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REDUCTION OF DISCRETE ALGEBRAIC RICCATI EQUATIONS: ELIMINATION OF GENERALIZED EIGENVALUES ON THE UNIT CIRCLE

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Abstract. The purpose of this paper is to introduce a two-stage procedure that can be used to decompose a discrete-time algebraic Riccati equation into a trivial part, a part that is entirely arbitrary, and a part that can be obtained by computing the set of solutions of a reduced-order Riccati equation whose associated symplectic pencil has no generalized eigenvalues on the unit circle.

1. Introduction

In the past fifty years, Riccati equations have been found to emerge as fundamental tools in several branches of engineering and applied mathematics, including network analysis, optimal control and filtering, spectral factorization, stochastic realization to name only a few. Several monographs have been entirely devoted to the study of Riccati equations, [20, 21, 15, 14, 1].

In particular, many techniques have appeared in the literature on the issue of the reduction of the order of Riccati equations. These contributions include – but are far from being limited to – [17, 11, 12, 13, 4, 9, 18]. The development of these techniques has been even more intense for the case of discrete-time algebraic (and difference) Riccati equations, because the structure of these equations is richer and more challenging than the structure of their continuous-time counterpart. Two main theoretical/computational difficulties arise in the determination of the set of solutions of a discrete-time algebraic Riccati equation. The first is the case in which the symplectic pencil and/or the closed-loop matrix is singular. The second is the one where some generalized eigenvalues of the symplectic pencil lie on the unit circle.

Some results that have been published on this topic have focussed on reduction techniques that are tailored to the task of computing the stabilizing solution of the Riccati equation. Some others, which include [4, 9, 18], can be employed to reduce the order of the Riccati equation to the end of obtaining the full set of Hermitian solutions of the original Riccati equation.

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In particular, in [4] a method was presented which, differently from earlier contributions presented on this topic, aimed at iteratively decomposing DARE(Σ) into a trivial part and a reduced DARE whose associated closed-loop matrix is non-singular. The subsequent contribution [9] achieves a similar goal by avoiding the need for an iterative procedure. A further important advantage of [9] over [4] lies in the fact that the technique in [9] can also be applied in the case of an indefinite Popov matrix. In [18], the method of [4] was revisited and extended to the case of the so-called generalized discrete algebraic Riccati equation, which has been the object of intensive studies in the past twenty years, because it provides an important generalization of the classic Riccati equation and, as shown e.g. in [7] and [8], it represents the most natural tool to use in the solution of indefinite/semidefinite, finite/infinite horizon discrete-time linear quadratic optimal control problems, see also [1, 5, 6, 13, 14, 19]. For the dual version in filtering problems we refer the reader to [23, 24, 25]. The framework associated with the constrained generalized Riccati equation is the one that corresponds to the case in which the symplectic pencil is singular. The procedure developed in [18] hinges on the idea of decomposing the generalized Riccati equations into two parts, which correspond to an additive decomposition $X = X_0 + \Delta$ of each solution X of the Riccati equation.

The first part provides an explicit expression of the term X_0, which is fixed and independent of the particular solution X. The second part can be either a reduced-order discrete-time standard algebraic Riccati equation whose associated closed-loop matrix is non-singular, or a symmetric Stein equation. However, regardless of the structure of the original discrete-time algebraic Riccati equation, the reduced-order regular Riccati equation obtained as a result of the application of any of the methods in [4, 9, 18] still corresponds to a closed-loop matrix which may contain eigenvalues on the unit circle, and this represents a major computational issue in the calculation of the set of solutions to this equation, see for example the MATLAB® routine `dare.m` for the computation of the stabilizing solution of the discrete-time algebraic Riccati equation.[1] The main purpose of this paper is to address this issue, by proposing a reduction whose aim is to decompose the Riccati equation that one obtains by applying one of the procedures outlined in [4, 9, 18] (which is characterized by the fact that the closed-loop matrix is non-singular) into a trivial part, a part which is arbitrary, a part that can be obtained by solving a reduced-order discrete algebraic Riccati equation, and a part that can come from the solution of a reduced-order continuous-time algebraic Riccati equation.

2. Preliminaries

This paper is concerned with the problem of computing the set of Hermitian solutions of the so-called *discrete-time algebraic Riccati equation* DARE(Σ)

$$X = A^*XA - (A^*XB + S)(R + B^*XB)^{-1}(B^*XA + S^*) + Q, \tag{1}$$

where A, B, Q, R and S are given matrices of sizes $n \times n$, $n \times m$, $n \times n$, $m \times m$ and $n \times m$, respectively, and are such that the *Popov matrix*, here denoted by Π, is

[1] When running the `dare.m` command in MATLAB®, in such case one obtains an error message which warns the user that "the symplectic spectrum is too close to the unit circle".

Hermitian and positive semidefinite, i.e., it satisfies

$$\Pi \stackrel{\text{def}}{=} \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} = \Pi^* \geqslant 0. \tag{2}$$

The set of matrices $\Sigma = (A, B; \Pi)$ is often referred to as *Popov triple*. For any matrix $X = X^* \in \mathbb{C}^{n \times n}$, we define the gain matrix

$$K_X \stackrel{\text{def}}{=} (R + B^* X B)^{-1} (S^* + B^* X A) \tag{3}$$

as well as the closed-loop matrix

$$A_X \stackrel{\text{def}}{=} A - B K_X. \tag{4}$$

As recalled in Section 1, DARE(Σ) is generalized by the so-called *constrained generalized discrete-time algebraic Riccati equation*, herein denoted by CGDARE(Σ), given by

$$X = A^*XA - (A^*XB + S)(R + B^*XB)^\dagger (S^* + B^*XA) + Q, \tag{5}$$

$$\ker(R + B^* X B) \subseteq \ker(A^* X B + S), \tag{6}$$

where the symbol $\dagger$ in (5) denotes the Moore-Penrose pseudo-inverse operation.[2]

CGDARE(Σ) – rather than DARE(Σ) – represents the natural equation arising in the solution of Linear Quadratic optimal control and filtering problems, [19, 8]. In fact, it is only when the underlying linear system (obtained by the full-rank factorization $\Pi = \begin{bmatrix} C^* \\ D^* \end{bmatrix} [\, C \; D \,]$ and considering a system described by the quadruple (A, B, C, D)) is left invertible that the standard DARE(Σ) admits solutions. The dynamic optimization problem, however, may still admit solutions in the more general setting where the underlying linear system is not left-invertible so that the corresponding Popov function $\Phi(z) \stackrel{\text{def}}{=} [G(z^{-1})]^* \Pi \, G(z)$, with $G(z) \stackrel{\text{def}}{=} \begin{bmatrix} (zI - A)^{-1} B \\ I_m \end{bmatrix}$, is singular. In these cases, however, the standard DARE(Σ) does not admit solutions and the correct equation that must be used to address the original optimization problem is CGDARE(Σ), see e.g. [5]. As discussed in [1, Chapt. 6], these general situations are particularly relevant in the context of stochastic control problems, see also [3, 10] and the references cited therein. It was also observed in [7] that generalized Riccati equations appear to be a more direct and natural way than the standard Riccati equations in the solution of indefinite Linear Quadratic optimal control problems. On the other hand, whenever the standard DARE(Σ) admits solutions, the set of its solutions coincides with the set of solutions of CGDARE(Σ). This means that CGDARE(Σ) is a genuine generalization of DARE(Σ). As already mentioned, in [18] two iterative procedures were presented that reduce a general CGDARE(Σ) to a DARE(Σ) of smaller order featuring a non-singular closed-loop matrix and a non-singular matrix R. Both these reduction procedures can

[2] We recall that given an *arbitrary* matrix $M \in \mathbb{C}^{h \times k}$, there exists a unique matrix $M^\dagger \in \mathbb{C}^{k \times h}$ that satisfies the following four properties: **(1)** $M M^\dagger M = M$; **(2)** $M^\dagger M M^\dagger = M^\dagger$; **(3)** $(M^\dagger M)^* = M^\dagger M$; **(4)** $(M M^\dagger)^* = M M^\dagger$. By definition, the matrix $M^\dagger$ is the *Moore-Penrose pseudo-inverse* of the matrix M.

be carried out only using the problem data A,B,Q,R,S. This means that these two procedures can be performed without the need to compute a particular solution of the Riccati equation. The fact that, when R is non-singular, CGDARE(Σ) reduces to a DARE(Σ) is a consequence of the inclusion $\ker(R+B^* X B) \subseteq \ker R$, see [18, Proposition 1] and [5, Lemma 4.1]. This paper presents an additional iterative procedure – to be carried out after the two aforementioned procedures have been applied to a Riccati equation to obtain a DARE(Σ) with non-singular matrices A_X and R – that at each step delivers a reduced order DARE where the eigenvalues on the unit circle of the closed-loop matrix have been eliminated.

Since we are considering that at each iteration of the procedure presented here we first perform the procedure in [18], we eliminate the closed-loop eigenvalues on the unit circle assuming without loss of generality that DARE(Σ) under consideration is such that A_X and R are invertible.

The procedures in [4, 9, 18], together with the technique presented in this paper, enable us to obtain the entire set of Hermitian solutions of any generalized discrete-time algebraic Riccati equation by resorting to the computation of the set of solutions of well-behaved reduced order Riccati equations or Stein equations.

We recall that the so-called *symplectic pencil* is defined as the matrix pencil $N_\Sigma - zM_\Sigma$, where

$$M_\Sigma \stackrel{\text{def}}{=} \begin{bmatrix} I_n & 0 & 0 \\ 0 & -A^* & 0 \\ 0 & -B^* & 0 \end{bmatrix} \qquad \text{and} \qquad N_\Sigma \stackrel{\text{def}}{=} \begin{bmatrix} A & 0 & B \\ Q & -I_n & S \\ S^* & 0 & R \end{bmatrix}.$$

When the matrix pencil $N_\Sigma - zM_\Sigma$ is regular (i.e., when there exists $z \in \mathbb{C}$ such that $\det(N_\Sigma - zM_\Sigma) \neq 0$), CGDARE($\Sigma$) becomes indeed a DARE($\Sigma$), whereas the case where $N_\Sigma - zM_\Sigma$ is singular (i.e., the determinant of $N_\Sigma - zM_\Sigma$ is the zero polynomial) corresponds to a case in which DARE(Σ) does not admit solutions. It is shown in [4] for DARE(Σ) and in [6] for CGDARE(Σ) that if A_X is singular, the Jordan structure of A_X associated with the eigenvalue $\lambda = 0$ is completely determined by the matrix pencil $N_\Sigma - zM_\Sigma$ (and therefore by the parameters of the problem), and is independent of the particular solution X of DARE(Σ) or CGDARE(Σ). It is also shown in [4] that in the case where the matrix pencil $N_\Sigma - zM_\Sigma$ is regular (or, equivalently, the CGDARE(Σ) and the standard DARE(Σ) have the same solutions) the following statements are equivalent:

(1) N_Σ is singular;

(2) $N_\Sigma - zM_\Sigma$ has a generalized eigenvalue at zero;

(3) there exists a solution X of CGDARE(Σ) such that the closed-loop matrix A_X is singular;

(4) for any solution X of CGDARE(Σ), the corresponding closed-loop matrix A_X is singular;

(5) at least one of the two matrices R and $A - BR^\dagger S^*$ is singular.

The following result [18] is a well-known result of the classic Riccati theory, which shows how to eliminate the cross-penalty matrix S.

LEMMA 1. *Let* $A_0 \stackrel{def}{=} A - BR^{-1}S^*$ *and* $Q_0 \stackrel{def}{=} Q - SR^{-1}S^*$. *Moreover, let* $\Pi_0 \stackrel{def}{=} \begin{bmatrix} Q_0 & 0 \\ 0 & R \end{bmatrix}$ *and* $\Sigma_0 \stackrel{def}{=} (A_0, B, \Pi_0)$. *Then, the following statements hold true:*

(i) DARE(Σ*) has the same set of Hermitian solutions as DARE(*Σ_0*)*

$$X = A_0^* X A_0 - A_0^* X B (R + B^* X B)^{-1} B^* X A_0 + Q_0; \tag{7}$$

(ii) for any Hermitian solution X *of DARE(*Σ*), we have*

$$A_X = A_{0X} \stackrel{def}{=} A_0 - B\,(R + B^* X B)^{-1} B^* X A_0;$$

(iii) $Q_0 \geqslant 0$.

Another useful result that can be established by direct computation is the following.

LEMMA 2. *Let* $T \in \mathbb{C}^{n \times n}$ *be unitary. Let* $\tilde{A}_0 \stackrel{def}{=} T^* A_0 T$, $\tilde{B} \stackrel{def}{=} T^* B$, *and* $\tilde{Q}_0 \stackrel{def}{=} T^* Q_0 T$. *Let also* $\Pi_T \stackrel{def}{=} \begin{bmatrix} Q_T & 0 \\ 0 & R \end{bmatrix}$ *and* $\Sigma_T \stackrel{def}{=} (A_T, B_T, \Pi_T)$. *Then,* X *is a Hermitian solution of DARE(*Σ*) – and therefore also of DARE(*Σ_0*) – if and only if* $\tilde{X} = T^* X T$ *is a Hermitian solution of DARE(*Σ_T*)*

$$\tilde{X} = \tilde{A}_0^* \tilde{X} \tilde{A}_0 - \tilde{A}_0^* \tilde{X} \tilde{B} (R + \tilde{B}^* \tilde{X} \tilde{B})^{-1} \tilde{B}^* \tilde{X} \tilde{A}_0 + \tilde{Q}_0. \tag{8}$$

The following lemma presents a useful decomposition of the symplectic pencil, see [6] for a proof.

LEMMA 3. *Let* X *be a symmetric solution of DARE(*Σ*). Let* $R_X = R + B^* X B$ *and let* K_X *be the associated gain and* A_X *be the associated closed-loop matrix. Two invertible matrices* U_X *and* V_X *of suitable sizes exist such that*

$$U_X\,(N_\Sigma - zM_\Sigma)\,V_X = \begin{bmatrix} A_X - zI_n & 0 & B \\ 0 & I_n - zA_X^* & 0 \\ 0 & -zB^* & R_X \end{bmatrix}. \tag{9}$$

Since R_X is non-singular, the dynamics represented by the symplectic matrix pencil $N_\Sigma - zM_\Sigma$ are decomposed into a part governed by the generalized eigenstructure of $A_X - zI_n$, a part governed by the finite generalized eigenstructure of $I_n - zA_X^*$, and a part which corresponds to the dynamics of the eigenvalue at infinity. Thus, the generalized eigenvalues[3] of $N_\Sigma - zM_\Sigma$ are given by the eigenvalues of A_X, the reciprocal of the eigenvalues of A_X, and a generalized eigenvalue at infinity whose algebraic multiplicity is equal to m.

[3]Recall that a generalized eigenvalue of a matrix pencil $N - zM$ is a value of $z \in \mathbb{C}$ for which the rank of the matrix pencil $N - zM$ is lower than its normal rank.

3. Main results

Given a solution X of DARE(Σ), the spectrum corresponding to closed-loop matrix A_X may contain eigenvalues on the unit circle

$$\mathfrak{D} \stackrel{\text{def}}{=} \{z \in \mathbb{C} \,:\, |z| = 1\}.$$

In this section, we show how DARE(Σ) can be decomposed into a part that has a solution which is completely arbitrary, and which is associated with the eigenvalues on the unit circle of A_X, and a part that can be computed by solving a reduced-order Riccati equation.

In particular, from now on we will refer to DARE(Σ_0), where we recall that $\Sigma_0 = (A_0, B, \Pi_0)$ as defined in Lemma 1, since its set of solutions coincides with that of DARE(Σ). The corresponding symplectic pencil is $z N_{\Sigma_0} - M_{\Sigma_0}$. First, if A_X contains eigenvalues on the unit circle, the symplectic pencil $z N_{\Sigma_0} - M_{\Sigma_0}$ also contains generalized eigenvalues on the unit circle in view of Lemma 3. Let $\theta \in \mathfrak{D}$ be an eigenvalue of A_X on the unit circle. The matrix pencil

$$N_{\Sigma_0} - \theta\, M_{\Sigma_0} = \begin{bmatrix} A_0 - \theta\, I_n & 0 & B \\ Q_0 & \theta\, A_0^* - I_n & 0 \\ 0 & B^*\,\theta & R \end{bmatrix}$$

loses rank with respect to the normal rank of $N_{\Sigma_0} - z M_{\Sigma_0}$. Since R is invertible, this implies that its Schur complement

$$\begin{aligned} W_\theta &\stackrel{\text{def}}{=} \begin{bmatrix} A_0 - \theta\, I_n & -\theta\, B R^{-1} B^* \\ Q_0 & \theta\, A_0^* - I_n \end{bmatrix} \\ &= \begin{bmatrix} A - \theta\, I_n & 0 \\ Q & \theta\, A^* - I_n \end{bmatrix} - \begin{bmatrix} B \\ S \end{bmatrix} R^{-1} [\, S^* \; B^*\,\theta \,] \end{aligned}$$

loses rank. We now investigate a very important property of the null-space of W_θ.

LEMMA 4. *Let* $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ *with* $v_1, v_2 \in \mathbb{C}^n$. *Let* $\theta \in \mathfrak{D}$ *be such that* $\operatorname{rank} W_\theta < \operatorname{normrank} W$. *Then,* $v \in \ker W_\theta$ *if and only if* $\begin{bmatrix} v_1 \\ 0 \end{bmatrix} \in \ker W_\theta$ *and* $\begin{bmatrix} 0 \\ v_2 \end{bmatrix} \in \ker W_\theta$.

Proof. Sufficiency is obvious. Let us prove necessity. Let $v \in \ker W_\theta$. We can write

$$\begin{bmatrix} -\theta\, B R^{-1} B^* & A_0 - \theta\, I_n \\ \theta\, A_0^* - I_n & Q_0 \end{bmatrix} \begin{bmatrix} v_2 \\ v_1 \end{bmatrix} = 0 \tag{10}$$

In a suitable basis of the state space, Q_0 and A_0 can be written as

$$Q_0 = \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_0 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where Λ is invertible. Let q denote its order. Let

$$K_\theta \stackrel{\text{def}}{=} \begin{bmatrix} A_{11} - \theta I_q \\ A_{21} \end{bmatrix}$$

and

$$H_\theta \stackrel{\text{def}}{=} \begin{bmatrix} A_{12} \\ A_{22} - \theta I_{n-q} \end{bmatrix},$$

so that, taking into account that $\theta\,\theta^* = 1$ because $\theta \in \mathfrak{D}$, we can rewrite (10) as

$$\left[\begin{array}{c|cc} -\theta\, B R^{-1} B^* & K_\theta & H_\theta \\ \hline \theta\, K_\theta^* & \Lambda & 0 \\ \theta\, H_\theta^* & 0 & 0 \end{array}\right] \left[\begin{array}{c} v_2 \\ \hline v_{11} \\ v_{12} \end{array}\right] = 0, \tag{11}$$

where $v_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$ has been partitioned conformably with Q_0. From the second we find $v_{11} = -\theta\,\Lambda^{-1} K_\theta^*\, v_2$. Substituting this expression into the first equation obtained by expanding (11) gives

$$-\,\theta\, B R^{-1} B^*\, v_2 + K_\theta\, v_{11} + H_\theta\, v_{12} = 0. \tag{12}$$

Premultiplying both sides of (12) by v_2^*, and taking into account that $H_\theta^*\, v_2 = 0$ using the third equation obtained from (11), yields $\theta\, v_2^* L_\theta\, v_2 = 0$, where $L_\theta \stackrel{\text{def}}{=} B R^{-1} B^* + K_\theta \Lambda^{-1} K_\theta^*$. Since both R and Λ are positive definite, then $B^*\, v_2 = 0$ and $K_\theta^*\, v_2 = 0$. Since we have also $H_\theta^*\, v_2 = 0$, we can conclude that

$$v_2 \in \ker \begin{bmatrix} B^* \\ \theta\, A_0^* - I_n \end{bmatrix} = \ker \begin{bmatrix} \theta\, B R^{-1} B^* \\ \theta\, A_0^* - I_n \end{bmatrix},$$

which also implies that $\begin{bmatrix} 0 \\ v_2 \end{bmatrix} \in \ker W_\theta$. Moreover, from $v_{11} = -\theta\,\Lambda^{-1} K_\theta^*\, v_2$ and $K_\theta^*\, v_2 = 0$ we obtain $v_{11} = 0$ which, together with $H_\theta\, v_{12} = 0$, leads to $(A_0 - \theta\, I_n)\, v_1 = 0$ and $Q_0\, v_1 = 0$, so that indeed $\begin{bmatrix} v_1 \\ 0 \end{bmatrix} \in \ker W_\theta$. □

Thanks to Lemma 4, we can always consider as a basis matrix for the null-space of W_θ a block matrix in the form $\begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}$, where V_1 is a basis matrix of the kernel of $\begin{bmatrix} A_0 - \theta\, I_n \\ Q_0 \end{bmatrix}$ and V_2 is a basis of the kernel of $\begin{bmatrix} \theta\, B R^{-1} B^* \\ \theta\, A_0^* - I_n \end{bmatrix}$. This enables us to introduce two separate and independent reduction procedures for DARE(Σ_0). The first aims at eliminating vectors from the null-space of W_θ that are in the range of V_2. The second is a reduction that eliminates vectors from $\ker W_\theta$ that are in the range of V_1. Differently from the problem of eliminating the singularity from the closed-loop matrix, where the Jordan structure of the zero eigenvalue of the closed-loop is completely determined by the symplectic pencil $N_\Sigma - z M_\Sigma$ (which in turn is an explicit function of the problem data A, B, Q, R, S), here we have no *a priori* information on the Jordan structure of the eigenvalues of A_X in $\mathfrak{D}$. The iterative nature cannot be avoided by adapting in a straightforward manner the techniques such as the one discussed in [9].

4. Reduction associated with V_2

As already observed, we begin by examining the first reduction technique, which can be carried out if V_2 is non-zero. We need to distinguish between two cases.

4.1. Case 1: $\theta \in \{-1,1\}$

We now consider a change of basis in the original DARE(Σ_0), using the result of Lemma 2, where the change of coordinate T is real-valued. In particular, we define the change of coordinate matrix $T = [\,T_1\ T_2\,]$, where T_1 is an orthonormal basis for $\operatorname{im} V_2$ and T is orthogonal (so that $T^{-1} = T^\top = T^*$). Thus, the subspace $\operatorname{im} V_2$, whose dimension is denoted by v, is written in the new basis as $\operatorname{im}\begin{bmatrix} I_v \\ 0 \end{bmatrix}$. We define the matrices $\tilde{A}_0$, $\tilde{B}$ and $\tilde{Q}_0$ as in Lemma 2.

Since $(\theta A_0^* - I_n) V_2 = 0$, we have also $\theta A_0^* V_2 = V_2$, which can be expressed in the new basis as $\theta \tilde{A}_0^* T^* V_2 = T^* V_2$. Thus, in the new basis we can write

$$\tilde{A}_0^* = \begin{bmatrix} I_v\,\theta & A_{21}^* \\ 0 & A_{22}^* \end{bmatrix}, \tag{13}$$

so that indeed $\theta \begin{bmatrix} I_v\,\theta & A_{21}^* \\ 0 & A_{22}^* \end{bmatrix} \begin{bmatrix} I_v \\ 0 \end{bmatrix} = \begin{bmatrix} I_v \\ 0 \end{bmatrix}$. From $B^* V_2 = 0$, we find $\tilde{B}^* = [\,0\ B_2^*\,]$. Consider the decomposition of $\tilde{X} = T^* X\, T$ and $\tilde{Q}_0 = T^* Q_0\, T$ into block matrices whose sizes are compatible with the decomposition in (13), i.e.,

$$\tilde{X} = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}, \qquad \tilde{Q}_0 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix}.$$

One can verify by direct inspection that the following equalities hold:

$$\begin{aligned}
\tilde{A}_0^* \tilde{X} \tilde{A}_0 &= \begin{bmatrix} X_{11} + X_{12} A_{21}\,\theta + A_{21}^* X_{12}^*\,\theta + A_{21}^* X_{22} A_{21} & X_{12} A_{22}\,\theta + A_{21}^* X_{22} A_{22} \\ A_{22}^* X_{12}^*\,\theta + A_{22}^* X_{22} A_{21} & A_{22}^* X_{22} A_{22} \end{bmatrix}, \\
\tilde{A}_0^* \tilde{X} \tilde{B} &= \begin{bmatrix} \theta^* X_{12} B_2 + A_{21}^* X_{22} B_2 \\ A_{22}^* X_{22} B_2 \end{bmatrix}, \\
R + \tilde{B}^* \tilde{X} \tilde{B} &= R + [\,0\ B_2^*\,] \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = R + B_2^* X_{22} B_2.
\end{aligned}$$

We define $R_2 \stackrel{\text{def}}{=} R + B_2^* X_{22} B_2$ to simplify the notation. Using these expressions, we can write (8) as

$$\begin{aligned}
\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} &= \begin{bmatrix} X_{11} + X_{12} A_{21}\,\theta + A_{21}^* X_{12}^*\,\theta + A_{21}^* X_{22} A_{21} & X_{12} A_{22}\,\theta + A_{21}^* X_{22} A_{22} \\ A_{22}^* X_{12}^*\,\theta + A_{22}^* X_{22} A_{21} & A_{22}^* X_{22} A_{22} \end{bmatrix} \\
&\quad - \begin{bmatrix} \theta X_{12} B_2 + A_{21}^* X_{22} B_2 \\ A_{22}^* X_{22} B_2 \end{bmatrix} R_2^{-1} [\,\theta B_2^* X_{12}^* + B_2^* X_{22} A_{21}\ \ B_2^* X_{22} A_{22}\,] \\
&\quad + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix},
\end{aligned}$$

which leads to the three equations

$$0 = X_{12}A_{21}\,\theta + A_{21}^* X_{12}^*\,\theta + A_{21}^* X_{22}A_{21} \\ \quad -(\theta X_{12}B_2 + A_{21}^* X_{22}B_2)R_2^{-1}(\theta B_2^* X_{12}^* + B_2^* X_{22}A_{21}) + Q_{11}, \tag{14}$$

$$X_{12} = X_{12}A_{22}\,\theta + A_{21}^* X_{22}A_{22} - (\theta X_{12}B_2 + A_{21}^* X_{22}B_2)R_2^{-1}B_2^* X_{22}A_{22} + Q_{12}, \tag{15}$$

$$X_{22} = A_{22}^* X_{22}A_{22} - A_{22}^* X_{22}B_2R_2^{-1}B_2^* X_{22}A_{22} + Q_{22}. \tag{16}$$

We notice the following facts:

- None of these equations depend on X_{11}. Thus, X_{11} is completely arbitrary.
- The third equation (16) is decoupled from the previous two (14–15), and is a reduced-order DARE. This equation can be solved independently of X_{12}. If (16) does not admit solutions, the original DARE has no solutions.
- Once X_{22} is computed using (16), it can be substituted into (15), which then becomes a linear equation in X_{12}:

$$X_{12} = X_{12}\,\theta\,(A_{22} - B_2R_2^{-1}B_2^* X_{22}A_{22}) + (A_{21}^* X_{22}A_{22} - A_{21}^* X_{22}B_2R_2^{-1}B_2^* X_{22}A_{22} + Q_{12}) \\ = X_{12}\,\theta A_{X_{22}} + (A_{21}^* X_{22}A_{X_{22}} + Q_{12}), \tag{17}$$

where the matrix $A_{X_{22}} \overset{\text{def}}{=} A_{22} - B_2(R + B_2^* X_{22}B_2)^{-1}B_2^* X_{22}A_{22}$ is the closed-loop matrix relative to the subsystem 22. Thus, (17) can be written as

$$X_{12}(I - \theta A_{X_{22}}) = A_{21}^* X_{22}A_{X_{22}} + Q_{12}.$$

This equation admits solutions if and only if [4]

$$\ker(I - \theta A_{X_{22}}) \subseteq \ker(A_{21}^* X_{22}A_{X_{22}} + Q_{12}). \tag{18}$$

If this condition is not satisfied, then (15) does not admit solutions. Thus, also the original DARE does not admit solutions. If (18) is satisfied and $A_{X_{22}}$ has no eigenvalues at θ, matrix $I - \theta A_{X_{22}}$ is invertible, and (15) has only one solution

$$X_{12}^\circ = (A_{21}^* X_{22}A_{X_{22}} + Q_{12})(I - \theta A_{X_{22}})^{-1}. \tag{19}$$

It is sufficient to check whether this solution also satisfies (14). If it does not, again, the original DARE does not admit solutions, while if the only solution X_{12}° of (15) also solves (14), we have parameterized the solutions of DARE into

$$\begin{bmatrix} X_{11} & X_{12}^\circ \\ (X_{12}^\circ)^* & X_{22} \end{bmatrix},$$

where X_{11} is arbitrary, X_{22} is the solution of a reduced-order DARE and X_{12}° is the only solution that satisfies simultaneously (14) and (15).

[4]This condition can equivalently expressed by saying that for any matrix Ξ such that $(I - \theta A_{X_{22}})\Xi = 0$, we also have $(A_{21}^* X_{22}A_{X_{22}} + Q_{12})\Xi = 0$, i.e., $(A_{21}^* X_{22}\,\theta + Q_{12})\Xi = 0$.

We may also have the case in which (15) has infinite solutions. The set of its solutions is parameterized in terms of a matrix of suitable size K as

$$X_{12} = \widehat{X}_{12} + K\Delta, \quad \text{where} \quad \widehat{X}_{12} \stackrel{\text{def}}{=} (A_{21}^* X_{22} A_{X_{22}} + Q_{12})\,(I - \theta A_{X_{22}})^{\dagger},$$

with the rows of Δ span the null-space of $\ker(I - \theta A_{X_{22}}^*)$, i.e., $\Delta = \theta\,\Delta A_{X,22}$. By substitution of $X_{12} = \widehat{X}_{12} + K\Delta$ into (14) we obtain a new equation in Δ, which reads as

$$K\Delta\theta[A_{21} - B_2 R_2^{-1} B_2^* (\theta \widehat{X}_{12}^* + X_{22} A_{21})] + [A_{21}^* - (\theta \widehat{X}_{21} + A_{21}^* X_{22}) B_2 R_2^{-1} B_2^*]\theta \Delta^* K^* \\ - K\,\Delta B_2 R_2^{-1} B_2^* \Delta^* K^* + \Omega = 0, \tag{20}$$

where

$$\Omega \stackrel{\text{def}}{=} \widehat{X}_{12} A_{21}\,\theta + A_{21}^* \widehat{X}_{12}^* \theta + A_{21}^* X_{22} A_{21} \\ - (\theta\,\widehat{X}_{12} + A_{21}^* X_{22})\,B_2 R_2^{-1} B_2^*\,(\theta\,\widehat{X}_{12}^* + X_{22} A_{21}) + Q_{11} \geqslant 0.$$

Interestingly, (20) is a reduced-order non-square continuous-time Riccati equation, for which a rich literature is available, see e.g. [16, 2] and the references cited therein.

4.2. Case 2: $\theta \in \mathfrak{D} \setminus \{1, -1\}$

We now consider a change of basis given by $T = [\,T_1\ T_2\ T_3\,]$, where T_1 is an orthonormal basis for $\operatorname{im} V_2$, each entry in T_2 is the complex conjugate of the corresponding entry in T_1 and T is unitary (so that $T^{-1} = T^*$). Again, the subspace $\operatorname{im} V_2$ in the new basis is written as $\operatorname{im}\begin{bmatrix} I_v \\ 0 \end{bmatrix}$. In this case, partitioning A_0 conformably with this basis, we find

$$\tilde{A}_0^* = T^* A_0^* T = \begin{bmatrix} \theta^* I_v & 0 & A_{31}^* \\ 0 & \theta I_v & A_{32}^* \\ 0 & 0 & A_{33}^* \end{bmatrix} \quad \text{and} \quad \tilde{B}^* = B^* T = [\,0\ 0\ B_3^*\,].$$

Let us also partition the matrices $\tilde{X} = T^* X T$ and $\tilde{Q}_0 = T^* Q_0 T$ accordingly as

$$\tilde{X} = T^* X T = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^* & X_{22} & X_{23} \\ X_{13}^* & X_{23}^* & X_{33} \end{bmatrix} \quad \text{and} \quad \tilde{Q}_0 = T^* Q_0 T = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^* & Q_{22} & Q_{23} \\ Q_{13}^* & Q_{23}^* & Q_{33} \end{bmatrix}.$$

One can directly check that

$$\tilde{A}_0^* \tilde{X} \tilde{A}_0 = \\ \begin{bmatrix} X_{11} + \theta^* X_{13} A_{31} + \theta A_{31}^* X_{13}^* + A_{31}^* X_{33} A_{31} & (\theta^*)^2 X_{12} + \theta^* X_{13} A_{32} + \theta^* A_{31}^* X_{23}^* + A_{31}^* X_{33} A_{32} & \theta^* X_{13} A_{33} + A_{31}^* X_{33} A_{33} \\ \star & X_{22} + \theta X_{23} A_{32} + \theta^* A_{32}^* X_{23}^* + A_{32}^* X_{33} A_{32} & \theta X_{23} A_{33} + A_{32}^* X_{33} A_{33} \\ \star & \star & A_{33}^* X_{33} A_{33} \end{bmatrix},$$

where the submatrices indicated by $\star$ are obtained by taking the complex conjugate of the block entries that are symmetric with respect to the main diagonal. Then,

$$\tilde{A}_0^* \tilde{X} \tilde{B} = \begin{bmatrix} \theta^* X_{13} B_3 + A_{31}^* X_{33} B_3 \\ \theta X_{23} B_3 + A_{32}^* X_{33} B_3 \\ A_{33}^* X_{33} B_3 \end{bmatrix} \quad \text{and} \quad R_3 \stackrel{\text{def}}{=} R + \tilde{B}^* \tilde{X} \tilde{B} = R + B_3^* X_{33} B_3.$$

We obtain the following 6 matrix equations:

$$\begin{aligned} 0 &= \theta^* X_{13} A_{31} + \theta A_{31}^* X_{13}^* + A_{31}^* X_{33} A_{31} \\ &\quad -(\theta^* X_{13} B_3 + A_{31}^* X_{33} B_3) R_3^{-1} (\theta B_3^* X_{13}^* + B_3^* X_{33}^* A_{31}) + Q_{11}, && (21) \\ X_{12} &= (\theta^*)^2 X_{12} + \theta^* X_{13} A_{32} + \theta^* A_{31}^* X_{23}^* + A_{31}^* X_{33} A_{32} \\ &\quad -(\theta^* X_{13} B_3 + A_{31}^* X_{33} B_3) R_3^{-1} (\theta^* B_3^* X_{23}^* + B_3^* X_{33} A_{32}) + Q_{12}, && (22) \\ X_{13} &= \theta^* X_{13} A_{33} + A_{31}^* X_{33} A_{33} - (\theta^* X_{13} B_3 + A_{31}^* X_{33} B_3) R_3^{-1} B_3^* X_{33} A_{33} + Q_{13}, && (23) \\ 0 &= \theta X_{23} A_{32} + \theta^* A_{32}^* X_{23}^* + A_{32}^* X_{33} A_{32} \\ &\quad -(\theta X_{23} B_3 + A_{32}^* X_{33} B_3) R_3^{-1} (\theta^* B_3^* X_{23}^* + B_3^* X_{33} A_{32}) + Q_{22}, && (24) \\ X_{23} &= \theta X_{23} A_{33} + A_{32}^* X_{33} A_{33} - (\theta X_{23} B_3 + A_{32}^* X_{33} B_3) R_3^{-1} B_3^* X_{33} A_{33} + Q_{23} && (25) \\ X_{33} &= A_{33}^* X_{33} A_{33} - A_{33}^* X_{33} B_3 R_3^{-1} B_3^* X_{33} A_{33} + Q_{33}. && (26) \end{aligned}$$

None of these equations depends on X_{11} and X_{22}, which are therefore completely arbitrary. Moreover, the last equation (which is a reduced-order DARE with complex coefficients) can be solved in X_{33} independently of the others. Denoting by

$$A_{X_{33}} \stackrel{\text{def}}{=} A_{33} - B_3 R_3^{-1} B_3^* X_{33} A_{33}$$

the closed-loop matrix that corresponds to the solution X_{33} of the reduced-order DARE (26), equations (23) and (25) can respectively be written as

$$X_{13} (I - \theta^* A_{X_{33}}) = A_{31}^* X_{33} A_{X_{33}} + Q_{13} \tag{27}$$

$$X_{23} (I - \theta A_{X_{33}}) = A_{32}^* X_{33} A_{X_{33}} + Q_{23} \tag{28}$$

which are linear in X_{13} and X_{23}, respectively. They admit solutions if and only if $\ker(I - \theta^* A_{X_{33}}) \subseteq \ker(A_{31}^* X_{33} A_{X_{33}} + Q_{13})$ and $\ker(I - \theta A_{X_{33}}) \subseteq \ker(A_{32}^* X_{33} A_{X_{33}} + Q_{23})$, respectively. We can parameterize the set of solutions of (27) as $X_{13} = \widehat{X}_{13} + K_{13} \Delta_{13}$, where $\widehat{X}_{13} \stackrel{\text{def}}{=} (A_{31}^* X_{33} A_{X_{33}} + Q_{13}) (I - \theta^* A_{X_{33}})^\dagger$ and the rows of Δ_{13} span the null-space of $\ker(I - \theta^* A_{X_{33}})$, so that $\operatorname{im} \Delta_{13}^* = \ker(I - \theta^* A_{X_{33}})$. Similarly, the set of solutions of (28) can be written as $X_{23} = \widehat{X}_{23} + K_{23} \Delta_{23}$, where $\widehat{X}_{23} \stackrel{\text{def}}{=} (A_{32}^* X_{33} A_{X_{33}} + Q_{23}) (I - \theta A_{X_{33}})^\dagger$ and $\operatorname{im} \Delta_{23}^* = \ker(I - \theta A_{X_{33}})$.

Substitution of $X_{13} = \widehat{X}_{13} + K_{13} \Delta_{13}$ and $X_{23} = \widehat{X}_{23} + K_{23} \Delta_{23}$ into (23) and (25) yields

$$\begin{aligned} & K_{13}\Delta_{13}\theta^*[A_{31} - B_3 R_3^{-1} B_3^* (\theta \widehat{X}_{13}^* + X_{33} A_{31})] + [A_{31}^* - (\theta^* \widehat{X}_{31} + A_{31}^* X_{33}) B_3 R_3^{-1} B_3^*] \theta \Delta_{13}^* K_{13}^* \\ & \qquad - K_{13}\Delta_{13} B_3 R_3^{-1} B_3^* \Delta_{13}^* K_{13}^* + \Omega_{13} = 0, && (29) \\ & K_{23}\Delta_{23}\theta[A_{32} - B_3 R_3^{-1} B_3^* (\theta^* \widehat{X}_{23}^* + X_{33} A_{32})] + [A_{32}^* - (\theta \widehat{X}_{32} + A_{32}^* X_{33}) B_3 R_3^{-1} B_3^*] \theta^* \Delta_{23}^* K_{23}^* \\ & \qquad - K_{23}\Delta_{23} B_3 R_3^{-1} B_3^* \Delta_{23}^* K_{23}^* + \Omega_{23} = 0, && (30) \end{aligned}$$

respectively, where

$$\begin{aligned}\Omega_{13} &\stackrel{\text{def}}{=} \widehat{X}_{13} A_{31}\, \theta^* + A_{31}^* \widehat{X}_{13}^* \theta + A_{31}^* X_{33} A_{31} \\ &\quad -(\theta^* \widehat{X}_{13} + A_{31}^* X_{33}) B_3 R_3^{-1} B_3^* (\theta \widehat{X}_{13}^* + X_{33} A_{31}) + Q_{13}, \\ \Omega_{23} &\stackrel{\text{def}}{=} \widehat{X}_{23} A_{32}\, \theta + A_{32}^* \widehat{X}_{23}^* \theta^* + A_{32}^* X_{33} A_{32} \\ &\quad -(\theta \widehat{X}_{23} + A_{32}^* X_{33}) B_3 R_3^{-1} B_3^* (\theta^* \widehat{X}_{23}^* + X_{33} A_{32}) + Q_{23}.\end{aligned}$$

Once X_{13} and X_{23} have been computed, and one verifies that they also satisfy the first and the fourth equation, X_{12} can be computed from the second equation, which is linear in X_{12}, and always admits solutions if $\theta \neq \pm 1$.

EXAMPLE 4.1. Consider DARE(Σ) with

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 2 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 12 \\ 0 \\ 0 \end{bmatrix}, \quad R = 36.$$

It is easily seen that $A_0 = \begin{bmatrix} 0 & 0 & -1 \\ 5/3 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $Q_0 = 0$. It is also easy to see that W_θ loses rank for $\theta = \pm i$. Consider $\theta = -i$. Then

$$\ker W_{-i} = \operatorname{im} \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ i & 0 \\ \hline 0 & 3 \\ 0 & 2+i \\ 0 & 3i \end{array}\right]$$

An orthonormal basis matrix for the upper block of the latter is given by $V_2 = \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ i\sqrt{2}/2 \end{bmatrix}$, so that we have a unitary change of coordinates $T = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & -1 \\ i\sqrt{2}/2 & -i\sqrt{2}/2 & 0 \end{bmatrix}$. We easily find

$$\tilde{A}_0^* = T^* A_0^* T = \begin{bmatrix} i & 0 & -\frac{5}{3\sqrt{2}} \\ 0 & -i & -\frac{5}{3\sqrt{2}} \\ 0 & 0 & -2 \end{bmatrix},$$

from which we find $A_{31} = A_{32} = -\frac{5}{3\sqrt{2}}$ and $A_{33} = -2$. Let us also partition $T^* X T$ accordingly as

$$T^* X T = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{12}^* & x_{22} & x_{23} \\ x_{13}^* & x_{23}^* & x_{33} \end{bmatrix}.$$

Equation (26) in this case reduces to

$$x_{33} = 4 x_{33} - \frac{4 x_{33}^2}{36 + x_{33}},$$

whose solutions are $x_{33}=0$ and $x_{33}=108$. Let us first consider the solution $x_{33}=0$. In this case, $A_{X_{33}}=A_{33}$, and (23) reduces to $x_{13}=-2ix_{13}$, whose unique solution is $x_{13}=0$. Similarly, equation (24) becomes $(1-2i)x_{23}=0$, whose solution is $x_{23}=0$. Notice that $x_{13}=x_{23}=0$ satisfy the first and the fourth equations. We only need to compute x_{12} using (22), which in this case becomes $x_{12}=-x_{12}$, so that $x_{12}=0$. It follows that the set of all solutions of the transformed DARE that arise from $x_{33}=0$ can be written as $\operatorname{diag}\{\alpha,\beta,0\}$, where $\alpha,\beta\in\mathbb{R}$ are arbitrary. Thus, the corresponding solution in the original basis is

$$X=T\begin{bmatrix}\alpha&0&0\\0&\beta&0\\0&0&0\end{bmatrix}T^*=\begin{bmatrix}\frac{1}{2}(\alpha+\beta)&0&\frac{i}{2}(\beta-\alpha)\\0&0&0\\\frac{i}{2}(\alpha-\beta)&0&\frac{1}{2}(\alpha+\beta)\end{bmatrix}.$$

The first set of Hermitian solutions of DARE is therefore given by

$$\mathscr{X}_{x_{33}=0}=\left\{\begin{bmatrix}p&0&iq\\0&0&0\\-iq&0&p\end{bmatrix}:p,q\in\mathbb{R}\right\}.$$

Let us now consider $x_{33}=108$. In this case, $A_{X_{33}}=-1/2$, and (23) reduces to $x_{13}=-\frac{1}{2}ix_{13}+\frac{5}{6\sqrt{2}}x_{33}$, whose solution is $x_{13}=36\sqrt{2}-18\sqrt{2}i$. Similarly, (25) gives $x_{23}=36\sqrt{2}+18\sqrt{2}i$. It is easily seen that x_{13} satisfies the first equation, and x_{23} satisfies the fourth equation. Finally, the second equation gives $x_{12}=18-24i$. It follows that

$$\tilde{X}=\begin{bmatrix}\alpha&18-24i&36\sqrt{2}-18\sqrt{2}i\\18+24i&\beta&36\sqrt{2}+18\sqrt{2}i\\36\sqrt{2}+18\sqrt{2}i&36\sqrt{2}-18\sqrt{2}i&108\end{bmatrix}$$

is another solution of the transformed DARE for any $\alpha,\beta\in\mathbb{R}$. In the original basis we have

$$X=\begin{bmatrix}\frac{1}{2}(\alpha+\beta)+18&-72&24-\frac{i}{2}(\alpha-\beta)\\-72&108&-36\\24+\frac{i}{2}(\alpha-\beta)&-36&-18+\frac{1}{2}(\alpha+\beta)\end{bmatrix}.$$

The second set of Hermitian solutions of DARE is therefore given by

$$\mathscr{X}_{x_{33}=108}=\left\{\begin{bmatrix}r+18&-72&24-is\\-72&108&-36\\24+is&-36&-18+r\end{bmatrix}:r,s\in\mathbb{R}\right\},$$

and the complete set of Hermitian solutions of the original DARE is given by $\mathscr{X}_{x_{33}=0}\cup\mathscr{X}_{x_{33}=108}$.

4.3. Reduction associated with V_1

In this section we examine the second procedure aimed at the elimination of V_1. We assume that we have carried out the reduction procedure associated with the presence of V_2. As in the previous case, we need to distinguish between two cases.

4.4. Case 1: $\theta \in \{1,-1\}$

We now consider a change of basis in $\mathbb{R}^n$ given by $T = [\,T_1\ T_2\,]$, where T_1 is an orthonormal basis for V_1 and T is orthogonal. Thus, the subspace $\operatorname{im} V_1$, whose dimension is denoted by μ, in the new basis is written as $\operatorname{im}\begin{bmatrix} I_\mu \\ 0 \end{bmatrix}$. Since $(A_0-\theta\, I_n)\,V_1 = 0$, we have also $\theta\, A_0\, V_1 = V_1$, which can be written in the new basis as $\theta T^* A_0 T\, T^* V_1 = T^* V_1$. Thus, in the new basis

$$T^* A_0 T = \begin{bmatrix} I_\mu\, \theta & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

so that indeed $\theta \begin{bmatrix} I_\nu\, \theta & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} I_\mu \\ 0 \end{bmatrix} = \begin{bmatrix} I_\mu \\ 0 \end{bmatrix}$. From $Q_0\, V_1 = 0$, we find that in this basis $T^* Q_0\, T = \operatorname{diag}\{0, Q_{22}\}$. Let us consider the DARE in this new basis

$$\begin{aligned} T^*\, X\, T &= (T^*\, A_0^*\, T)\,(T^*\, X\, T)\,(T^*\, A_0\, T) - (T^*\, A_0^*\, T)\,(T^*\, X\, T)\,(T^*\, B) \\ &\quad \times \big(R + (B^*\, T)\,(T^*\, X\, T)\,(T^*\, B)\big)^{\dagger}\,(B^*\, T)\,(T^*\, X\, T)\,(T^*\, A_0\, T) + (T^*\, Q_0\, T). \end{aligned} \tag{31}$$

Let us also introduce the partitioning

$$T^* X\, T = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}, \qquad T^* B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

One can immediately verify that

$$\begin{aligned} &(T^* A_0^* T)\,(T^* X T)\,(T^* A_0\, T) \\ &\quad = \begin{bmatrix} X_{11} & \theta\,(X_{11}\, A_{12} + X_{12}\, A_{22}) \\ \theta\,(A_{12}^*\, X_{11} + A_{22}^*\, X_{12}) & A_{12}^*\, X_{11}\, A_{12} + A_{22}^*\, X_{12}^*\, A_{12} + A_{12}^*\, X_{12}\, A_{22} + A_{22}^*\, X_{22}\, A_{22}. \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} &(B\, T)\,(T^*\, X\, T)\,(T^*\, A_0\, T) \\ &\quad = [\,\theta\,(B_1^*\, X_{11} + B_2^*\, X_{12}^*)\ \ B_1^*\, X_{11}\, A_{12} + B_1^*\, X_{12}\, A_{22} + B_2^*\, X_{12}^*\, A_{12} + B_2^*\, X_{22}\, A_{22}\,]. \end{aligned}$$

Finally,

$$R_X \stackrel{\text{def}}{=} R + (B^*\, T)\,(T^*\, X\, T)\,(T^*\, B) = R + B_1^*\, X_{11}\, B_1 + B_1^*\, X_{12}\, B_2 + B_2^*\, X_{12}^*\, B_1 + B_2^*\, X_{22}\, B_2.$$

In this case, the Riccati equation can be written as the three equations

$$\begin{aligned} 0 &= -(X_{11}\, B_1 + X_{12}\, B_2)\, R_X^{-1}\,(B_1^*\, X_{11} + B_2^*\, X_{12}^*), \\ X_{12} &= \theta\, X_{11}\, A_{12} + \theta\, X_{12}\, A_{22} \\ &\qquad -\theta\,(X_{11}\, B_1 + X_{12}\, B_2)\, R_X^{-1}\,(B_1^*\, X_{11}\, A_{12} + B_1^*\, X_{12}\, A_{22} + B_2^*\, X_{12}^*\, A_{12} + B_2^*\, X_{22}\, A_{22}), \\ X_{22} &= A_{12}^*\, X_{11}\, A_{12} + A_{22}^*\, X_{12}^*\, A_{12} + A_{12}^*\, X_{12}\, A_{22} + A_{22}^*\, X_{22}\, A_{22} \\ &\qquad -(A_{12}^*\, X_{11}\, B_1 + A_{22}^*\, X_{12}^*\, B_1 + A_{12}^*\, X_{12}\, B_2 + A_{22}^*\, X_{22}\, B_2)\, R_X^{-1} \\ &\qquad \times (B_1^*\, X_{11}\, A_{12} + B_1^*\, X_{12}\, A_{22} + B_2^*\, X_{12}^*\, A_{12} + B_2^*\, X_{22}\, A_{22}) + Q_{22}. \end{aligned}$$

The first yields $X_{11}B_1 + X_{12}B_2 = 0$, which once substituted into the second yields $X_{12} = \theta X_{11}A_{12} + \theta X_{12}A_{22}$. These two equations can be written together as

$$\begin{bmatrix} \theta A_{12}^* & \theta A_{22} - I_{n-\mu} \\ B_1^* & B_2^* \end{bmatrix} \begin{bmatrix} X_{11} \\ X_{12}^* \end{bmatrix} = 0.$$

On the other hand, since the first elimination procedure has already been carried out, the nullspace of the matrix

$$\begin{bmatrix} \theta A_0^* - I \\ B^* \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \theta A_{12}^* & \theta A_{22} - I_{n-\mu} \\ B_1^* & B_2^* \end{bmatrix}$$

is the origin. This implies that the submatrices X_{11} and X_{12} are zero. Therefore, the third equation can be written as

$$X_{22} = A_{22}^* X_{22} A_{22} - A_{22}^* X_{22} B_2 (R + B_2^* X_{22} B_2)^{-1} B_2^* X_{22} A_{22} + Q_{22},$$

which is a reduced-order Riccati equation.

4.5. Case 2: $\theta \in \mathfrak{D} \setminus \{1, -1\}$

We now consider a change of basis given by $T = [\,T_1\ T_2\ T_3\,]$, where T_1 is an orthonormal basis for $\operatorname{im} V_1$, $T_2 = \overline{T}_1$ and T is unitary. Thus, $\operatorname{im} V_1$, whose dimension is denoted by μ, in the new basis is written as $\operatorname{im}\begin{bmatrix} I_\mu \\ 0 \end{bmatrix}$. In this case we find

$$T^* A_0 T = \begin{bmatrix} \theta I_\mu & 0 & A_{13} \\ 0 & \theta^* I_\mu & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \qquad \text{and} \qquad T^* Q_0 T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Q_{33} \end{bmatrix}.$$

We partition T^*B and $T^*X\,T$ conformably as

$$T^* X\, T = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{12}^* & X_{22} & X_{23} \\ X_{13}^* & X_{23}^* & X_{33} \end{bmatrix} \qquad \text{and} \qquad T^* B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}.$$

One can easily verify that

$$(T^* A_0^* T)(T^* X T)(T^* A_0 T) = \begin{bmatrix} X_{11} & X_{12}(\theta^*)^2 & \theta^*(X_{11}A_{13} + X_{12}A_{23} + X_{13}A_{33}) \\ \star & X_{22} & \theta(X_{12}^* A_{13} + X_{22}A_{23} + X_{23}A_{33}) \\ \star & \star & \Xi \end{bmatrix},$$

where the submatrices indicated by $\star$ are obvious from the context and where

$$\begin{aligned} \Xi = {} & A_{13}^* X_{11} A_{13} + A_{23}^* X_{12} A_{13} + A_{33}^* X_{13}^* A_{13} + A_{13}^* X_{12} A_{23} \\ & + A_{23}^* X_{22} A_{23} + A_{33}^* X_{23}^* A_{23} + A_{13}^* X_{13} A_{33} + A_{23}^* X_{23} A_{33} + A_{33}^* X_{33}^* A_{33}. \end{aligned}$$

Moreover,

$$(B\,T)(T^* X\, T)(T^* A_0 T) = [\,\theta(B_1^* X_{11} + B_2^* X_{12} + B_3^* X_{13})\ \ \theta^*(B_1^* X_{12} + B_2^* X_{22} + B_3^* X_{23})\ \ \Phi\,],$$

where

$$\begin{aligned}\Phi \stackrel{\text{def}}{=} & B_1^* X_{11} A_{13} + B_1^* X_{12} A_{23} + B_1^* X_{13} A_{33} \\ & + B_2^* X_{12} A_{13} + B_2^* X_{22} A_{23} + B_2^* X_{23} A_{33} \\ & + B_3^* X_{13} A_{13} + B_3^* X_{23} A_{23} + B_3^* X_{33} A_{33}.\end{aligned}$$

Again, $R_X \stackrel{\text{def}}{=} R + (B^* T)(T^* X T)(T^* B)$. Writing the block submatrix in position (1,1) of DARE written in this basis, we find

$$X_{11} = X_{11} - (X_{11} B_1 + X_{12} B_2 + X_{13} B_3) R_X^{-1} (B_1^* X_{11} + B_2^* X_{12} + B_3^* X_{13}),$$

which yields $X_{11} B_1 + X_{12} B_2 + X_{13} B_3 = 0$. Writing the block submatrix in position (1,1) of DARE written in this basis, we find

$$X_{12} = (\theta^*)^2 \left(X_{12} - (X_{11} B_1 + X_{12} B_2 + X_{13} B_3) R_X^{-1} (B_1^* X_{12} + B_2^* X_{22} + B_3^* X_{23}) \right)$$

Using the identity $X_{11} B_1 + X_{12} B_2 + X_{13} B_3 = 0$ found above in the block submatrix in position (1,2) of DARE gives the equation $X_{12} = X_{12} (\theta^*)^2$. Since $\theta \notin \{1, -1\}$, the only solution is $X_{12} = 0$. The block submatrix in position (1,3) of DARE with respect to this basis is

$$X_{13} = \theta^* X_{11} A_{13} + \theta^* X_{12} A_{23} + \theta^* X_{13} A_{33} - (X_{11} B_1 + X_{12} B_2 + X_{13} B_3) R_X^{-1} \Phi.$$

Using $X_{11} B_1 + X_{12} B_2 + X_{13} B_3 = X_{11} B_1 + X_{13} B_3 = 0$ and $X_{12} = 0$ into the latter gives $X_{13} = \theta^* X_{11} A_{13} + \theta^* X_{13} A_{33}$. This equation and $X_{11} B_1 + X_{12} B_2 + X_{13} B_3 = 0$ can be written together as

$$\begin{bmatrix} \theta A_{13}^* & \theta A_{33} - I \\ B_1^* & B_3^* \end{bmatrix} \begin{bmatrix} X_{11} \\ X_{13}^* \end{bmatrix} = 0.$$

Since the first elimination procedure has already been carried out, the null-space of the matrix $\begin{bmatrix} \theta A_0^* - I \\ B^* \end{bmatrix}$ is the origin. Thus, X_{11} and X_{13} are zero. In a similar way, the block submatrix in position (2,2) is

$$X_{22} = X_{22} - (X_{12}^* B_1 + X_{22} B_2 + X_{23} B_3) R_X^{-1} (B_1^* X_{12} + B_2^* X_{22} + B_3^* X_{23}),$$

from which we find $X_{12}^* B_1 + X_{22} B_2 + X_{23} B_3 = 0$. It follows that the submatrix in position (2,3) is

$$X_{23} = \theta X_{22} A_{23} + \theta X_{23} A_{33}.$$

With the same argument used above, we find that X_{23} and X_{22} are zero. As a result of this discussion, we can write the submatrix in position (3,3) as

$$X_{33} = A_{33}^* X_{33} A_{33} - A_{33}^* X_{33} B_3 (R + B_3^* X_{33} B_3)^{-1} B_3^* X_{33} A_{33} + Q_{33},$$

which is again a reduced-order Riccati equation.

5. Numerical examples

EXAMPLE 5.1. Consider the DARE with

$$A = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 9 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad R = 16.$$

Matrix W_θ loses rank at $\theta = -1$. Then $\ker W_{-1} = \operatorname{span}[0\ 0\ 0 | 0\ 1\ 0]^\top$. Let $V_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Consider the change of coordinate matrix

$$T = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right],$$

which gives $T^* A_0 T = \begin{bmatrix} -1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then, $A_{12} = [-2\ 0]$ and $A_{22} = 0$. Moreover $T^* B = \begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix}$, which implies $B_1 = 9$ and $B_2 = 0$. It follows that X_{11} and X_{12} are zero, and the reduced-order Riccati equation is simply

$$X_{22} = Q_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, the only solution of the original Riccati equation is

$$X = T \left[\begin{array}{c|cc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] T^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

EXAMPLE 5.2. Consider the DARE with

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ -7 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 67 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \quad S = \begin{bmatrix} -72 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad R = 81.$$

It is easily seen that $Q_0 = \operatorname{diag}\{3, 0, 0, 2\}$ and

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & -2 & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{56}{9} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Matrix W_θ loses rank at $\theta = 0$ and $\theta = \pm i$. Let $\theta = i$. Then $\ker W_i = [0\ 0\ 0\ 0 | 0\ 1\ i\ 0]^\top$. Let $V_1 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ i/\sqrt{2} \\ 0 \end{bmatrix}$. The null-space of $\begin{bmatrix} iBR^{-1}B^* \\ iA_0^* - I \end{bmatrix}$ is $\{0\}$. Therefore, only the reduction relative to V_1 needs to be carried out. Consider the change of coordinates

$$T = \left[\begin{array}{c|c|cc} 0 & 0 & 1 & 0 \\ \hline \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \hline \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right],$$

which leads to

$$T^* A_0 T = \left[\begin{array}{c|c|cc} i & 0 & -\sqrt{2}+\frac{28}{9}\sqrt{2}i & -\sqrt{2}i \\ \hline 0 & -i & -\sqrt{2}-\frac{28}{9}\sqrt{2}i & \sqrt{2}i \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad T^* B = \left[\begin{array}{c} \frac{7}{\sqrt{2}}i \\ \hline -\frac{7}{\sqrt{2}}i \\ \hline 0 \\ 0 \end{array}\right]$$

and

$$T^* Q_0 T = \operatorname{diag}\{0,0,3,2\}.$$

It follows that A_{33} is the zero matrix, so that $X_{33} = Q_{33} = \operatorname{diag}\{3,2\}$. Thus, the only solution of the original DARE is $X = T \operatorname{diag}\{0,0,3,2\} T^* = \operatorname{diag}\{3,0,0,2\}$.

Concluding remarks

In this paper we have presented a reduction technique aimed at decomposing a discrete algebraic Riccati equation into a part that is arbitrary, and a part that can be obtained by computing the set of solutions of a reduced-order Riccati equation whose associated symplectic pencil has no generalized eigenvalues on the unit circle. A delicate computational issue, which was not discussed in this paper, is the following. In practice, the generalized eigenvalues of the symplectic pencil must be computed numerically. Thus, it will very rarely occur that their modulus is exactly one. Therefore, it is necessary to select a threshold that can be used to discriminate between the generalized eigenvalues that can be numerically considered to be on the unit circle from those that are structurally outside it.

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