Operators and Matrices
Volume 12, Number 1 (2018), 157–168 doi:10.7153/oam-2018-12-10

DISTANCE ESTIMATES, NORM OF HANKEL OPERATORS AND RELATED QUESTIONS

NAJLA ALTWAIJRY, AMANI S. BAAZEEM AND MUBARIZ GARAYEV

(*Communicated by G. Misra*)

Abstract. We consider Berezin symbols and Hankel operators on the Hardy space $H^2(\mathbb{D})$ over the unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ and give their some applications. Namely, we estimate in terms of Hankel operators and Berezin symbols the distances from a given operator to the algebra of all analytic Toeplitz operators and to the set of all Toeplitz operators on $H^2(\mathbb{D})$. We use Hankel operator also to prove some lower estimate for the so-called Berezin number of bounded linear operators on H^2. Some other related questions are also discussed.

1. Introduction and notations

Recall that if H is a Hilbert space and $\mathfrak{B}(H)$ is the algebra of all bounded linear operators on H, then for $A\in\mathfrak{B}(H)$ the so-called derivation Δ_A of the algebra $\mathfrak{B}(H)$ is defined by

$$\Delta_A(B) := AB - BA.$$

If $\mathscr{R}$ is a subalgebra of $\mathfrak{B}(H)$, then the norm of the restriction of the operator Δ_A to $\mathscr{R}$ numerically characterizes the "degree of concommutativity" of A with $\mathscr{R}$. The following two inequalities are proved in [7, Theorem 2 and Corollary 2].

(**i**) For any $A\in\mathfrak{B}(H)$ the following inequality holds:

$$\operatorname{dist}(A,\Lambda_+) \leqslant 2\left\|\Delta_A|\Lambda_+\right\|,$$

where $\Lambda_+=\{B\in\mathfrak{B}(H): VB=BV\}$, here V is the forward shift operator on H.

(**ii**) If an operator $K\in\mathfrak{B}(H)$ is compact, then

$$\operatorname{dist}(K,\Lambda_+) \geqslant \frac{1}{2}\|K\|. \tag{1}$$

Note that in case of $H=H^2$ (Hardy space), Λ_+ is the algebra of all analytic Toeplitz operators T_φ, $\varphi\in H^\infty$, defined by $T_\varphi f=\varphi f$, $f\in H^2$ and V is defined by $Vf(z)=zf(z)$. So, it is natural to prove in particular lower estimate for $\operatorname{dist}(A,\Lambda_+)$ in case of

Mathematics subject classification (2010): 47A05, 47B10.
Keywords and phrases: Berezin symbol, Hankel operator, Toeplitz operator, compact operator, Berezin number, distance estimate, model operator.
This research project was supported by a grant from the "Research Center of the Female Scientific and Medical Colleges", Deanship of Scientific Research, King Saud University.

noncompact operators $A \in \mathfrak{B}(H^2)$. Here we solve this problem (see Theorem 1 below). Namely, we use Berezin symbols and Hankel operators in estimating of distance from a given noncompact operator to the algebra of analytic Toeplitz operators on the Hardy Hilbert space $H^2 = H^2(\mathbb{D})$ over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and in obtaining a lower estimate for the Berezin number of linear bounded operator on H^2. We also give a uniform estimate of the distance from the set of compact operators to some subset of Λ_+ (see Theorems 3 and 4 below). Moreover, we estimate the distance to the set of all Toeplitz operators on H^2 and the set of all functions from model operators on the model space K_θ. Further, we prove in terms of the norms of Hankel operators a lower estimate for the Berezin number of operators in $\mathfrak{B}(H^2)$ (see Theorem 9 in Section 3).

Before giving our results, let us introduce some necessary definitions and notations.

Recall that by a Reproducing Kernel Hilbert Space (RKHS) (see Aronzajn [1]) we mean a Hilbert space $\mathscr{H} = \mathscr{H}(\Omega)$ of complex-valued functions on some set Ω such that evaluation $f \to f(\lambda)$ at any point λ of Ω is a continuous linear functional on $\mathscr{H}$. The classical Riesz representation theorem ensures that a functional Hilbert space $\mathscr{H}$ has a reproducing kernel, that is, a function $k_{\mathscr{H},\lambda} : \Omega \times \Omega \to \mathbb{C}$ with defining property

$$\left\langle f, k_{\mathscr{H},\lambda} \right\rangle = f(\lambda)$$

for all $f \in \mathscr{H}$ and $\lambda \in \Omega$, where $k_{\mathscr{H},\lambda} = k_{\mathscr{H}}(.,\lambda) \in \mathscr{H}$. Let $\widehat{k}_{\mathscr{H},\lambda} := \frac{k_{\mathscr{H},\lambda}}{\left\| k_{\mathscr{H},\lambda} \right\|_{\mathscr{H}}}$ be the normalized reproducing kernel of $\mathscr{H}$. For any bounded linear operator A on $\mathscr{H}$, its Berezin symbol $\widetilde{A}$ is defined by (see [8])

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_{\mathscr{H},\lambda}, \widehat{k}_{\mathscr{H},\lambda} \right\rangle, \ \lambda \in \Omega.$$

Clearly, $\widetilde{A}$ is a bounded function on Ω and $\sup_{\lambda \in \Omega} \left| \widetilde{A}(\lambda) \right| \leqslant \|A\|$.

For a bounded linear operator on a RKHS its Berezin set and Berezin number are defined, respectively, as follows:

$$\mathrm{Ber}(A) := \mathrm{Range}(\widetilde{A}) = \left\{ \widetilde{A}(\lambda) : \lambda \in \Omega \right\}$$

$$\mathrm{ber}(A) := \sup \{|\eta| : \eta \in \mathrm{Ber}(A)\} = \left\| \widetilde{A} \right\|_{L^\infty(\Omega)}$$

Before giving our results, let us introduce some more necessary definitions and notations. The Hardy-Hilbert space $H^2 = H^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on the unit disc $\mathbb{D}$ satisfying

$$\|f\|^2 := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{it}) \right|^2 dt < +\infty.$$

The symbol $H^\infty = H^\infty(\mathbb{D})$ denotes the Banach algebra of functions bounded and analytic on the unit disc $\mathbb{D}$ equipped with the norm $\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|$. A function

$\theta \in H^\infty$ such that $\left|\theta(e^{it})\right| = 1$ for almost all $t \in [0, 2\pi)$, is called an inner function. It is convenient to establish a natural embedding of the space H^2 in the space $L^2 = L^2(\mathbb{T})$ by associating to each function $f \in H^2$ its radial boundary values $(bf)(\xi) := \lim_{r\to 1^-} f(r\xi)$, which (by the classical Fatou Theorem) exist for $m-$almost all $\xi \in \mathbb{T}$ (unit circle), where m is the Lebesgue measure on $\mathbb{T}$. Then we have

$$H^2 = \left\{ f \in L^2(\mathbb{T}) : \widehat{f}(n) = 0,\ n < 0 \right\},$$

where $\widehat{f}(n) := \int_{\mathbb{T}} \overline{\xi}^n f(\xi) dm(\xi)$ is the Fourier coefficient of the function f. We denote

$$H^2_- = \left\{ f \in L^2(\mathbb{T}) : \widehat{f}(n) = 0,\ n > 0 \right\}.$$

For $\varphi \in L^\infty =.\ L^\infty(\mathbb{T})$, the Toeplitz operator T_φ with symbol φ is the operator on H^2 defined by $T_\varphi(f) = P_+(\varphi f)$; here P_+ is the orthogonal projection from $L^2(\mathbb{T})$ onto H^2. If $\varphi \in H^\infty$, then the corresponding Toeplitz operator $T_\varphi,\ T_\varphi(f) = \varphi f,\ f \in H^2$, is called analytic Toeplitz operator. It is clear that $T_\varphi T_\psi = T_\psi T_\varphi = T_{\varphi\psi}$ for all $\varphi, \psi \in H^\infty$. The Hankel operator H_φ is defined by $H_\varphi(f) = P_-(\varphi f),\ f \in H^2$, where $P_- := I - P_+$.

It is obvious that $H_\varphi = 0$ for any $\varphi \in H^\infty$, and $\left\|H_\varphi\right\| \leqslant \|\varphi\|_{L^\infty}$ for any $\varphi \in L^\infty$.

2. Distance estimates and norm of Hankel operators

In this section, we prove some distance estimates from a given operator to the sets of analytic Toeplitz operators, Toeplitz operators and functions of model operator.

Let us denote by Λ_+ the set(algebra) of all analytic Toeplitz operators on H^2 :

$$\Lambda_+ = \left\{ T_\varphi \in \mathfrak{B}(H^2) : \varphi \in H^\infty \right\},$$

where $\mathfrak{B}(H^2)$ is the Banach algebra of bounded linear operators on the Hardy space H^2.

THEOREM 1. *For any noncompact operator A on the Hardy space H^2, we have*

$$\operatorname{dist}(A, \Lambda_+) \geqslant \left\| H_{\widetilde{A}} \right\|,$$

where $\widetilde{A}$ is the Berezin symbol of A and $H_{\widetilde{A}}$ is the corresponding Hankel operator.

Proof. Indeed, let $\widehat{k}_{H^2,\lambda} = \frac{\left(1-|\lambda|^2\right)^{\frac{1}{2}}}{1-\overline{\lambda} z}$ be a normalized reproducing kernel of H^2, and let $\varphi \in H^\infty$ be any function. Then by considering that $T^*_\varphi k_{H^2,\lambda} = T_{\overline{\varphi}} k_{H^2,\lambda} = \overline{\varphi(\lambda)} k_{H^2,\lambda},\ \lambda \in \mathbb{D}$, and hence $\widetilde{T}_\varphi(\lambda) = \varphi(\lambda),\ \lambda \in \mathbb{D}$, we have:

$$\begin{aligned}\|A - T_\varphi\| &\geqslant \left\| (A - T_\varphi)\, \widehat{k}_{H^2,\lambda} \right\| \geqslant \left| \left\langle (A - T_\varphi)\, \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \right\rangle \right| \\ &= \left| (A - T_\varphi)^{\sim}(\lambda) \right| = \left| \widetilde{A}(\lambda) - \widetilde{T}_\varphi(\lambda) \right| = \left| \widetilde{A}(\lambda) - \varphi(\lambda) \right|\end{aligned}$$

for all $\lambda \in \mathbb{D}$, and hence

$$\begin{aligned} \left\|A - T_\varphi\right\| &\geqslant \sup_{\lambda \in \mathbb{D}} \left|\widetilde{A}(\lambda) - \varphi(\lambda)\right| \\ &= \left\|\widetilde{A} - \varphi\right\|_{L^\infty(\mathbb{D})}. \end{aligned}$$

So, it follows taking infimum in φ, that

$$\operatorname{dist}(A, \Lambda_+) \geqslant \operatorname{dist}\left(\widetilde{A}, H^\infty\right) = \left\|H_{\widetilde{A}}\right\|,$$

as desired. □

Now we will give a uniform distance estimate from the set of all compact operators to some subset of the set of Λ of all Toeplitz operators T_φ, $\varphi \in L^\infty$, which, apparently, can not be proved by the method of the proof of inequality (1) of Mustafaev and Shulman [7]. Our proof uses the following well known result of Englis [2], see also Karaev [4], which shows that the normalized reproducing kernel $\widehat{k}_{H^2,\lambda} = \frac{(1-|\lambda|^2)^{\frac{1}{2}}}{1-\overline{\lambda}z}$ of H^2 are, loosely speaking, asymptotic eigenfunctions for any Toeplitz operator T_φ, $\varphi \in L^\infty(\mathbb{T})$ (this generalizes the well-known fact that $T_{\overline{\varphi}}\widehat{k}_{H^2,\lambda} = \overline{\varphi(\lambda)}\widehat{k}_{H^2,\lambda}$ when φ is a bounded analytic function on $\mathbb{D}$, i.e. $\varphi \in H^\infty$).

LEMMA 1. *Let $\varphi \in L^\infty(\mathbb{T})$, and denote by $\widetilde{\varphi}$ its harmonic extension (by the Poisson formula) into $\mathbb{D}$. Then $T_\varphi \widehat{k}_{H^2,\lambda} - \widetilde{\varphi}(\lambda)\widehat{k}_{H^2,\lambda} \to 0$ radially, i.e.,*

$$\lim_{r \to 1^-} \left\|T_\varphi \widehat{k}_{H^2,re^{it}} - \widetilde{\varphi}(re^{it})\widehat{k}_{H^2,re^{it}}\right\| = 0$$

for almost all $t \in [0, 2\pi)$.

We set $\Lambda_\delta := \{T_\varphi \in \mathfrak{B}(H^2) : \varphi \in L^\infty$ and there exists $\delta > 0$ such that $|\widetilde{\varphi}(z)| \geqslant \delta$, $\forall z \in \mathbb{D}\}$ and $\mathfrak{G}_\infty(H^2) := \left\{K \in \mathfrak{B}(H^2) : K \text{ is compact}\right\}$.

THEOREM 2. *We have*

$$\operatorname{dist}\left(\mathfrak{G}_\infty(H^2), \Lambda_\delta\right) \geqslant \delta > 0. \tag{2}$$

Proof. Let $T_\varphi \in \Lambda_\delta$ and $K \in \mathfrak{G}_\infty(H^2)$ be any two operators. Then we have:

$$\begin{aligned} \left\|K - T_\varphi\right\| &\geqslant \left\|\left(K - T_\varphi\right)\widehat{k}_{H^2,\lambda}\right\| \geqslant \left\|T_\varphi \widehat{k}_{H^2,\lambda}\right\| - \left\|K\widehat{k}_{H^2,\lambda}\right\| \\ &= \left\|\left(T_\varphi \widehat{k}_{H^2,\lambda} - \widetilde{\varphi}(\lambda)\widehat{k}_{H^2,\lambda}\right) + \widetilde{\varphi}(\lambda)\widehat{k}_{H^2,\lambda}\right\| - \left\|K\widehat{k}_{H^2,\lambda}\right\| \\ &\geqslant |\widetilde{\varphi}(\lambda)| - \left\|T_\varphi \widehat{k}_{H^2,\lambda} - \widetilde{\varphi}(\lambda)\widehat{k}_{H^2,\lambda}\right\| - \left\|K\widehat{k}_{H^2,\lambda}\right\| \\ &\geqslant \delta - \left[\left\|T_\varphi \widehat{k}_{H^2,\lambda} - \widetilde{\varphi}(\lambda)\widehat{k}_{H^2,\lambda}\right\| + \left\|K\widehat{k}_{H^2,\lambda}\right\|\right] \end{aligned}$$

for every $\lambda \in \mathbb{D}$. Since $\left\|K\widehat{k}_{H^2,\lambda}\right\| \to 0$ as $\lambda \to \mathbb{T}$ (because $\widehat{k}_{H^2,\lambda} \overset{weakly}{\to} 0$ when $\lambda \to \mathbb{T}$ and K is compact), by applying Lemma 1 we deduce from the last inequality that

$$\left\|K - T_\varphi\right\| \geqslant \delta$$

for any $T_\varphi \in \Lambda_\delta$ and $K \in \mathfrak{S}_\infty(H^2)$, which obviously implies inequality (2). The theorem is proved. □

Let us denote

$$\mathfrak{A}^0 := \left\{A \in \mathfrak{B}(\mathscr{H}(\Omega)) : \left\|A\widehat{k}_{\mathscr{H},\lambda} - \widetilde{A}(\lambda)\widehat{k}_{\mathscr{H},\lambda}\right\| \to 0 \text{ whenever } \lambda \to \partial\Omega\right\}$$

and

$$\mathfrak{A}^0_\delta(\mathscr{H}) := \left\{A \in \mathfrak{A}^0 : \inf_{z\in\Omega}\left|\widetilde{A}(z)\right| = \delta > 0\right\}.$$

By Lemma 1, it is clear that $\mathfrak{T} := \left\{T_\varphi \in \mathfrak{B}(H^2) : \varphi \in L^\infty\right\} \subset \mathfrak{A}^0(H^2)$, also $\mathfrak{T} \cap \mathfrak{A}^0(H^2) \neq \varnothing$ and $\mathfrak{T} \cap \mathfrak{A}^0_\delta(H^2) \subset \mathfrak{A}^0_\delta(H^2)$. Now the following result, which generalizes Theorem 2, can be proved by the same arguments, as in the proof of Theorem 2, and therefore we omit its proof. Recall that the RKHS $\mathscr{H}(\Omega)$ is standard in sense of Nordgren and Rosenthal [8], if $\widehat{k}_{\mathscr{H},\lambda} \overset{weakly}{\to} 0$ whenever $\lambda \to \xi \in \partial\Omega$.

THEOREM 3. *Let $\mathscr{H} = \mathscr{H}(\Omega)$ be a standard RKHS in sense of Nordgren and Rosenthal. Then*

$$\operatorname{dist}\left(\mathfrak{S}_\infty(\mathscr{H}), \mathfrak{A}^0_\delta(\mathscr{H})\right) \geqslant \delta.$$

Let now $\Lambda := \left\{T_\varphi \in \mathscr{B}\left(H^2\right) : \varphi \in L^\infty(\mathbb{T})\right\}$ denote the set of all Toeplitz operators on the Hardy space H^2, and let $\mathscr{H}_{hh}$ denote the set of Hankel operators with harmonic symbols:

$$\mathscr{H}_{hh} := \left\{H_\psi : \psi \in h^\infty(\mathbb{D}) \text{ and } H_\psi : H^2 \to H^2_- \text{ is the Hankel operator}\right\},$$

where $h^\infty(\mathbb{D})$ is the set of all bounded harmonic functions on $\mathbb{D}$.

THEOREM 4. *If $A \in \mathscr{B}\left(H^2\right)$ and $\tilde{A}$ is its Berezin symbol, then*

$$\operatorname{dist}(A, \Lambda) \geqslant \operatorname{dist}(H_{\tilde{A}}, \mathscr{H}_{hh}).$$

Proof. Indeed, by using obvious inequalities $\|B\| \geqslant \left\|\tilde{B}\right\|_{L^\infty(\mathbb{D})}$ and $\|\psi\|_\infty \geqslant \left\|H_\psi\right\|$ for any operator $B \in \mathscr{B}\left(H^2\right)$ and function $\psi \in L^\infty(\mathbb{T})$, we have for any function $\varphi \in L^\infty(\mathbb{T})$ that

$$\begin{aligned}\left\|A - T_\varphi\right\| \geqslant \left\|\widetilde{A - T_\varphi}\right\|_{L^\infty(\mathbb{D})} &\geqslant \left\|H_{\widetilde{A-T_\varphi}}\right\| \\ &= \left\|H_{\widetilde{A}} - H_{\widetilde{T_\varphi}}\right\| = \left\|H_{\widetilde{A}} - H_{\widetilde{\varphi}}\right\|,\end{aligned}$$

where $\widetilde{\varphi}$ is the harmonic extension of φ into the unit disc $\mathbb{D}$ (see, for instance, Engliš [2]). Since $\widetilde{\varphi} \in h^{\infty}(\mathbb{D})$, the last inequality implies that

$$\left\|A - T_{\varphi}\right\| \geqslant \inf_{h \in h^{\infty}(\mathbb{D})} \left\|H_{\widetilde{A}} - H_h\right\| = \operatorname{dist}\left(H_{\widetilde{A}}, \mathscr{H}_{hh}\right),$$

as desired. □

Recall that the functional model operator of Sz.-Nagy and Foias (see, for instance, in [10]) is defined by

$$M_{\theta} := P_{\theta} S \mid K_{\theta},$$

where $K_{\theta} := \left(\theta H^2\right)^{\perp} = H^2 \ominus \theta H^2$ is the so-called model space associated with the inner function θ, $Sf(z) = zf(z)$, is the shift operator on H^2 and $P_{\theta} := I - T_{\theta} T_{\theta}^* = I - T_{\theta} T_{\overline{\theta}}$ is the orthogonal projection of H^2 onto K_{θ}. For any function $\varphi \in H^{\infty}$, the model operator M_{θ} admits functional calculus by the formula $\varphi(M_{\theta}) = P_{\theta} \varphi \mid K_{\theta}$, that is $\varphi(M_{\theta}) f = P_{\theta}(\varphi f)$ for any $f \in K_{\theta}$.

Our next result estimates distance from arbitrary operator $A \in \mathscr{B}(K_{\theta})$ to the following set $\mathscr{F}_{\theta}$ of all function from model operator M_{θ}:

$$\mathscr{F}_{\theta} := \{\varphi(M_{\theta}) : \varphi \in H^{\infty}\}.$$

THEOREM 5. *If θ is an inner function and $A \in \mathscr{B}(K_{\theta})$, then*

$$\inf_{\varphi \in H^{\infty}} \left[\left\|A - \varphi(M_{\theta})\right\| + \left\| \frac{\widetilde{\varphi \overline{\theta}} - \varphi \overline{\theta}}{1 - |\theta|^2} \right\|_{L^{\infty}(\mathbb{D})} \right] \geqslant \max\left(\left\|H_{\widetilde{A}}\right\|, \operatorname{dist}(A, \mathscr{F}_{\theta})\right).$$

Proof. Let $\varphi \in H^{\infty}$ be arbitrary. Since $k_{H^2,\lambda}(z) = \dfrac{1}{1 - \overline{\lambda} z}$ is the reproducing kernel of the space H^2, the reproducing kernel of the closed subspace $K_{\theta} \subset H^2$ will be $P_{\theta} k_{H^2,\lambda}$, because for any $f \in K_{\theta}$ and $\lambda \in \mathbb{D}$

$$\left\langle f, P_{\theta} k_{H^2,\lambda} \right\rangle = \left\langle P_{\theta} f, k_{H^2,\lambda} \right\rangle = \left\langle f, k_{H^2,\lambda} \right\rangle = f(\lambda).$$

Since P_{θ} has the form $P_{\theta} = I - T_{\theta} T_{\overline{\theta}}$ and $T_{\overline{\theta}} k_{H^2,\lambda} = \overline{\theta(\lambda)} k_{H^2,\lambda}$ for any $\lambda \in \Omega$, it is easy to see that the normalized reproducing kernel of the subspace K_{θ} has the form

$$\hat{k}_{\theta,\lambda}(z) := \frac{1 - |\lambda|^2}{1 - |\theta(\lambda)|^2} \frac{1 - \overline{\theta(\lambda)} \theta(z)}{1 - \overline{\lambda} z}, \ \lambda \in \mathbb{D}.$$

Then by using the well known fact that (see Engliš [2]) $\widetilde{T_{\psi}}(\lambda) = \widetilde{\psi}(\lambda)$ for all $\lambda \in \mathbb{D}$, it can be shown (see Karaev [5]) that the Berezin symbol

$$\widetilde{\varphi(M_{\theta})}(\lambda) := \left\langle \varphi(M_{\theta}) \hat{k}_{\theta,\lambda}, \hat{k}_{\theta,\lambda} \right\rangle, \ \lambda \in \mathbb{D},$$

is the following function:

$$\widetilde{\varphi(M_\theta)}(\lambda)=\frac{\varphi(\lambda)-\theta(\lambda)\widetilde{\varphi\overline{\theta}}(\lambda)}{1-|\theta(\lambda)|^2},\ \lambda\in\mathbb{D}, \tag{3}$$

where $\widetilde{\varphi\overline{\theta}}$ is the harmonic extension of the function $\varphi\overline{\theta}$ into $\mathbb{D}$. Now by using (3), we have for any $\varphi\in H^\infty$ that

$$\begin{aligned}\|A-\varphi(M_\theta)\| &\geqslant \left\|\widetilde{A-\varphi(M_\theta)}\right\|_{L^\infty(\mathbb{D})}=\left\|\widetilde{A}-\widetilde{\varphi(M_\theta)}\right\|_{L^\infty(\mathbb{D})}\\ &=\left\|\widetilde{A}-\frac{\varphi-\theta\widetilde{\varphi\overline{\theta}}}{1-|\theta|^2}\right\|_{L^\infty(\mathbb{D})}=\left\|\widetilde{A}-\varphi+\left(\varphi-\frac{\varphi}{1-|\theta|^2}\right)+\frac{\theta\widetilde{\varphi\overline{\theta}}}{1-|\theta|^2}\right\|_{L^\infty(\mathbb{D})}\\ &=\left\|\left(\widetilde{A}-\varphi\right)-\frac{|\theta|^2\varphi-\theta\widetilde{\varphi\overline{\theta}}}{1-|\theta|^2}\right\|_{L^\infty(\mathbb{D})}\\ &\geqslant\left\|\widetilde{A}-\varphi\right\|_{L^\infty(\mathbb{D})}-\left\|\frac{\theta\left(\varphi\overline{\theta}-\widetilde{\varphi\overline{\theta}}\right)}{1-|\theta|^2}\right\|_{L^\infty(\mathbb{D})},\end{aligned}$$

and hence

$$\left\|\widetilde{A}-\varphi\right\|_{L^\infty(\mathbb{D})}\leqslant\|A-\varphi(M_\theta)\|+\left\|\frac{\varphi\overline{\theta}-\widetilde{\varphi\overline{\theta}}}{1-|\theta|^2}\right\|_{L^\infty(\mathbb{D})}, \tag{4}$$

because $|\theta(\lambda)|\leqslant 1$ for all $\lambda\in\mathbb{D}$. It follows from (4) that

$$\begin{aligned}\inf_{\varphi\in H^\infty}\left[\|A-\varphi(M_\theta)\|+\left\|\frac{\varphi\overline{\theta}-\widetilde{\varphi\overline{\theta}}}{1-|\theta|^2}\right\|_{L^\infty(\mathbb{D})}\right]&\geqslant\inf_{\varphi\in H^\infty}\left\|\widetilde{A}-\varphi\right\|_{L^\infty(\mathbb{D})} \\ &=\operatorname{dist}\left(\widetilde{A},H^\infty\right)=\left\|H_{\widetilde{A}}\right\|\ \text{(Nehari formula)}.\end{aligned} \tag{5}$$

On the other hand, it is obvious that

$$\begin{aligned}&\inf_{\varphi\in H^\infty}\left[\|A-\varphi(M_\theta)\|+\left\|\frac{\varphi\overline{\theta}-\widetilde{\varphi\overline{\theta}}}{1-|\theta|^2}\right\|_{L^\infty(\mathbb{D})}\right]\\ &\geqslant\inf_{\varphi\in H^\infty}\|A-\varphi(M_\theta)\|+\inf_{\varphi\in H^\infty}\left\|\frac{\varphi\overline{\theta}-\widetilde{\varphi\overline{\theta}}}{1-|\theta|^2}\right\|_{L^\infty(\mathbb{D})}\end{aligned}$$

and $\inf_{\varphi\in H^\infty}\left\|\frac{\varphi\overline{\theta}-\widetilde{\varphi\overline{\theta}}}{1-|\theta|^2}\right\|_{L^\infty(\mathbb{D})}=0$. Therefore we obtain from inequality (4) that

$$\begin{aligned}&\inf_{\varphi\in H^\infty}\left[\|A-\varphi(M_\theta)\|+\left\|\frac{\varphi\overline{\theta}-\widetilde{\varphi\overline{\theta}}}{1-|\theta|^2}\right\|_{L^\infty(\mathbb{D})}\right] \\ &\geqslant \inf_{\varphi\in H^\infty}\|A-\varphi(M_\theta)\|=\operatorname{dist}(A,\mathscr{F}_\theta).\end{aligned} \tag{6}$$

Now (5) and (6) prove the theorem. □

In conclusion of this section, we prove some lower inequality for $\|H_\psi\|$ in terms of values of corresponding Toeplitz operator T_ψ. In order to formulate our result, we need to some notation.

Let $(\lambda_n)_{n\geqslant 1}\subset\mathbb{D}$ be a fixed sequence, and $\mathscr{E}$ is a (closed) subspace defined by

$$\mathscr{E}^\lambda:=\left\{f\in H^2:(\lambda_n)_{n\geqslant 1}\subset \operatorname{Null}(f)\right\},$$

where $\operatorname{Null}(f)$ is the set of nulls of f in $\mathbb{D}$. We denote by $\mathscr{E}_1^\lambda$ the unit sphere of $\mathscr{E}^\lambda$.

THEOREM 6. *Let $\psi\in L^\infty(\mathbb{T})$ be a function. Then*

$$\|H_\psi\|\geqslant\sup_{f\in\mathscr{E}_1^\lambda}\sup_{n\geqslant 1}\left(1-|\lambda_n|^2\right)^{\frac{1}{2}}\left|\left(T_\psi f\right)(\lambda_n)\right|,$$

where T_ψ is the Toeplitz operator on H^2.

Proof. Let $f\in\mathscr{E}_1^\lambda$ be arbitrary. Then we have for any $h\in H^\infty$ that $\left(T_\psi f\right)(\lambda_n)=\left(T_{\psi-h}f\right)(\lambda_n)$ for all $n=1,2,\ldots$. By using this, we have:

$$\begin{aligned}\left(1-|\lambda_n|^2\right)^{\frac{1}{2}}\left(T_\psi f\right)(\lambda_n)&=\left(1-|\lambda_n|^2\right)^{\frac{1}{2}}\left(T_{\psi-h}f\right)(\lambda_n)\\ &=\left\langle T_{\psi-h}f,\hat{k}_{\lambda_n}\right\rangle=\left\langle f,T^*_{\psi-h}\hat{k}_{\lambda_n}\right\rangle\\ &=\left\langle f,T_{\overline{\psi-h}}\hat{k}_{\lambda_n}\right\rangle=\left\langle f,P_+\left(\overline{\psi-h}\right)\frac{\left(1-|\lambda_n|^2\right)^{\frac{1}{2}}}{1-\overline{\lambda}_n z}\right\rangle\\ &=\left\langle f,\overline{\psi-h}\frac{\left(1-|\lambda_n|^2\right)^{\frac{1}{2}}}{1-\overline{\lambda}_n z}\right\rangle\\ &=\int_{\mathbb{T}}\frac{\left(1-|\lambda_n|^2\right)^{\frac{1}{2}}}{1-\lambda_n\overline{\zeta}}(\psi-h)(\zeta)f(\zeta)\,dm(\zeta),\end{aligned}$$

where $dm(\zeta)$ is the normalized Lebesgue measure on the unit circle $\mathbb{T}$. From this, by using that $\left|1-\lambda_n\overline{\zeta}\right| = |\zeta-\lambda_n|$ and applying the Hölder inequality, we have that

$$\begin{aligned}\left(1-|\lambda_n|^2\right)^{\frac{1}{2}}\left|\left(T_\psi f\right)(\lambda_n)\right| &\leqslant \left(\operatorname*{ess\,sup}_{\mathbb{T}}|\psi-h|\right)\left(\int_{\mathbb{T}}\frac{1-|\lambda_n|^2}{|\zeta-\lambda_n|^2}dm(\zeta)\right)^{\frac{1}{2}} \\ &\quad\times\left(\int_{\mathbb{T}}|f(\zeta)|^2\,dm(\zeta)\right)^{\frac{1}{2}} \\ &= \|\psi-h\|_{L^\infty(\mathbb{T})} \text{ for all } h\in H^\infty,\end{aligned}$$

because $\|f\|_2 = 1$, and $\frac{1-|\lambda_n|^2}{|\zeta-\lambda_n|^2} = P_{\lambda_n}(\zeta)$ is the Poisson kernel, and therefore

$$\int_{\mathbb{T}}\frac{1-|\lambda_n|^2}{|\zeta-\lambda_n|^2}dm(\zeta) = 1.$$

Hence by virtue of Nehari formula, we obtain

$$\left\|H_\psi\right\| = \operatorname{dist}(\psi, H^\infty) = \inf_{h\in H^\infty}\|\psi-h\|_{L^\infty(\mathbb{T})} \geqslant \left(1-|\lambda_n|^2\right)^{\frac{1}{2}}\left|\left(T_\psi f\right)(\lambda_n)\right|$$

for all $f\in\mathscr{E}_1^\lambda$ and $n\in\mathbb{N}$, which implies that

$$\left\|H_\psi\right\| = \sup_{f\in\mathscr{E}_1^\lambda}\sup_{n\in\mathbb{N}}\left(1-|\lambda_n|^2\right)^{\frac{1}{2}}\left|\left(T_\psi f\right)(\lambda_n)\right|.$$

This proves the theorem. □

We set $\mathscr{E}_1^0 = \left\{f\in H^2 : f(0) = 0 \text{ and } \|f\|_2 = 1\right\}$.
The following apparently must be known.

COROLLARY 1. *If $\psi\in L^\infty(\mathbb{T})$ is a function such that $\sup_{f\in\mathscr{E}_1^0}|\langle f,\overline{\psi}\rangle| = \|\psi\|_2$, then $\|\psi\|_2 \leqslant \left\|H_\psi\right\| \leqslant \|\psi\|_\infty$. In particular, if ψ is unimodular, then $\left\|H_\psi\right\| = \|\psi\|_\infty = 1$.*

Proof. Since $\sup_{f\in\mathscr{E}_1^0}|\langle f,\overline{\psi}\rangle| = \|\psi\|_2$, the proof is immediate from Theorem 6. □

3. An inequality for the Berezin number of operators

Let A be any nonscalar (i.e. $A \neq cI$) bounded linear operator on the RKHS $\mathscr{H} = \mathscr{H}(\Omega)$. It is easy to see that

$$\operatorname{ber}(A) \leqslant \|A\|.$$

However, it is well known that in general there is no a universal constant $C > 0$ such that $\|A\| \leqslant C \operatorname{ber}(A)$ (see Karaev and Iskenderov [6] and Karaev [5], and references therein). So, it will be interesting to find a positive number α, depending from a given operator A (i.e. a number $\alpha = \alpha(A) > 0$), such that

$$\operatorname{ber}(A) \geqslant \alpha(A). \tag{7}$$

In the present section, we will prove an inequality of the type (7) for any operator $A \in \mathfrak{B}(H^2)$ in terms of Hankel operators. For the related results, see, for instance, Karaev and Iskenderov [6] and Gürdal and etc. [3].

Below $(H^\infty)_1 := \{\psi \in H^\infty : \|\psi\|_\infty \leqslant 1\}$ will denote the closed unit ball of the space H^∞.

THEOREM 7. *Let $A \in \mathfrak{B}(H^2)$ be any fixed nonscalar operator on the Hardy space H^2, such that radial boundary values $\widetilde{A}^{rad}(\xi)$ of $\widetilde{A}$ exist and are finite almost everywhere on $\mathbb{T}$. For any $\varphi \in H^\infty$, let us define the following operator on H^2:*

$$N_{\varphi,A} := T_\varphi (I - A).$$

Then

$$\operatorname{ber}(A) \geqslant \sup_{\varphi \in (H^\infty)_1} \left\| H_{\widetilde{N}^{rad}_{\varphi,A}} \right\|. \tag{8}$$

Proof. Arguing in the same manner, as in the Theorem 2 of [6] and Theorem 4 of [3], we obtain:

$$\begin{aligned}
\widetilde{N}_{\varphi,A}(\lambda) &= \left\langle N_{\varphi,A} \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \right\rangle \\
&= \left\langle T_\varphi (I - A) \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \right\rangle \\
&= \left\langle (I - A) \widehat{k}_{H^2,\lambda}, T_\varphi^* \widehat{k}_{H^2,\lambda} \right\rangle \\
&= \left\langle (I - A) \frac{\left(1 - |\lambda|^2\right)^{\frac{1}{2}}}{1 - \overline{\lambda} z}, T_\varphi^* \frac{\left(1 - |\lambda|^2\right)^{\frac{1}{2}}}{1 - \overline{\lambda} z} \right\rangle \\
&= \left\langle (I - A) \frac{\left(1 - |\lambda|^2\right)^{\frac{1}{2}}}{1 - \overline{\lambda} z}, \overline{\varphi(\lambda)} \frac{\left(1 - |\lambda|^2\right)^{\frac{1}{2}}}{1 - \overline{\lambda} z} \right\rangle \\
&= \varphi(\lambda) \left\langle (I - A) \widehat{k}_{H^2,\lambda}, \widehat{k}_{H^2,\lambda} \right\rangle = \varphi(\lambda) (I - A)^\sim (\lambda) \\
&= \varphi(\lambda) \left(1 - \widetilde{A}\right)(\lambda) = \varphi(\lambda) - \varphi(\lambda) \widetilde{A}(\lambda)
\end{aligned}$$

for all $\lambda \in \mathbb{D}$. This implies that

$$\left|\widetilde{N}_{\varphi,A}(\lambda)-\varphi(\lambda)\right| = |\varphi(\lambda)|\left|\widetilde{A}(\lambda)\right|$$

for all $\lambda \in \mathbb{D}$, and thus

$$\left|\widetilde{N}_{\varphi,A}(\lambda)-\varphi(\lambda)\right| \leqslant \sup_{\lambda\in\mathbb{D}}|\varphi(\lambda)|\sup_{\lambda\in\mathbb{D}}\left|\widetilde{A}(\lambda)\right| = \|\varphi\|_\infty \operatorname{ber}(A)$$

for any function $\varphi \in H^\infty$. Since $\widetilde{N}_{\varphi,A}(\lambda) = \varphi(\lambda)-\varphi(\lambda)\widetilde{A}(\lambda)$ and $\varphi(\lambda)$ and $\widetilde{A}(\lambda)$ have radial boundary values almost everywhere on $\mathbb{T}$, we have that $\widetilde{N}^{rad}_{\varphi,A}(\xi)$ exists for almost all $\xi \in \mathbb{T}$. Then from the last inequality we have for any $\varphi \in (H^\infty)_1$ that

$$\operatorname{ess\,sup}_{\xi\in\mathbb{T}}\left|\widetilde{N}^{rad}_{\varphi,A}(\xi)-\varphi(\xi)\right| \leqslant \operatorname{ber}(A),$$

or

$$\left\|\widetilde{N}^{rad}_{\varphi,A}-\varphi\right\|_{L^\infty(\mathbb{T})} \leqslant \operatorname{ber}(A)$$

for any $\varphi \in (H^\infty)_1$, which implies by virtue of the inequality $\|H_\psi\| \leqslant \|\psi\|_{L^\infty}$, $\psi \in L^\infty$, that (since $\widetilde{A}$ has the finite radial limits almost everywhere on $\mathbb{T}$, $\widetilde{N}^{rad}_{\varphi,A} \in L^\infty(\mathbb{T})$)

$$\operatorname{ber}(A) \geqslant \left\|H_{\widetilde{N}^{rad}_{\varphi,A}-\varphi}\right\| = \left\|H_{\widetilde{N}^{rad}_{\varphi,A}} - H_\varphi\right\| = \left\|H_{\widetilde{N}^{rad}_{\varphi,A}}\right\|$$

for any $\varphi \in (H^\infty)_1$, which obviously proves inequality (8), as desired. □

Acknowledgement. The authors thank the referee for useful remarks and suggestions which improved the presentation of the paper.

REFERENCES

[1] N. ARONZAJN, *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.

[2] M. ENGLIS, *Toeplitz operators and the Berezin transform on H^2*, Linear Algebra Appl. **223/224** (1995), 171–204.

[3] M. GÜRDAL, M. T. GARAYEV, S. SALTAN AND U. YAMANCI, *On some numerical characteristics of operators*, Arab J. Math. Sci. **21** (1) (2015), 118–126.

[4] M. T. KARAEV, *On the Riccati equations*, Monatsh. Math. **155** (2008), 161–166.

[5] M. T. KARAEV, *Reproducing kernels and Berezin symbols techniques in various questions of operator theory*, Complex Anal. Oper. Theory **7** (2013), 983–1018.

[6] M. T. KARAEV AND N. SH. ISKENDEROV, *Berezin number of operators and related questions*, Methods of Funct. Anal. Topology **19** (1) (2013), 68–72.

[7] G. S. MUSTAFAEV AND V. S. SHULMAN, *Estimates for the norms of inner derivations in some operator algebras*, Math. Notes **45** (1989), 337–341.

[8] E. NORDGREN AND P. ROSENTHAL, *Boundary values of Berezin symbols*, Oper Theory: Advances and applications **73** (1994), 362–368.

[9] V. V. PELLER, *Hankel operators and their applications*, Springer, 2003.
[10] B. SZÖKELFALVI-NAGY AND C. FOIAS, *Harmonic Analysis of Operators on Hilbert space*, North Holland, New York (1970).

(Received April 10, 2017)

Najla Altwaijry
Department of Mathematics
College of Science, King Saud University
P. O. Box 2455, Riyadh 11451, Saudi Arabia
e-mail: najla@ksu.edu.sa

Amani S. Baazeem
Department of Mathematics
College of Science, King Saud University
P. O.Box 2455, Riyadh 11451, Saudi Arabia
e-mail: amani-sb@hotmail.com

Mubariz Garayev
Department of Mathematics
College of Science, King Saud University
P. O.Box 2455, Riyadh 11451, Saudi Arabia
e-mail: mgarayev@ksu.edu.sa