Operators
and
Matrices
Volume 12, Number 1 (2018), 189–203 doi:10.7153/oam-2018-12-12

WEIGHTED COMPOSITION AND BLOCK SHIFT MATRIX OPERATORS ON $\ell^2(\mathbb{N})$ SPACE

M. R. JABBARZADEH AND Z. MOAYYERIZADEH

(*Communicated by G. Misra*)

Abstract. In this paper we investigate some clasees of weighted composition operators on the Hilbert space of complex valued functions on the natural numbers. Next we introduce a new model of a block matrix operator $M(\alpha,\beta)$ induced by two sequences α and β and characterize its p-paranormality. Then we give examples of these operators to show that the p-paranormal classes are distinct.

1. Introduction and preliminaries

Let (X,Σ,μ) be a complete σ-finite measure space and let $\mathscr{A}$ be a sub-σ-finite algebra of Σ. We use the notation $L^2(\mathscr{A})$ for $L^2(X,\mathscr{A},\mu_{|_{\mathscr{A}}})$ and henceforth we write μ in place of $\mu_{|_{\mathscr{A}}}$. All comparisons between two functions or two sets are to be interpreted as holding up to a μ-null set. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. The support of $f\in L^0(\Sigma)$ is defined by $\sigma(f)=\{x\in X : f(x)\neq 0\}$. Let $T:X\to X$ be a transformation such that $T^{-1}(\Sigma)\subseteq\Sigma$ and $\mu\circ T^{-1}\ll\mu$. It is assumed that the Radon-Nikodym derivative $h=d\mu\circ T^{-1}/d\mu$ is finite-valued or equivalently $(X,T^{-1}(\Sigma),\mu)$ is σ-finite.

The associated conditional expectation with respect to $\mathscr{A}$ is denoted by $E^{\mathscr{A}}_{\mu}$, or when no confusion will arise, simply $E^{\mathscr{A}}$. We recall that $E^{\mathscr{A}}: L^2(\Sigma)\to L^2(\mathscr{A})$ is a surjective, positive, contractive orthogonal projection which satisfies $E^{\mathscr{A}}(E^{\mathscr{A}}(f)g)=E^{\mathscr{A}}(f)E^{\mathscr{A}}(g)$ for $f,g\in L^2(\Sigma)$. For more details see [13]. For a non-negative finite valued measurable function $u\in L^0(\Sigma)$, the weighted composition operator W on $L^2(\Sigma)$ induced by T and u is given by $W=M_u\circ C_T$ where M_u is a multiplication operator and C_T is a composition operator on $L^2(\Sigma)$ defined by $M_uf=uf$ and $C_Tf=f\circ T$, respectively. The assumption of non-negativity for u guarantees the existence of $E(u)$. The interested reader will see how generalizations for complex-valued u may be made. Put $\mathscr{A}=T^{-1}(\Sigma)$ and $E^{T^{-1}(\Sigma)}=E$. It is a classical fact that W is a bounded linear operator on $L^2(\Sigma)$ if and only if $J:=hE(u^2)\circ T^{-1}\in L^\infty(\Sigma)$ (see [7]). Throughout this paper we assume that $J\in L^\infty(\Sigma)$. Let $\mathscr{H}$ be the infinite dimensional complex Hilbert space and $\mathscr{L}(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$. Let $A=U|A|$ be the canonical polar decomposition for $A\in\mathscr{L}(\mathscr{H})$.

Mathematics subject classification (2010): 47B20, 47B38.
Keywords and phrases: Weighted composition operator, conditional expectation, hyponormal operator, block matrix operator.

Paper OaM-12-12

- A is hyponormal if $A^*A - AA^* \geqslant 0$.
- A is quasinormal if $AA^*A = A^*AA$.
- A is quasihyponormal if $A^*(A^*A)A \geqslant A^*(AA^*)A$.
- A is p-paranormal if $\||A|^pU|A|^px\| \geqslant \||A|^px\|^2$, for all unit vectors $x \in \mathscr{H}$.
- A is defined to be of class $(M;k)$ if $(A^*)^kA^k \geqslant (A^*A)^k$, for all integer $k \geqslant 2$.
- A is M-paranormal operator if for unit vectors $x \in \mathscr{H}$, $\|Ax\|^2 \leqslant M\|A^2x\|$

Put $p = 1$, then it is clear that A is paranormal if $\||A|U|A|x\| \geqslant \||A|x\|^2$, for all unit vectors $x \in \mathscr{H}$, moreover by using the property of read quadratic forms A is paranormal operator if and only if for all integer $k \geqslant 0$, $|A|U^*|A|^2U|A| - 2k|A|^2 + k^2 \geqslant 0$, see [9]. The hierarchical relationship between the classes is as follows: quasinormal $\Longrightarrow$ hyponormal $\Longrightarrow$ $(M;2)$ class $\Longrightarrow$ paranormal; quasihyponormal $\Longrightarrow$ paranormal, (see[5]).

The fundamental properties of weighted composition operators on measurable function spaces are studied by many mathematicians see [1, 3, 6, 7, 8, 10, 14, 15].

In this article we will restrict ourselves to the Hilbert space $\ell^2(\mathbb{N})$ of complex-valued functions on the natural numbers. The space of $\ell^2(\mathbb{N})$ can also be denoted by $L^2(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where $2^{\mathbb{N}}$ is the power set of natural numbers and μ is a measure on $2^{\mathbb{N}}$ define by $\mu(\{n\}) = m_n$ where $\{m_n\}_{n=1}^{\infty}$ be a sequance of positive real numbers.

This paper consists of three sections. In section 2 we discuss measure theoretic characterizations for weighted composition operators in some operator classes on $\ell^2(\mathbb{N})$ such as cohyponormal, coquasinormal, quasihyponormal, paranormal and M-paranormal. A key tool in this case is the notion of conditional expectation. In [4] Exner, Jung and Lee introduced an interesting block matrix operator and characterized its p-hyponrmality. In section 3 we define a new block shift matrix such that in the special case its corresponding operator on $\ell^2(\mathbb{N})$ is a shift operator. We obtain the p-paranormality criteria of these type block matrices. Finally, we give examples to show that block shift matrix operators can separate these classes.

2. Some Characterizations

Let $\{e_n\}_{n\in\mathbb{N}}$ is an orthornormal basis for $\ell^2(\mathbb{N})$, and $f = \sum_{n\in\mathbb{N}} f_n e_n$ be in $\ell^2(\mathbb{N})$. Put $J = hE(u^2) \circ T^{-1}$. Then some direct computations show that for each $k \in \mathbb{N}$:

$$h(k) = \frac{1}{m_k} \sum_{j\in T^{-1}(k)} m_j; \tag{2.1}$$

$$E(f)(k) = \frac{\sum_{j\in T^{-1}(T(k))} f_j m_j}{\sum_{j\in T^{-1}(T(k))} m_j}; \tag{2.2}$$

$$J(k) = \frac{1}{m_k} \sum_{j\in T^{-1}(k)} (u(j))^2 m_j. \tag{2.3}$$

We shall make use of the following general properties of E and W see [2, 7, 11]. For each $f \in L^2(\Sigma)$,

- $W^*f = hE(uf)\circ T^{-1}$.
- $W^*Wf = hE(u^2)\circ T^{-1}f$.
- $WW^*f = u(h\circ T)E(uf)$.

Let $U|W|$ be the polar decomposition of W. It is easy to check that $U(f) = \frac{u.f\circ T}{\sqrt{h\circ TE(u^2)}}$, (see [9]).

LEMMA 2.1. *Let $T:\mathbb{N}\to\mathbb{N}$, $u:\mathbb{N}\to[0,\infty)$, and $W\in\mathscr{L}(\ell^2(\mathbb{N}))$. Then we have*

(1) $Wf = \sum_{n=1}^{\infty} u(n)f_{T(n)}e_n$.

(2) $We_k = \sum_{n\in T^{-1}(k)} u(n)e_n$.

(3) $W^*f = \sum_{n=1}^{\infty}\frac{1}{m_n}\sum_{j\in T^{-1}(n)} u(j)f_jm_je_n$.

(4) $W^*e_k = \frac{m_k}{m_{T(k)}}u(k)e_{T(k)}$.

(5) $WW^*e_n = \frac{m_n}{m_{T(n)}}\sum_{k\in T^{-1}(T(n))} u(k)u(n)e_k$.

(6) $W^*We_n = \frac{1}{m_n}\sum_{j\in T^{-1}(n)} (u(j))^2m_je_n$.

(7) $WW^*We_n = \chi_{T^{-1}(n)}\sum_{k\in T^{-1}(n)}\frac{1}{m_{T(k)}}\sum_{j\in T^{-1}(n)} (u(j))^2u(k)m_je_k$.

(8) $W^*WWe_n = \chi_{T^{-1}(n)}\sum_{k\in T^{-1}(n)}\frac{1}{m_k}\sum_{j\in T^{-1}(k)} (u(j))^2u(k)m_je_k$.

Proof.

(1) Since for each $n\in\mathbb{N}$, $Wf(n) = u(n)f_{T(n)}$, so $Wf = \sum_{n=1}^{\infty}u(n)f_{T(n)}e_n$. Put $f = e_k$. Then $We_k = \sum_{n=1}^{\infty} u(n)\langle e_k, e_{T(n)}\rangle e_n$, and so $We_k = \sum_{n\in T^{-1}(k)} u(n)e_n$.

(3) For each $n\in\mathbb{N}$, by using (2.3) we get that

$$W^*f(n) = (hE(uf)\circ T^{-1})(n) = \frac{1}{m_n}\sum_{j\in T^{-1}(n)} u(j)f_jm_j.$$

It follows that,

$$W^*f = \sum_{n=1}^{\infty}\frac{1}{m_n}\sum_{j\in T^{-1}(n)} u(j)f_jm_je_n.$$

In particular,

$$W^*e_k = \sum_{n=1}^{\infty} \frac{1}{m_n} \sum_{j\in T^{-1}(n)} m_j \langle e_k, e_j\rangle u(j)e_n = \frac{m_k}{m_{T(k)}} u(k)e_{T(k)}.$$

(5) By (2.1) and (2.2), for each $n,k \in \mathbb{N}$, we have

$$\begin{aligned}(WW^*e_n)(k) &= u(k)(h\circ T)(k)E(ue_n)(k)\\ &= u(k)\frac{1}{m_{T(k)}} \sum_{j\in T^{-1}(T(k))} m_j \frac{\sum_{j\in T^{-1}(T(k))} u(j)e_n(j)m_j}{\sum_{j\in T^{-1}(T(k))} m_j}\\ &= \frac{u(k)}{m_{T(k)}} \sum_{j\in T^{-1}(T(k))} u(j)e_{n(j)m_j}.\end{aligned}$$

Hence

$$WW^*e_n = \frac{m_n}{m_{T(n)}} \sum_{k\in T^{-1}(T(n))} u(k)u(n)e_k.$$

(6) Since $W^*Wf = Jf$, then it follows by (2.3).
(7) Let $T^{-1}(n) \neq \emptyset$. It is easy to see that

$$WW^*We_n = WW^*(ue_n\circ T) = u(h\circ T)E(u^2(e_n\circ T)).$$

Hence

$$\begin{aligned}WW^*We_n(k) &= \frac{u(k)}{m_{T(k)}} \sum_{j\in T^{-1}(T(k))} m_j \frac{\sum_{j\in T^{-1}(T(k))} (u(j))^2 m_j e_n(T(j))}{\sum_{j\in T^{-1}(T(k))} m_j}\\ &= \frac{u(k)}{m_{T(k)}} \sum_{j\in T^{-1}(n)} (u(j))^2 m_j,\end{aligned}$$

and so

$$WW^*We_n = \sum_{k\in T^{-1}(n)} \sum_{j\in T^{-1}(n)} \frac{1}{m_{T(k)}} u(k)(u(j))^2 m_j e_k.$$

(8) By (2.3), for each $n,k \in \mathbb{N}$ with $T^{-1}(n) \neq \emptyset$, we have

$$\begin{aligned}W^*WWe_n(k) &= h(k)E(uu_2(e_n\circ T^2))\circ T^{-1}(k)\\ &= \frac{1}{m_k} \sum_{j\in T^{-1}(k)} (u(j))^2 u(T(j))e_n(T^2(j))m_j,\end{aligned}$$

and hence $W^*WWe_n = \sum_{k\in T^{-1}(n)} \frac{1}{m_k} \sum_{j\in T^{-1}(k)} (u(j))^2 u(k) m_j e_k.$ $\square$

THEOREM 2.2. ([11]) *Let W be a weighted composition operator on $L^2(\Sigma)$. Then W is hyponormal if and only if*

(i) $\sigma(u) \subseteq \sigma(J)$ and

(ii) $(h \circ T)E(\frac{u^2}{J}) \leqslant \chi_{\sigma(E(u))}$ (the fraction is interpreted as 0 off $\sigma(J)$).

By definition of J, we have $\mathbb{N} \setminus \sigma(J) = \{j \in \mathbb{N} : T^{-1}(j) = \emptyset \text{ or } u(T^{-1}(j)) = \{0\}\}$. Hence $\sigma(J) = \{n \in \mathbb{N} : T^{-1}(\{n\}) \cap \sigma(u) \neq \emptyset\} = T(\sigma(u))$. Also, by (2.1), (2.2), (2.3) and $\sigma(u) \subseteq \sigma(E(u))$ for each $k \in \mathbb{N}$, we have

$$h \circ T(k) = \frac{1}{m_{T(k)}} \sum_{j \in T^{-1}(T(k))} m_j,$$

$$E\left(\frac{u^2}{J}\right)(k) = \frac{\sum_{j \in T^{-1}(T(k))} \frac{u^2(j)}{J(j)} m_j}{\sum_{j \in T^{-1}(T(k))} m_j}.$$

These observations establish the following theorem.

THEOREM 2.3. *Let W is bounded weighted composition operator on $\ell^2(\mathbb{N})$. Then W is hyponormal if and only if $\sigma(u) \subseteq T(\sigma(u))$ and for each $k \in \sigma(u)$,*

$$\frac{1}{m_{T(k)}} \sum_{j \in T^{-1}(T(k))} \frac{(u(j))^2 m_j^2}{\sum_{s \in T^{-1}(j)} (u(s))^2 m_s} \leqslant 1.$$

THEOREM 2.4. *W^* is hyponormal if and only if the restriction T to $\sigma(J)$ is injective and for each $k \in \sigma(J)$,*

$$\frac{1}{m_k} \sum_{j \in T^{-1}(k)} (u(j))^2 m_j \leqslant \frac{1}{m_{T(k)}} (u(k))^2 m_k.$$

Proof. From [2, Theorem 4.2], W^* is hyponormal on $L^2(\Sigma)$ if and only if

(i) $J \leqslant J \circ T$

(ii) $\Sigma \cap \sigma(J) \subseteq T^{-1}(\Sigma) \cap \sigma(u)$.

From equation (2.3), (i) is equivalent to $\sigma(J) \subseteq \sigma(E(u))$ and

$$\frac{1}{m_k} \sum_{j \in T^{-1}(k)} (u(j))^2 m_j \leqslant \frac{1}{m_{T(k)}} \sum_{j \in T^{-1}(T(k))} (u(j))^2 m_j \tag{2.4}$$

for each $k \in \sigma(J)$. On $\ell^2(\mathbb{N})$, condition (ii) is equivalent to: if $k \in \sigma(J)$, then $\{k\} = T^{-1}(A) \cap \sigma(u)$ for some $A \subseteq \mathbb{N}$. Since $k \in T^{-1}(T(k))$ and $k \notin T^{-1}(n)$ for any $n \neq T(k)$, so that condition (ii) is equivalent to: if $k \in \sigma(J)$, then $\{k\} = T^{-1}(T(k)) \cap \sigma(u)$. This is clearly equivalent to the injectivity of the restriction T to $\sigma(J) \subseteq \sigma(u)$. Thus the right-hand side of (2.4) is equal to $\frac{1}{m_{T(k)}}(u(k))^2 m_k$, which completes the proof. □

THEOREM 2.5. *For $W \in \mathscr{L}(\ell^2(\mathbb{N}))$, the following assertions hold.*
(i) W is quasinormal if and only if for each $k \in \sigma(u)$,

$$\frac{1}{m_k} \sum_{j \in T^{-1}(k)} (u(j))^2 m_j = \frac{1}{m_{T(k)}} \sum_{j \in T^{-1}(T(k))} (u(j))^2 m_j.$$

(ii) W^* is quasinormal if and only if the restriction T to $\sigma(J)$ is injective and for each $k \in \sigma(J)$,

$$\frac{1}{m_k} \sum_{j \in T^{-1}(k)} (u(j))^2 m_j = \frac{m_k}{m_{T(k)}} (u(k))^2 m_k.$$

Proof. (i) From [11], W is quasinormal on $L^2(\Sigma)$ if and only if $J = J \circ T$ on $\sigma(u)$. Now, the desired conclusion follows from (2.3). Note that

$$\sigma(h) \supseteq \sigma(J) \supseteq \sigma(J) \cap \sigma(u) = \sigma(J \circ T) \cap \sigma(u) = \sigma(E(u)) \cap \sigma(u) = \sigma(u),$$

and so for each $k \in \sigma(u)$, $T^{-1}(k) \neq \emptyset$.

(ii) By [2, Theorem 4.4 Simplified], W^* is quasinormal on $L^2(\Sigma)$ if and only if $J \circ T = J$ on $\sigma(J)$ and $\Sigma \cap \sigma(J) \subseteq T^{-1}\Sigma \cap \sigma(u)$. The result follows immediately from Theorem 2.4. $\square$

THEOREM 2.6. *Let W be a weighted composition operator on $\ell^2(\mathbb{N})$. Then the following statements are equivalent*

(i) *W is quasihyponormal.*

(ii) *W is paranormal.*

(iii) $\frac{1}{m_n} \left(\sum_{s \in T^{-1}(n)} (u(s))^2 m_s \right)^2 \leqslant \sum_{s \in T^{-1}(n)} (u(s))^2 \sum_{j \in T^{-1}(s)} (u(j))^2 m_j$, *for each $n \in \mathbb{N}$.*

Proof. $(i) \Leftrightarrow (iii)$ By Lemma 2.1, we get that

$$\begin{aligned} W^*(W^*W)We_n &= W^* \left(\sum_{k \in T^{-1}(n)} \frac{1}{m_k} \sum_{j \in T^{-1}(k)} (u(j))^2 u(k) m_j e_k \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{m_n} \sum_{s \in T^{-1}(n)} u(s) \left(\sum_{j \in T^{-1}(s)} \frac{1}{m_s} u(s)(u(j))^2 m_j \right) m_s e_n \\ &= \sum_{n=1}^{\infty} \sum_{s \in T^{-1}(n)} \sum_{j \in T^{-1}(s)} \frac{1}{m_n} (u(s))^2 (u(j))^2 m_j e_n, \end{aligned}$$

and

$$W^*(WW^*)We_n = W^*\left(\sum_{k\in T^{-1}(n)}\sum_{j\in T^{-1}(n)}(u(j))^2\frac{u(k)}{m_{T(k)}}m_je_k\right)$$
$$= \sum_{n=1}^{\infty}\frac{1}{m_n}\sum_{s\in T^{-1}(n)}\sum_{j\in T^{-1}(n)}\frac{m_s}{m_{T(s)}}(u(s))^2(u(j))^2m_je_n,$$

for each $n\in\mathbb{N}$. Now, (iii) follows from the inequality $W^*(W^*W)W \geqslant W^*(WW^*)W$.

$(ii) \Leftrightarrow (iii)$ It is easy to see that for each $n\in\mathbb{N}$, $|W|^2(e_n) = Je_n$ and

$$|W|U^*|W|^2U|W|(e_n) = W^*(W^*WW(e_n))$$
$$= \sum_{n=1}^{\infty}\sum_{s\in T^{-1}(n)}\sum_{j\in T^{-1}(s)}\frac{1}{m_n}(u(s))^2(u(j))^2m_je_n.$$

Thus W is paranormal if and only if (iii) holds. □

For each $n\in\mathbb{N}$, let $T^{-n}(\Sigma)$ is a σ-finite algebra of Σ. Put $J_n = h_nE_n(u_n^2)\circ T^{-n}$, where $u_n = u(u\circ T)(u\circ T^2)\cdots(u\circ T^{n-1})$, $E^{T^{-n}(\Sigma)} = E_n$ and $h_n = d\mu\circ T^{-n}/d\mu$. Set $J_1 = J$, $h_1 = h$, $E_1 = E$. Also by relations (2.1) and (2.3) for each $k\in\mathbb{N}$, we obtain

$$h_n(k) = \frac{1}{m_k}\sum_{j\in T^{-n}(k)}m_j;$$

and

$$J_n(k) = \frac{1}{m_k}\sum_{j\in T^{-n}(k)}(u_n(j))^2m_j.$$

THEOREM 2.7. *Let $W\in\mathscr{L}(\ell^2(\mathbb{N}))$. Then*
(i) W is of class (M,k) with $k\geqslant 2$ if and only if

$$\left(\frac{1}{m_n}\right)^k\left(\sum_{j\in T^{-1}(n)}(u(j))^2m_j\right)^k \leqslant \frac{1}{m_n}\sum_{j\in T^{-k}(n)}u_k(j)^2m_j.$$

(ii) W is M-paranormal if and only if

$$\left(\frac{1}{m_n}\right)^2\left(\sum_{j\in T^{-1}(n)}(u(j))^2m_j\right)^2 \leqslant M^2\frac{1}{m_n}\sum_{j\in T^{-2}(n)}u_2(j)^2m_j.$$

Proof. According to [12, Theorem 2.1], W is of class (M,k) if and only if $J^k \leqslant J_k$. Also by [12, Theorem 2.3], W is M-paranormal if and only $J^2 \leqslant M^2J_2$. Now, (i) and (ii) follows from (2.3) □

3. Block shift matrix operators

Let $\alpha := \{a_i^n\}_{\substack{1\leqslant i\leqslant r\\ 0\leqslant n<\infty}}$ and $\beta := \{b_j^n\}_{\substack{1\leqslant j\leqslant s\\ 0\leqslant n<\infty}}$ be bounded sequences of positive real numbers. Let $M(\alpha,\beta) := [A_{ij}]_{0\leqslant i,j<\infty}$ be a block matrix operator whose blocks are $(r+s)\times(r+2)$ matrices such that $A_{ij}=0$, $i\neq j$, and

$$A_n := A_{nn} = \begin{bmatrix} 0 & a_1^{(n)} & & O & \\ & & \ddots & & \\ & & & a_r^{(n)} & \\ & & & & b_1^{(n)} \\ & O & & & \vdots \\ & & & & b_s^{(n)} \end{bmatrix}. \tag{3.1}$$

where other entries are 0 except a_*^n and b_*^n in (3.1)

DEFINITION 3.1. For two bounded sequences $\alpha:=\{a_i^n\}_{\substack{1\leqslant i\leqslant r\\ 0\leqslant n<\infty}}$ and $\beta:=\{b_j^n\}_{\substack{1\leqslant j\leqslant s\\ 0\leqslant n<\infty}}$, the block matrix operator $M := M(\alpha,\beta)$ satisfying in (3.1) is called a block shift matrix operator with weight sequence (α,β).

Let M be a block shift matrix operator with weight sequence (α,β) and let $W_{\alpha,\beta}$ be its corresponding operator on ℓ^2 relative to some orthornormal basis. Then $W_{\alpha,\beta}$ may provide a repetitive form; for example $r=2$, $s=3$ and $a_i^{(n)}=b_j^{(n)}=1$ for all $i,j,n\in\mathbb{N}$, then the block matrix operator with (α,β) is unitarily equivalent to the following operator $W_{\alpha,\beta}$ on ℓ^2 defined by

$$W_{\alpha,\beta}(x_1,x_2,x_3,x_4.x_5,\ldots) = (x_2,x_3,x_4,x_4,x_4,x_5,\ldots).$$

Now we put $X=\mathbb{N}_0$ and the power set $\mathscr{P}(X)$ of X for the σ-algebra Σ. Define a non-singular measurable transformation T on $\mathbb{N}_0$ such that

$$\begin{aligned} T^{-1}(k(r+1)+r+1) &= \{k(r+s)+i+r-1 : 1\leqslant i\leqslant s\}, \quad k=0,1,2,\ldots, \\ T^{-1}(k(r+1)+i) &= k(r+s)+i-1, \quad 1\leqslant i\leqslant r, \quad k=0,1,2,\ldots \end{aligned} \tag{3.2}$$

We write $m(\{i\}) := m_i, i\in\mathbb{N}_0$, for the underlying point mass measure on X, and we assume throughout that each m_i is strictly positive.

PROPOSITION 3.2. *The composition operator C_T on ℓ^2 defined by $C_T f = f\circ T$ is unitarily equivalent to the block shift matrix operator $M(\alpha,\beta)$, where $\alpha := \{a_i^n\}_{\substack{1\leqslant i\leqslant r\\ 0\leqslant n<\infty}}$ and $\beta := \{b_j^n\}_{\substack{1\leqslant j\leqslant s\\ 0\leqslant n<\infty}}$ and for each $n\in\mathbb{N}_0$*

$$a_i^{(n)} = \sqrt{\frac{m_{n(r+s)+i-1}}{m_{n(r+1)+i}}} \quad (1\leqslant i\leqslant r),$$

$$b_j^{(n)} = \sqrt{\frac{m_{n(r+s)+j+r-1}}{m_{n(r+1)+r+1}}} \quad (1 \leqslant j \leqslant s).$$

Proof. Let $e_i = \frac{1}{\sqrt{m_i}}\chi_i$ $(i \in \mathbb{N}_0)$. Then $\{e_i\}_{i\in\mathbb{N}_0}$ is an orthornormal basis for ℓ^2. We have

$$C_T e_j = e_j \circ T = \frac{1}{\sqrt{m_j}}\chi_{T^{-1}\{j\}} = \frac{1}{\sqrt{m_j}} \sum_{i\in T^{-1}(j)} e_i\sqrt{m_i}.$$

Hence, we obtain that

$$C_T e_j = \begin{cases} \sum\limits_{1\leqslant i\leqslant s} \sqrt{\dfrac{m_{k(r+s)+i+r-1}}{m_{k(r+1)+r+1}}}\, e_{k(r+s)+i+r-1} & j = k(r+1)+r+1,\ k \in \mathbb{N}_0, \\ \sqrt{\frac{m_{k(r+s)+i-1}}{m_{k(r+1)+i}}}\, e_{k(r+s)+i-1} & j = k(r+1)+i,\ 1 \leqslant i \leqslant r,\ k \in \mathbb{N}_0. \end{cases}$$

Now, we set weight sequences $\alpha := \{a_i^n\}_{\substack{1\leqslant i\leqslant r \\ 0\leqslant n<\infty}}$ and $\beta := \{b_j^n\}_{\substack{1\leqslant j\leqslant s \\ 0\leqslant n<\infty}}$, where

$$a_i^{(n)} = \sqrt{\frac{m_{n(r+s)+i-1}}{m_{n(r+1)+i}}} \quad 1 \leqslant i \leqslant r,\ 0 \leqslant n < \infty,$$

$$b_j^{(n)} = \sqrt{\frac{m_{n(r+s)+j+r-1}}{m_{n(r+1)+r+1}}} \quad 1 \leqslant j \leqslant s,\ 0 \leqslant n < \infty.$$

Therefore it is easy to check that C_T is unitarily equivalent to the block shift matrix operator $M(\alpha,\beta)$ with weight sequence (α,β). $\square$

PROPOSITION 3.3. [4] *Let $M(\alpha,\beta)$ be a block matrix operator with weight sequence (α,β), where $\alpha := \{a_i^n\}_{\substack{1\leqslant i\leqslant r \\ 0\leqslant n<\infty}}$ and $\beta := \{b_j^n\}_{\substack{1\leqslant j\leqslant s \\ 0\leqslant n<\infty}}$. Then there exists a measurable transformation T on a σ-finite measure space $(\mathbb{N}_0, \mathscr{P}(\mathbb{N}_0), m)$ such that $M(\alpha,\beta)$ is unitarily equivalent to the composition operator C_T on ℓ^2.*

REMARK 3.4. Let $m := \{m_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers. Put the space $\ell^2(m) = L^2(\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where $2^{\mathbb{N}}$ is the power set of natural numbers and μ is a measure on $2^{\mathbb{N}}$ defined by $\mu(\{n\}) = m_n$. Let $f = \{f_n\}_{n=1}^{\infty}$. Also assume $T : \mathbb{N} \to \mathbb{N}$ be a measurable transformation. Then for each $k \in \mathbb{N}$ we have

$$h(k) = \frac{1}{m_k} \sum_{j\in T^{-1}(k)} m_j; \quad E(f)(k) = \frac{\sum_{j\in T^{-1}(T(k))} f_j m_j}{\sum_{j\in T^{-1}(T(k))} m_j};$$

THEOREM 3.5. *Let T be a non-singular measurable transformation on ℓ^2 as in (3.2) and let $p \in (0,\infty)$. Then the following assertions are equivalent*

(i) C_T is p-paranormal on ℓ^2;

(ii) the block shift matrix operator $M(\alpha,\beta)$ as in Proposition 3.2 is p-paranormal;
(iii) $h^p\circ T(n)\leqslant E(h^p)(n)$, $n\in\mathbb{N}_0$, where $h=d\mu\circ T^{-1}/d\mu$;
(iv) the following inequality holds

$$\left(\frac{m(T^{-1}(T(n)))}{m_{T(n)}}\right)^p\leqslant\frac{1}{m(T^{-1}(T(n)))}\sum_{l\in T^{-1}(T(n))}\frac{m(T^{-1}(l))^p}{m_l^p}m_l,\quad n\in\mathbb{N}_0. \tag{3.3}$$

Proof. By [9, Theorem 2.2], we get that (i) and (iii) are equivalent. Also by using Proposition 3.2, we have (i) and (ii) are equivalent. Hence, it is sufficient to show that (i) and (iv) are equivalent. To compute $h^p\circ T(n)$, we require the consideration of two cases.

Case 1: $n=k(r+s)+i+r-1\ \ (1\leqslant i\leqslant s)$, then $n\in T^{-1}(k(r+1)+r+1)$. By (2) we obtain that

$$(h^p\circ T)(n)=\left(\frac{m(T^{-1}(T(n)))}{m_{T(n)}}\right)^p,\quad n=k(r+s)+i+r-1,\quad 1\leqslant i\leqslant s.$$

Case 2: $n=k(r+s)+i-1\ \ ((1\leqslant i\leqslant r)$, we have $n\in T^{-1}(k(r+1)+i)$, and

$$(h^p\circ T)(n)=\left(\frac{m(T^{-1}(T(n)))}{m_{T(n)}}\right)^p,\quad n=k(r+s)+i-1,\quad 1\leqslant i\leqslant r.$$

We now turn to the computation of $E(h^p)$. This also will be considered in two cases as above. For $n=k(r+s)+i+r-1\ \ (1\leqslant i\leqslant s)$, we have $n\in T^{-1}(k(r+1)+r+1)$ and using (3.2) we have

$$\begin{aligned}E(h^p)(n)&=\frac{1}{m(T^{-1}(k(r+1)+r+1)}\sum_{l\in T^{-1}(k(r+1)+r+1)}h^P(l)m_l\\&=\frac{1}{m(T^{-1}(T(n)))}\sum_{l\in T^{-1}(T(n))}\Big(\frac{1}{m_l}\sum_{s\in T^{-1}(l)}m_s\Big)^p m_l\\&=\frac{1}{m(T^{-1}(T(n)))}\sum_{l\in T^{-1}(T(n))}\frac{m(T^{-1}(l))^p m_l}{m_l^p}\end{aligned}$$

On the other hand, for $n=k(r+s)+i-1$, $((1\leqslant i\leqslant r)$, we have $n\in T^{-1}(k(r+1)+i)$, by using (3.2) we can arrive at the last line above, or in this case we may simplify to

$$E(h^p)(n)=\frac{1}{m_{k(r+s)+i-1}}.h^P(n).m_n=\left(\frac{m_{T^{-1}(n)}}{m_n}\right)^p.$$

Therefore, we deduce the inequality $h^p\circ T(n)\leqslant E(h^p)$ is equivalent to that

$$\left(\frac{m(T^{-1}(T(n)))}{m_{T(n)}}\right)^p\leqslant\frac{1}{m(T^{-1}(T(n)))}\sum_{l\in T^{-1}(T(n))}\frac{m(T^{-1}(l))^p}{m_l^p}m_l.\quad\square$$

The conditions above simplify considerably if we specialize to the case of a repeated block. Let $M(\alpha,\beta)$ be a block shift matrix operator where $\alpha := \{a_i^n\}_{\substack{1\leqslant i\leqslant r\\ 0\leqslant n<\infty}}$ and $\beta := \{b_j^n\}_{\substack{1\leqslant j\leqslant s\\ 0\leqslant n<\infty}}$ as follows:

$$\begin{aligned} M(\alpha,\beta) : A \equiv A_1 = A_2 = \dots \\ \alpha : a_i^{(n)} = a_i,\ \ n\in\mathbb{N}_0,\ \ 1\leqslant i\leqslant r; \\ \beta : b_j^{(n)} = b_j,\ \ n\in\mathbb{N}_0,\ \ 1\leqslant j\leqslant s. \end{aligned} \tag{3.4}$$

For any $n\in\mathbb{N}_0$, let t_n denote the solution to the conditions $1\leqslant t_n\leqslant r+s$ and $n=k(r+s)+t_n+r-1$ for some $k\in\mathbb{N}_0$. Similarly, let i_n satisfy $1\leqslant i_n\leqslant r$ and $n=k_1(r+1)+i_n$ for some $k_1\in\mathbb{N}_0$.

THEOREM 3.6. *Let $M(\alpha,\beta)$ be as in (3.4). Then the block matrix operator $M(\alpha,\beta)$ is p-paranormal if and only if the following two conditions hold:*

(i) if $n=k(r+s)+i+r-1$ for $1\leqslant i\leqslant s$, then for all $1\leqslant i_l\leqslant r$ and $1\leqslant t_l\leqslant s$ we have

$$\begin{aligned} \left(\sum_{1\leqslant i\leqslant s} b_i^2\right)^p \leqslant & \sum_{\substack{l\in T^{-1}(T(n))\\ l\equiv r+1 \operatorname{mod}(r+1)}} \left(\sum_{1\leqslant i\leqslant s} b_i^2\right)^p \left(\frac{b_{t_l}^2}{\sum_{1\leqslant i\leqslant s} b_i^2}\right) \\ & + \sum_{\substack{l\in T^{-1}(T(n))\\ l\equiv i_l \operatorname{mod}(r+1)}} (a_{i_l})^{2p} \left(\frac{b_{t_l}^2}{\sum_{1\leqslant i\leqslant s} b_i^2}\right) \end{aligned} \tag{3.5}$$

(ii) if $n=k(r+s)+m-1$ for $1\leqslant m\leqslant r$, then

$$\begin{aligned} &(ii-a) \quad a_m^2 \leqslant \sum_{1\leqslant i\leqslant s} b_i^2 \qquad n\equiv r+1 \quad \operatorname{mod}(r+1) \\ &(ii-b) \quad a_m^2 \leqslant a_{i_n}^2 \qquad\qquad n\equiv i_n \qquad \operatorname{mod}(r+1). \end{aligned}$$

Proof. *Case* 1: $n=k(r+s)+i+r-1$ for $1\leqslant i\leqslant s$. Thus $T(n)=k(r+1)+r+1$ and $T^{-1}(T(n))=\{k(r+s)+i+r-1 : 1\leqslant i\leqslant s\}$. By using Proposition 3.2 we get that

$$m(T^{-1}(T(n))) = \sum_{1\leqslant i\leqslant s} m_{k(r+s)+i+r-1} = \sum_{1\leqslant i\leqslant s} (b_i^{(k)})^2 m_{k(r+1)+r+1},$$

since for any $k\in\mathbb{N}_0$, $b_i^{(k)}=b_i$. So $m(T^{-1}(T(n)))=\sum_{1\leqslant i\leqslant s} b_i^2 m_{k(r+1)+r+1}$. Hence, we have

$$\left(\frac{m(T^{-1}(T(n)))}{m_{T(n)}}\right)^p = \left(\frac{\sum_{1\leqslant i\leqslant s} b_i^2 m_{T(n)}}{m_{T(n)}}\right)^p = \left(\sum_{1\leqslant i\leqslant s} b_i^2\right)^p.$$

To compute the right hand side (3.3), we first calculate $\frac{m_l}{m(T^{-1}(T(n)))}$ for $l \in T^{-1}(T(n))$. By using Proposition 3.2, $m_l = m_{k(r+s)+t_l+r-1} = b_{t_l}^2 m_{k(r+1)+r+1}$, then

$$\frac{m_l}{m(T^{-1}(T(n)))} = \frac{b_{t_l}^2 m_{k(r+1)+r+1}}{\sum_{1\leqslant i\leqslant s} b_i^2 m_{k(r+1)+r+1}} = \frac{b_{t_l}^2}{\sum_{1\leqslant i\leqslant s} b_i^2}, \quad 1 \leqslant t_l \leqslant s.$$

Now, we compute $\left(\frac{m(T^{-1}(l))}{m_l}\right)^p$ for $l \in T^{-1}(T(n))$. We consider two subcases.

Case $1a$: $l = k_1(r+1)+r+1$, $k_1 \in \mathbb{N}_0$, then we have $T^{-1}(l) = \{k_1(r+s)+i+r-1 : 1 \leqslant i \leqslant s\}$. By Proposition 3.2, we obtain that

$$m(T^{-1}(l)) = \sum_{1\leqslant i\leqslant s} m_{k_1(r+s)+i+r-1} = \sum_{1\leqslant i\leqslant s} b_i^2 m_{k_1(r+1)+r+1}.$$

Therefore

$$\left(\frac{m(T^{-1}(l))}{m_l}\right)^p = \left(\frac{\sum_{1\leqslant i\leqslant s} b_i^2 m_{k_1(r+1)+r+1}}{m_{k_1(r+1)+r}}\right)^p = \left(\sum_{1\leqslant i\leqslant s} b_i^2\right)^p.$$

Case $1b$: $l = k_1(r+1)+i_l$ for $k_1 \in \mathbb{N}_0$ and $1 \leqslant i_l \leqslant r$. In this case we get that $T^{-1}(l) = k_1(r+s)+i_l-1 : 1 \leqslant i_l \leqslant r$, so Proposition 3.2 implies that

$$m(T^{-1}(l)) = m_{k_1(r+s)+i_l} - 1 = a_{i_l}^2 m_{k_1(r+1)+i_l},$$

this follows that

$$\left(\frac{m(T^{-1}(l))}{m_l}\right)^p = (a_{i_l}^2)^p.$$

Therefore, for $l \in T^{-1}(T(n))$,

$$\left(\frac{m(T^{-1}(l))}{m_l}\right)^p = \begin{cases} \left(\sum_{1\leqslant i\leqslant r} b_i^2\right)^p & l \equiv r, \mod (r+1), \\ a_{i_l}^{2p} & l \equiv i_l, \mod (r+1). \end{cases}$$

Consequently, for $n = k(r+s)+i+r-1$ and $1 \leqslant i \leqslant s$, we deduce that (3.3) is equivalent to (3.5).

Case 2: $n = k(r+s)+m-1$ for $1 \leqslant m \leqslant r$. It is easy to see that $T(n) = k(r+1)+m-1$ and $T^{-1}(T(n)) = n$, by using Proposition 3.2, we get that

$$\frac{m(T^{-1}(T(n)))}{m(T(n))} = \frac{m_n}{m_{k(r+1)+m}} = \frac{m_{k(r+s)+m-1}}{m_{k(r+1)+m}} = \frac{a_m^2 m_{k(r+1)+m}}{m_{k(r+1)+m}} = a_m^2,$$

hence

$$\left(\frac{m(T^{-1}(T(n)))}{m(T(n))}\right)^p = a_m^{2p}.$$

Since $T^{-1}(T(n)) = n$ for $n = k(r+s)+m-1$, obviously $\frac{m(T^{-1}(T(n)))}{m_l} = 1$ for $l \in T^{-1}(T(n))$. Now we consider two subcases for computations of $\left(\frac{m(T^{-1}(l))}{m_l}\right)^p$, $l \in T^{-1}(T(n))$.

Case $2a$: $l(=n) = k_2(r+1)+r+1$ for some $k_2 \in \mathbb{N}_0$. Then we have $T^{-1}(l) = \{k_2(r+s)+i+r-1 : 1 \leqslant i \leqslant s\}$. By Proposition 3.2, we obtain that

$$m(T^{-1}(l)) = \sum_{1\leqslant i\leqslant s} m_{k_2(r+s)+i+r-1} = \sum_{1\leqslant i\leqslant s} b_i^2 m_{k_2(r+1)+r+1},$$

this implies that

$$\frac{m(T^{-1}(l))}{m_l} = \frac{\sum_{1\leqslant i\leqslant s} b_i^2 m_{k_2(r+1)+r+1}}{m_{k_2(r+1)+r+1}} = \sum_{1\leqslant i\leqslant s} b_i^2.$$

Case $2b$: $l(=n) = k_2(r+1)+i_n$ for some $k_2 \in \mathbb{N}_0$, with $1 \leqslant i_n \leqslant r$. Obviously $T^{-1}(l) = \{k_2(r+s)+i_n-1 : 1 \leqslant i_n \leqslant r\}$, by Proposition 3.2, we get that

$$m(T^{-1}(l)) = m_{k_2(r+s2)+i_n} - 1 = a_{i_n}^2 m_{k_2(r+1)+i_n},$$

consequently

$$\frac{m(T^{-1}(l))}{m_l} = \frac{a_{i_n}^2 m_{k_2(r+1)+i_n}}{m_{k_2(r+1)+i_n}} = a_{i_n}^2.$$

Therefore we deduce that for $l \in T^{-1}(T(n))$,

$$\left(\frac{m(T^{-1}(l))}{m_l}\right)^p = \begin{cases} \left(\sum_{1\leqslant i\leqslant r} b_i^2\right)^p & l(=n) \equiv r+1, \quad \mathrm{mod}\,(r+1), \\ a_{i_n}^{2p} & l(=n) \equiv i_n, \quad \mathrm{mod}\,(r+1). \end{cases}$$

Thus for $n = k(r+s)+m-1$ and $1 \leqslant m \leqslant r$, we get that (3.3) is equivalent to

$$\begin{cases} a_m^2 \leqslant \left(\sum_{1\leqslant i\leqslant r} b_i^2\right)^p & n \equiv r+1, \quad \mathrm{mod}\,(r+1), \\ a_m^2 \leqslant a_{i_n}^{2p} & n \equiv i_n, \quad \mathrm{mod}\,(r+1). \end{cases} \qquad \square$$

EXAMPLE 3.7. Let

$$D := \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad M := \begin{bmatrix} D & & \\ & D & \\ & & \ddots \end{bmatrix}.$$

Note that a and b are fixed positive real number. Then some direct computations show that the conditions for M to be p-paranormal in Theorem 3.5 is equivalent to the following condition:

$$a^{2p} + b^{2p} \geqslant 2^p \tag{3.6}$$

Let $0 < p < q$ and M be p-paranormal. Then by using (3.6) we can find a and b such that M is not q-paranormal. Namely for $a = 1$ and $b = 1$, it is easy to check that M is 1-paranormal but it is not 2-paranormal. Similarly for $a = 1.2$ and $a = 1.3$ it is easy to see that M is 2-paranormal but it is not 3-paranormal.

EXAMPLE 3.8. Let

$$E := \begin{bmatrix} 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad M := \begin{bmatrix} E & & \\ & E & \\ & & \ddots \end{bmatrix}.$$

Note that x and y are fixed positive real number. Then by using Theorem 3.6 it is easy to see that M is p-paranormal if and only if the following conditions hold:

$$9\left(\frac{x^2}{12}\right)^p + 2\left(\frac{y^2}{12}\right)^p \geqslant 11, \qquad 12 \geqslant x^2 \quad \text{and} \quad x^2 \geqslant y^2 \tag{3.7}$$

Let $0 < p < q$ and M be q-paranormal. Then by using (3.7) we can find x and y such that M is not p-paranormal. Put $x = 3.48$ and $y = 3.4$ by using (3.7) it is easy to see that M is 2-paranormal but it is not 1-paranormal.

REFERENCES

[1] C. BURNAP, I. JUNG AND A. LAMBERT, *Separating partial normality classes with composition operators*, J. Operator Theory, **53** (2005), 381–397.

[2] J. CAMPBELL AND W. HORNOR, *Seminormal composition operators*, J. Operator Theory, **29** (1993), 323–343.

[3] Y. ESTAREMI AND M. R. JABBARZADEH, *Weighted Lambert type operators on L^p spaces*, J. Operators and Matrices, **7** (2013), 101–116.

[4] G. EXNER, I. JUNG AND M. LEE, *Block matrix operators and weak hyponormalities*, J. Integr. Equ. Oper. Theory, **65** (2009), 345–362.

[5] T. FURUTA, *Invitation to Linear Operators*, Taylor and Francis Inc., 2001.

[6] D. HARRINGTON AND R. WHITLEY, *Seminormal composition operators*, J. Operator Theory, **11** (1984), 125–135.

[7] T. HOOVER, A. LAMBERT AND J. QUINN, *The Markov process determined by a weighted composition operator*, Studia Math. (Poland) **LXXII** (1982), 225–235.

[8] Y. CUI, H. HUDZIK, R. KUMAR AND L. MALIGRANDA, *Composition operators in Orlicz spaces*, J. Aust. Math. Soc. **76** (2004), 189–206.

[9] M. R. JABBARZADEH AND M. R. AZIMI, *Some weak hyponormal classes of weighted composition operators*, Bull. Korean. Math. Soc. **47** (2010), 793–803.

[10] M. R. JABBARZADEH, *A conditional expectation type operator on L^p spaces*, J. Operators and Matrices, **4** (2010), 445–453.

[11] A. LAMBERT, *Hyponormal composition operators*, Bull. London Math. Soc. **18** (1986), 395–400.

[12] S. PANAYAPPAN, *Non-hyponormal weighted composition operators*, Indian J. Pure Appl. Math. **27** (1996), 979–983.

[13] M. M. RAO, *Conditional measure and applications*, Marcel Dekker, New York, 1993.
[14] R. K. SINGH AND J. S. MANHAS, *Composition Operators on Function Spaces*, North Holland Math. Studies 179, Amsterdam, 1993.
[15] H. TAKAGI, *Compact weighted composition operators on L^p*, J. Proc. Amer. Math. Soc. **116** (1992), 505–511.

(Received April 28, 2016)

M. R. Jabbarzadeh
Faculty of Mathematical Sciences
University of Tabriz
Tabriz, Iran
e-mail: mjabbar@tabrizu.ac.ir

Z. Moayyerizadeh
Faculty of Mathematical Sciences
Lorestan University
Khorramabad, Iran
e-mail: moayerizadeh.za@lu.ac.ir