1, we have

$$\lim_{k\to\infty} \|T_{a,w}^n (f\chi_{E_n})\|_p = \lim_{k\to\infty} \|S_{a,w}^n (g\chi_{E_n})\|_p = 0.$$

For each $n \in \mathbb{N}$ and

$$v_n = f\chi_{E_n} + S_{a,w}^n (g\chi_{E_n}) \in L^p(K),$$

we have

$$\|v_n - f\|_p \leqslant \|f\chi_{C\setminus E_n}\|_p + \|S_{a,w}^n (g\chi_{E_n})\|_p,$$

and

$$\|T_{a,w}^n v_n - g\|_p \leqslant \|T_{a,w}^n (f\chi_{E_n})\|_p + \|g\chi_{C\setminus E_n}\|_p.$$

Therefore, $\lim_{n\to\infty} v_n = f$ and $\lim_{n\to\infty} T^n_{a,w} v_n = g$, which imply $T^n_{a,w}(U) \cap V \neq \emptyset$ from some n onwards.

(i) $\Rightarrow$ (ii). Let $T_{a,w}$ be topologically mixing and $C \subseteq K$ be a compact set with $\lambda(C) > 0$. Given $\varepsilon \in (0,1)$, by the topological mixing of $T_{a,w}$, there exist a vector $f \in L^p(K)$ and $N \in \mathbb{N}$ such that

$$\|f - \chi_C\|_p < \varepsilon^2 \quad \text{and} \quad \|T^n_{a,w} f + \chi_C\|_p < \varepsilon^2,$$

for all $n \geqslant N$. By a similar argument as in the proof of Theorem 3.1, one can obtain condition (ii). □

Based on the result in Theorem 3.1 and the property of aperiodicity of elements, we characterize chaotic weighted translation operators on hypergroups. Inspired by the study on aperiodic elements of a locally compact group in [5], the definition of aperiodic elements for hypergroups is formulated as below.

DEFINITION 3.3. An element $a \in \mathrm{Ma}(K)$ is called *aperiodic* if for each compact subset $C \subseteq K$ with $\lambda(C) > 0$, there exists $N \in \mathbb{N}$ such that $C \cap (C * \{a\}^n) = \emptyset$ for all $n \geqslant N$.

If K is a locally compact group, then $\mathrm{Ma}(K) = K$ and we get another equivalent condition for aperiodicity, that is, $a \in K$ is aperiodic if and only if, given compact subset $C \subseteq K$ with $\lambda(C) > 0$, there exists $N \in \mathbb{N}$ such that $Ca^{rn} \cap Ca^{sn} = \emptyset$, for all $n \geqslant N$ and $r,s \in \mathbb{Z}$ with $r \neq s$. Indeed, if a is aperiodic and $Ca^{rn} \cap Ca^{sn} \neq \emptyset$ with $r < s$, then for some $y \in C$ we have $ya^{rn} \in Ca^{sn}$, which says $y \in C \cap Ca^{(s-r)n}$, a contradiction. We formulate this condition for hypergroups as follows.

LEMMA 3.4. *An element $a \in Ma(K)$ is aperiodic if and only if for each compact subset $C \subseteq K$ with $\lambda(C) > 0$, there exists $N \in \mathbb{N}$ such that $(C * \{a\}^{rn}) \cap (C * \{a\}^{sn}) = \emptyset$ for all $n \geqslant N$ and $r,s \in \mathbb{Z}$ with $r \neq s$, where $\{a\}^{-n} := \{a^-\}^n$.*

Proof. Let $a \in \mathrm{Ma}(K)$ and $C \subseteq K$ be a compact set with $\lambda(C) > 0$. Assume that there exists $N \in \mathbb{N}$ such that $(C * \{a\}^{rn}) \cap (C * \{a\}^{sn}) = \emptyset$, for all $n \geqslant N$ and $r,s \in \mathbb{Z}$ with $r \neq s$. Taking $r = 0$ and $s = 1$, it follows that a is aperiodic.

For the converse, given a compact subset C of K with $\lambda(C) > 0$, by the assumption of aperiodicity of a, there exists $N \in \mathbb{N}$ such that $C \cap (C * \{a\}^n) = \emptyset$, for all $n \geqslant N$. Now take $x \in (C * \{a\}^{rn}) \cap (C * \{a\}^{sn})$, for some $n \geqslant N$ and distinct elements $r,s \in \mathbb{Z}$. By [12, 4.1B], for each $A,B,E \subseteq K$, $A \cap (B * E) \neq \emptyset$, if and only if, $(A * E^-) \cap B \neq \emptyset$, where $E^- := \{t^- : t \in E\}$. Using this property, one can deduce that $x \in C * \{a\}^{rn}$ and there exists an element $z \in C \cap (\{x\} * \{a\}^{-sn})$. These imply that

$$z \in C \cap (C * \{a\}^{rn} * \{a\}^{-sn}) = C \cap (C * \{a\}^{(r-s)n}),$$

which contracts the aperiodicity of a. □

Here, applying a similar idea as in the proof of [4, Theorem 2.1], we obtain the next result. We include the proof for the sake of completeness.

THEOREM 3.5. *Let K be a hypergroup, and $a \in Ma(K)$ be an aperiodic. Let $1 \leqslant p < \infty$ and $w, w^{-1} \in L^{\infty}(K)$. If $T_{a,w}$ is a weighted translation on $L^p(K)$, and $\mathscr{P}(T_{a,w})$ is the set of periodic elements of $T_{a,w}$, then the following are equivalent.*

(i) *$T_{a,w}$ is chaotic.*

(ii) *$\mathscr{P}(T_{a,w})$ is dense in $L^p(K)$.*

(iii) *For each compact subset $C \subseteq K$ with $\lambda(C) > 0$, there are a sequence of Borel sets (E_k) in C, and a sequence (n_k) of positive numbers such that $\lambda(C) = \lim_{k\to\infty} \lambda(E_k)$ and*

$$\lim_{k\to\infty}\left(\sum_{l=1}^{\infty} \nu_{ln_k}(E_k) + \sum_{l=1}^{\infty} \widetilde{\nu}_{ln_k}(E_k)\right) = 0.$$

Proof. We will show (ii) $\Rightarrow$ (iii), and (iii) $\Rightarrow$ (i).

(ii) $\Rightarrow$ (iii). Let $C \subseteq K$ be a compact set with $\lambda(C) > 0$. Since a is aperiodic, there exists $N \in \mathbb{N}$ such that $C \cap (C * \{a\}^m) = \emptyset$ for all $m > N$. Let $\chi_C \in L^p(K)$ be the characteristic function of C. By the density of $\mathscr{P}(T_{a,w})$, we can find a sequence (f_k) of periodic points of $T_{a,w}$ satisfying $\|f_k - \chi_C\|_p < \frac{1}{4^k}$, and a sequence $(n_k) \subset \mathbb{N}$ such that $T_{a,w}^{n_k} f_k = f_k = S_{a,w}^{n_k} f_k$, where we may assume $n_{k+1} > n_k > N$. Therefore, $(C * \{a\}^{rn_k}) \cap (C * \{a\}^{sn_k}) = \emptyset$, for all $r, s \in \mathbb{Z}$ with $r \neq s$. Let $A_k = \{x \in C : |f_k(x) - 1| \geqslant \frac{1}{2^k}\}$ and let $E_k = C \setminus A_k$. As in the proof of Theorem 3.1, we have

$$|f_k(x)| > 1 - \frac{1}{2^k} \qquad (x \in C \setminus A_k) \qquad \text{and} \qquad \lambda(C \setminus A_k) < \frac{1}{2^{pk}}.$$

Moreover, by the right invariance of the Haar measure λ, and $(C * \{a\}^{rn_k}) \cap (C * \{a\}^{sn_k}) = \emptyset$, for $r \neq s$,

$$\begin{aligned}
\frac{1}{4^{pk}} &> \|f_k - \chi_C\|_p^p = \int_G |f_k(x) - \chi_C(x)|^p d\lambda(x) \geqslant \int_{G\setminus C} |f_k(x)|^p d\lambda(x) \\
&\geqslant \sum_{l=1}^{\infty} \int_{C*\{a\}^{ln_k}} |f_k(x)|^p d\lambda(x) + \sum_{l=1}^{\infty} \int_{C*\{a^-\}^{ln_k}} |f_k(x)|^p d\lambda(x) \\
&= \sum_{l=1}^{\infty} \int_K |f_k(x)|^p \chi_{C*\{a\}^{ln_k}}(x) d\lambda(x) + \sum_{l=1}^{\infty} \int_K |f_k(x)|^p \chi_{C*\{a^-\}^{ln_k}}(x) d\lambda(x) \\
&= \sum_{l=1}^{\infty} \int_K |T_{a,w}^{ln_k} f_k(x)|^p \chi_{C*\{a\}^{ln_k}}(x) d\lambda(x) + \sum_{l=1}^{\infty} \int_K |S_{a,w}^{ln_k} f_k(x)|^p \chi_{C*\{a^-\}^{ln_k}}(x) d\lambda(x) \\
&= \sum_{l=1}^{\infty} \int_K |T_{a,w}^{ln_k} f_k(x * a^{ln_k})|^p \chi_{C*\{a\}^{ln_k}}(x * a^{ln_k}) d\lambda(x) \\
&\quad + \sum_{l=1}^{\infty} \int_K |S_{a,w}^{ln_k} f_k(x * (a^-)^{ln_k})|^p \chi_{C*\{a^-\}^{ln_k}}(x * (a^-)^{ln_k}) d\lambda(x) \\
&= \sum_{l=1}^{\infty} \int_C |T_{a,w}^{ln_k} f_k(x * a^{ln_k})|^p d\lambda(x) + \sum_{l=1}^{\infty} \int_C |S_{a,w}^{ln_k} f_k(x * (a^-)^{ln_k})|^p d\lambda(x)
\end{aligned}$$

$$\begin{aligned}
&\geqslant \sum_{l=1}^{\infty}\int_{E_k}|T_{a,w}^{ln_k}f_k(x*a^{ln_k})|^p d\lambda(x)+\sum_{l=1}^{\infty}\int_{E_k}|S_{a,w}^{ln_k}f_k(x*(a^-)^{ln_k})|^p d\lambda(x)\\
&=\sum_{l=1}^{\infty}\int_{E_k}|\varphi_{ln_k}(x)f_k(x)|^p d\lambda(x)+\sum_{l=1}^{\infty}\int_{E_k}|\widetilde{\varphi}_{ln_k}(x)f_k(x)|^p d\lambda(x)\\
&>\left(1-\frac{1}{2^k}\right)^p\sum_{l=1}^{\infty}\nu_{ln_k}(E_k)+\left(1-\frac{1}{2^k}\right)^p\sum_{l=1}^{\infty}\widetilde{\nu}_{ln_k}(E_k)
\end{aligned}$$

which proves condition (iii).

(iii) $\Rightarrow$ (i). It follows from Theorem 3.1 that $T_{a,w}$ is topologically transitive. Therefore, we only need to show that $\mathscr{P}(T_{a,w})$ is dense in $L^p(K)$. Let $f\in C_c(K)$ with compact support $C\subseteq K$. Then there is a sequence of Borel sets (E_k) in C such that $\lambda(C)=\lim_{k\to\infty}\lambda(E_k)$. As in the proof of Theorem 3.1, one has

$$\|T_{a,w}^{ln_k}(f\chi_{E_k})\|_p^p\leqslant \nu_{ln_k}(E_k)\|f\|_\infty^p \qquad \text{and} \qquad \|S_{a,w}^{ln_k}(f\chi_{E_k})\|_p^p\leqslant \widetilde{\nu}_{ln_k}(E_k)\|f\|_\infty^p.$$

Now let

$$v_k:=f\chi_{E_k}+\sum_{l=1}^{\infty}T_{a,w}^{ln_k}(f\chi_{E_k})+\sum_{l=1}^{\infty}S_{a,w}^{ln_k}(f\chi_{E_k})\in L^p(G).$$

Then, by $(C*\{a\}^{rn_k})\cap(C*\{a\}^{sn_k})=\emptyset$, we have

$$\begin{aligned}
\|v_k-f\|_p^p&\leqslant \|f\|_\infty^p\lambda(K\setminus E_k)+\sum_{l=1}^{\infty}\|T_{a,w}^{ln_k}(f\chi_{E_k})\|_p^p+\sum_{l=1}^{\infty}\|S_{a,w}^{ln_k}(f\chi_{E_k})\|_p^p\\
&\leqslant \|f\|_\infty^p\lambda(K\setminus E_k)+\|f\|_\infty^p\left(\sum_{l=1}^{\infty}\nu_{ln_k}(E_k)+\sum_{l=1}^{\infty}\widetilde{\nu}_{ln_k}(E_k)\right)
\end{aligned}$$

which implies that $v_k\to f$ as $k\to\infty$. Moreover,

$$\begin{aligned}
T_{a,w}^{n_k}v_k&=T_{a,w}^{n_k}(f\chi_{E_k})+\sum_{l=1}^{\infty}T_{a,w}^{n_k}T_{a,w}^{ln_k}(f\chi_{E_k})+\sum_{l=1}^{\infty}T_{a,w}^{n_k}S_{a,w}^{ln_k}(f\chi_{E_k})\\
&=\sum_{l=1}^{\infty}T_{a,w}^{ln_k}(f\chi_{E_k})+f\chi_{E_k}+\sum_{l=1}^{\infty}S_{a,w}^{ln_k}(f\chi_{E_k})=v_k.
\end{aligned}$$

Combing all these, $\mathscr{P}(T_{a,w})$ is dense in $L^p(K)$. $\square$

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