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SUBSPACE–HYPERCYCLIC WEIGHTED SHIFTS

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Abstract. Our aim in this paper is to obtain necessary and sufficient conditions for bilateral and unilateral weighted shift operators to be subspace-transitive. We show that the Herrero question [6] holds true even on a subspace of a Hilbert space, i.e. there exists an operator T such that both T and T^* are subspace-hypercyclic operators for some subspaces. We display the conditions on the direct sum of two invertible bilateral forward weighted shift operators to be subspace-hypercyclic.

1. Introduction

A bounded linear operator T on a separable Hilbert space $\mathscr{H}$ is hypercyclic if there is a vector $x \in \mathscr{H}$ such that $Orb(T,x) = \{T^n x : n \geqslant 0\}$ is dense in $\mathscr{H}$, such a vector x is called hypercyclic for T. The first example of a hypercyclic operator on a Banach space was constructed by Rolewicz in 1969 [12]. He showed that if B is the backward shift on $\ell^p(\mathbb{N})$ then λB is hypercyclic if and only if $|\lambda| > 1$. The hypercyclicity concept was probably born with the thesis of Kitai in 1982 [8] who introduced the hypercyclic criterion to show the existence of hypercyclic operators.

The study of the scaled orbit $\mathbb{C}Orb(T,x)$ and the disk orbit $\mathbb{D}Orb(T,x)$ of an operator T is motivated by the Rolewicz example [12]. In 1974, Hilden and Wallen [7] defined supercyclic operators as follows: An operator T is called supercyclic if there is a vector x such that its scaled orbit is dense in $\mathscr{H}$. Similarly, Zeana [14] defined diskcyclicity concept. An operator T is called diskcyclic if there is a vector $x \in \mathscr{H}$ such that its disk orbit is dense in $\mathscr{H}$. For more information about these operators, the reader may consult [3, 5].

In 2011, Madore and Martínez-Avendaño [10] introduced and studied the density of an orbit in a non-trivial subspace instead of the whole space and called that phenomenon subspace-hypercyclicity. For more details on subspace-hypercyclic operators, the reader may refer to [1, 9, 11].

In 1991, Herrero [6] asked whether there exists a hypercyclic operator T such that its adjoint is also hypercyclic. In 1995, Salas [13] characterized all hypercyclic bilateral weighted shift operators and consequently, he gave an example supporting Herrero's question. However, those characterizations were so complicated; therefore,

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Feldman [4] constructed simpler conditions that characterize hypercyclic invertible bilateral weighted shifts. Now, it is natural to ask: What kinds of weighted shift operators are subspace-hypercyclic?

In this paper, we follow the line of Salas's proofs [13] and Feldman's proofs [4] to characterize subspace-hypercyclic weighted shift operators for some subspaces. In particular, we give necessary and sufficient conditions for bilateral weighted shift operators to be subspace-transitive. We give some simpler conditions that characterize subspace-transitive invertible bilateral weighted shift operators in terms of their weight sequences. Then, we show that the same conditions hold true for a weaker property than invertibility. We use these characterization to show that the Herrero question [6] still holds for subspace-hypercyclic operators; i.e, there is an operator T such that both T and its adjoint are subspace-hypercyclic for some subspaces; however, we don't know whether they are subspace-hypercyclic for the same subspace or not. Moreover, we characterize subspace-hypercyclic unilateral backward weighted shift operators in term of their weight sequences. Also, we characterize the direct sum of weighted shifts that are subspace-hypercyclic.

We recall the following facts from the literature.

DEFINITION 1.1. [10] Let $T \in \mathscr{B}(\mathscr{H})$ and $\mathscr{M}$ be a closed subspace of $\mathscr{H}$. Then T is called $\mathscr{M}$-hypercyclic or a subspace-hypercyclic operator for a subspace $\mathscr{M}$ if there exists a vector $x \in \mathscr{H}$ such that $Orb(T,x) \cap \mathscr{M}$ is dense in $\mathscr{M}$. Such a vector x is called an $\mathscr{M}$-hypercyclic vector for T.

DEFINITION 1.2. [10] Let $T \in \mathscr{B}(\mathscr{H})$ and $\mathscr{M}$ be a closed subspace of $\mathscr{H}$. Then T is called $\mathscr{M}$-transitive or subspace-transitive for a subspace $\mathscr{M}$ if for each pair of non-empty open sets U_1, U_2 of $\mathscr{M}$ there exists an $n \in \mathscr{N}$ such that $T^{-n}U_1 \cap U_2$ contains a non-empty relatively open set in $\mathscr{M}$.

THEOREM 1.3. [10] *Every $\mathscr{M}$-transitive operator on $\mathscr{H}$ is $\mathscr{M}$-hypercyclic.*

PROPOSITION 1.4. [2] *Let $T \in \mathscr{B}(\mathscr{H})$, and $\mathscr{M}$ be a closed subspace of $\mathscr{H}$. The following statements are equivalent:*

1. *T is $\mathscr{M}$-transitive,*
2. *for each $x, y \in \mathscr{M}$, there exist sequences $\{x_k\}_{k\in\mathscr{N}} \subset \mathscr{M}$ and an increasing sequence of positive integers $\{n_k\}_{k\in\mathscr{N}}$ such that $T^{n_k}\mathscr{M} \subseteq \mathscr{M}$ for all $k \geqslant 1$, $x_k \to x$ and $T^{n_k}x_k \to y$ as $k \to \infty$,*
3. *for each $x, y \in \mathscr{M}$ and each 0-neighborhood W in $\mathscr{M}$, there exist $z \in \mathscr{M}$ and $n \in \mathscr{N}$ such that $x - z \in W$, $T^n z - y \in W$ and $T^n\mathscr{M} \subseteq \mathscr{M}$.*

2. Main results

All results in this section hold true for the Banach spaces $\ell^p(\mathbb{Z})$ and $\ell^p(\mathbb{N})$ $(1 < p < \infty)$; however, for the sake of simplicity we only deal with the Hilbert spaces $\ell^2(\mathbb{Z})$ and $\ell^2(\mathbb{N})$.

Let T be the bilateral forward weighted shift operator on $\ell^2(\mathbb{Z})$ with a weight sequence $\{w_n\}_{n\in\mathbb{Z}}$, then $T(e_r) = w_r e_{r+1}$ for all $r \in \mathbb{Z}$. Let S be the right inverse (backward shift) to T and be defined as follows: $S(e_r) = \frac{1}{w_{r-1}} e_{r-1}$. Observe that $TSe_r = e_r$ for all $r \in \mathbb{Z}$. If T is invertible then $T^{-1} = S$. Also, we have

$$T^k(e_{m_r}) = (\prod_{j=m_r}^{m_r+k-1} w_j) e_{m_r+k} \text{ and } S^k(e_{m_r}) = (\prod_{j=m_r-1}^{m_r-k} \frac{1}{w_j}) e_{m_r-k}.$$

The next theorem gives necessary and sufficient conditions for a bilateral weighted shift operator on $\ell^2(\mathbb{Z})$ to be $\mathscr{M}$-transitive. First, we suppose that all subspaces $\mathscr{M}$ in this section are non-trivial topologically closed and have some subset of $\{e_r\}$ as a basis, where $\{e_r\}$ is the canonical Schauder basis for $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{N})$ and $e_r(j) = \delta_{rj}$ (Kronecker delta). It follows that $\mathscr{M} \cap \{e_r\} = \{e_{m_j} : j \in \mathscr{N}\} \neq \phi$.

THEOREM 2.1. *Let T be a bilateral forward weighted shift in $\ell^2(\mathbb{Z})$ with a positive weight sequence $\{w_n\}_{n\in\mathbb{Z}}$, $\mathscr{M}$ be a subspace of $\ell^2(\mathbb{Z})$ and $\mathscr{F} = \{m_j : e_{m_j} \in \mathscr{M} \cap \{e_r\}\}$. Then T satisfies $\mathscr{M}$-hypercyclic criterion if and only if, for any $q \in \mathscr{N}$, we have*

(i) $\liminf_{n\to\infty} \max\{\prod_{k=1}^{n} \frac{1}{w_{m_j-k}} : m_j \in \mathscr{F} \text{ and } |m_j| \leqslant q\} = 0,$

(ii) $\liminf_{n\to\infty} \max\{\prod_{k=0}^{n-1} w_{m_j+k} : m_j \in \mathscr{F} \text{ and } |m_j| \leqslant q\} = 0,$

(iii) $T^{n_p}\mathscr{M} \subseteq \mathscr{M}$ *for an increasing sequence of positive integers* $\{n_p\}_{p\in\mathscr{N}}$,

Proof. Let T satisfy $\mathscr{M}$-hypercyclic criterion, then T is $\mathscr{M}$-transitive. Suppose that $q \in \mathscr{N}$ and $y = z = \sum_{\substack{|m_j|\leqslant q \\ m_j\in\mathscr{F}}} e_{m_j} \in \mathscr{M}$. Then by 1.4 there exist a vector $x \in \mathscr{M}$, a large positive integer $n > 2q$ and a small positive integer ε_n such that

$$\left\| x - \sum_{\substack{|m_j|\leqslant q \\ m_j\in\mathscr{F}}} e_{m_j} \right\| < \varepsilon_n, \tag{1}$$

$$\left\| T^n x - \sum_{\substack{|m_j|\leqslant q \\ m_j\in\mathscr{F}}} e_{m_j} \right\| < \varepsilon_n \tag{2}$$

and

$$T^n\mathscr{M} \subseteq \mathscr{M} \ (n \text{ has to be in } \mathscr{F}). \tag{3}$$

(1) implies that $|x_{m_j}| > 1-\varepsilon_n$ if $|m_j| \leqslant q$ and $|x_{m_j}| < \varepsilon_n$ otherwise. Since $n > 2q$, (2) implies that for all $|m_j| \leqslant q$, we have

$$|x_{m_j}| \left\| T^n e_{m_j} \right\| = |x_{m_j}| \left(\prod_{k=0}^{n-1} w_{k+m_j} \right) < \varepsilon_n.$$

It follows that

$$\left(\prod_{k=0}^{n-1} w_{k+m_j} \right) < \frac{\varepsilon_n}{|x_{m_j}|} < \frac{\varepsilon_n}{1-\varepsilon_n} = \delta_n, \tag{4}$$

Also (2) implies that

$$\left\| x_{m_j-n}(T^n e_{m_j-n}) - e_{m_j} \right\| < \varepsilon_n.$$

for all $|m_j| \leqslant q$. Thus

$$\left| x_{m_j-n} \right| \left| \prod_{k=0}^{n-1} w_{m_j-n+k} \right| - 1 = \left| x_{m_j-n} \right| \left| \prod_{k=1}^{n} w_{m_j-k} \right| - 1 < \varepsilon_n.$$

Therefore

$$\left(\prod_{k=1}^{n} \frac{1}{w_{m_j-k}} \right) < \frac{\left| x_{m_j-n} \right|}{1-\varepsilon_n} < \frac{\varepsilon_n}{1-\varepsilon_n} = \delta_n. \tag{5}$$

It is clear that $\delta_n \to 0$ when $n \to \infty$. The proof follows by (3), (4) and (5).

Conversely, we verify the $\mathscr{M}$-hypercyclic criterion with the dense subsets $D = D_1 = D_2$ of $\mathscr{M}$ consisting of all sequences with finite support. By hypothesis, there exists an increasing sequence of positive integers $\{n_p\}_{p\in\mathscr{N}}$ such that $T^{n_p}\mathscr{M} \subseteq \mathscr{M}$. Also, there exist $j \in \mathscr{N}$ such that $m_j \in \mathscr{F}; |m_j| \leqslant q$,

$$\lim_{p\to\infty} \prod_{k=1}^{n_p} \frac{1}{w_{m_j-k}} = 0 \tag{6}$$

and

$$\lim_{p\to\infty} \prod_{k=0}^{n_p-1} w_{m_i+k} = 0. \tag{7}$$

Let $x = \sum_{|m_i|\leqslant q} x_i e_{m_i} \in D$ and $y = \sum_{|m_j|\leqslant q} y_j e_{m_j} \in D$, and let B be the backward shift defined on D by $B(e_n) = \frac{1}{w_{n-1}} e_{n-1}$, then

$$\|B^{n_p} y\| \leqslant \frac{\|y\|}{\min\left\{ \prod_{k=1}^{n_p} w_{m_j-k} : |m_j| \leqslant q \right\}}, \tag{8}$$

and

$$\|T^{n_p} x\| \leqslant \max\left\{ \prod_{k=0}^{n_p-1} w_{m_i+k} : |m_i| \leqslant q \right\} \|x\| \tag{9}$$

By hypothesis, it is clear that $\lim_{p\to\infty}\|B^{n_p}y\| = 0$, $\lim_{p\to\infty}\|T^{n_p}x\| = 0$ and $T^{n_p}B^{n_p}y = y$. Thus, by taking $x_k = B^{n_p}y$, T satisfies $\mathscr{M}$-hypercyclic criterion. $\square$

For invertible bilateral weighted shifts, 2.1 can be simplified more, first we need the following lemma.

LEMMA 2.2. *Let T be an invertible bilateral weighted shift on $\ell^2(\mathbb{Z})$ and $\{n_k\}_{k\in\mathscr{N}}$ be an increasing sequence of positive integers. Suppose that $\mathscr{M}$ is a subspace of $\ell^2(\mathbb{Z})$ with standard basis $\{e_{m_i} : i \in \mathscr{N}, m_i \in \mathbb{Z}\}$ such that $T^{n_k}\mathscr{M} \subseteq \mathscr{M}$ for all $k \geqslant 1$. If there exists an $i \in \mathscr{N}$ such that $T^{n_k}e_{m_i} \to 0$ as $k \to \infty$, then $T^{n_k}e_{m_r} \to 0$ for all $r \in \mathscr{N}$.*

Proof. Since $T^{n_k}\mathscr{M} \subseteq \mathscr{M}$, the proof is similar to the proof of Lemma 3.1 of [4]. $\square$

THEOREM 2.3. *Let T be an invertible bilateral forward weighted shift in $\ell^2(\mathbb{Z})$ with a positive weight sequence $\{w_n\}_{n\in\mathbb{Z}}$, $\mathscr{M}$ be a subspace of $\ell^2(\mathbb{Z})$ and $\mathscr{F} = \{m_j : e_{m_j} \in \mathscr{M} \cap \{e_r\}\}$. Then T is $\mathscr{M}$-transitive if and only if there exist an increasing sequence of positive integers $\{n_k\}_{k\in\mathscr{N}}$ and $m_i \in \mathscr{F}$ such that $T^{n_k}\mathscr{M} \subseteq \mathscr{M}$ for all $k \in \mathscr{N}$ and*

$$\lim_{k\to\infty} \prod_{j=m_i}^{m_i+n_k-1} w_j = 0 \quad \text{and} \quad \lim_{k\to\infty} \prod_{j=1+m_i}^{n_k+m_i} \frac{1}{w_{-j}} = 0 \tag{10}$$

Proof. To prove the "if" part , we verify the $\mathscr{M}$-hypercyclic criterion with the dense subsets $D = D_1 = D_2$ of $\mathscr{M}$ consisting of all sequences with finite support. Let $x, y \in D$, then by 2.2 and triangle inequality it is enough to consider $x = y = e_{m_i}$ for some $m_i \in \mathscr{F}$. Let $x_k = B^{n_k}y$ where B is a bilateral weighted shift with weight sequence $\frac{1}{w_{n-1}}$. By hypothesis, we have

$$\|T^{n_k}e_{m_i}\| = \prod_{j=m_i}^{m_i+n_k-1} w_j \to 0$$

and

$$\|B^{n_k}e_{m_i}\| = \prod_{j=1+m_i}^{n_k+m_i} \frac{1}{w_{-j}} \to 0.$$

Moreover, it is clear that $T^{n_k}B^{n_k}x = x$. Thus, the conditions of $\mathscr{M}$-hypercyclic criterion are satisfied.

The proof of the "only if" part follows from 2.1. $\square$

The next theorem shows that the 2.3 still holds by assuming a weaker form of invertibility.

THEOREM 2.4. *Let $T \in \mathscr{B}(\ell^2(\mathbb{Z}))$ be a bilateral forward weighted shift with a positive weight sequence $\{w_n\}_{n\in\mathbb{Z}}$ such that for all $n < 0$, $w_n \geqslant b > 0$. Let $\mathscr{M}$ be a subspace of $\ell^2(\mathbb{Z})$ and $\mathscr{F} = \{m_j : e_{m_j} \in \mathscr{M} \cap \{e_r\}\}$. Then T is $\mathscr{M}$-transitive if and *only if there exist an increasing sequence of positive integers* $\{n_k\}_{k\in\mathcal{N}}$ *and* $m_i \in \mathcal{F}$ *such that* $T^{n_k}\mathcal{M} \subseteq \mathcal{M}$ *for all* $k \in \mathcal{N}$,

$$\lim_{k\to\infty} \prod_{j=m_i}^{m_i+n_k-1} w_j = 0 \quad \text{and} \quad \lim_{k\to\infty} \prod_{j=1+m_i}^{n_k+m_i} \frac{1}{w_{-j}} = 0. \tag{11}$$

Proof. For "if" part, we verify 2.1. Let $\varepsilon > 0$, $q \in \mathcal{N}$ and let $\delta_1, \delta_2 > 0$ (to be determined later). By hypothesis, there exists an increasing sequence of positive integers $\{n_r\}_{r\in\mathcal{N}}$ such that

$$T^{n_r}\mathcal{M} \subseteq \mathcal{M} \tag{12}$$

for all $r \in \mathcal{N}$, and there exists an arbitrary large $n_k \in \{n_r\}_{r\in\mathcal{N}}$ and $m_i \in \mathcal{F}$ such that

$$\prod_{j=m_i}^{m_i+n_k-1} w_j < \delta_1 \quad \text{and} \quad \prod_{j=1+m_i}^{n_k+m_i} \frac{1}{w_{-j}} < \delta_2$$

Suppose that $n = n_k + m_i + q + 1$ (which ensure that $m_p + n - 1 \geqslant n_k + m_i$ for all $|m_p| \leqslant q$). Now, for all $m_p \in \mathcal{F}$ with $|m_p| \leqslant q$, we have

$$\begin{aligned}
\prod_{j=m_p}^{n+m_p-1} w_j &= \left(\prod_{j=m_i}^{m_p-1} \frac{1}{w_j}\right)\left(\prod_{j=m_i}^{m_p-1} w_j\right)\left(\prod_{j=m_p}^{m_i+n_k-1} w_j\right)\left(\prod_{j=m_i+n_k}^{n+m_p-1} w_j\right) \\
&= \left(\prod_{j=m_i}^{m_p-1} \frac{1}{w_j}\right)\left(\prod_{j=m_i}^{m_i+n_k-1} w_j\right)\left(\prod_{j=m_i+n_k}^{n+m_p-1} w_j\right) \\
&\leqslant C\left(\prod_{j=m_i}^{m_i+n_k-1} w_j\right)\left\|T^{2q}\right\| \\
&\leqslant C\delta_1\left\|T^{2q}\right\|,
\end{aligned}$$

where C is a constant. If we assume that $\delta_1 < \frac{\varepsilon}{C\|T^{2q}\|}$, then

$$\prod_{j=m_p}^{n+m_p-1} w_j \leqslant \varepsilon \text{ for all } m_p \in \mathcal{F}; |m_p| \leqslant q. \tag{13}$$

Now, for all $m_p \in \mathcal{F}$ with $|m_p| \leqslant q$, we have

$$\begin{aligned}
\prod_{j=1+m_p}^{n+m_p} \frac{1}{w_{-j}} &= \left(\prod_{j=1+m_p}^{m_i} \frac{1}{w_{-j}}\right)\left(\prod_{j=1+m_i}^{n_k+m_i} \frac{1}{w_{-j}}\right)\left(\prod_{j=n_k+m_i+1}^{n+m_p} \frac{1}{w_{-j}}\right) \\
&\leqslant L\delta_2\left(\frac{1}{b}\right)^{2q},
\end{aligned}$$

where L is a constant. Hence, if $\delta_2 < \frac{b^{2q}\varepsilon}{L}$, then

$$\prod_{j=1+m_p}^{n+m_p} \frac{1}{w_{-j}} \leqslant \varepsilon \text{ for all } m_p \in \mathcal{F}; |m_p| \leqslant q. \tag{14}$$

It follows by (12), (13) and (14) that T is $\mathscr{M}$-transitive .

Conversely, follows immediately by 2.1. $\square$

The following Proposition can be proved by the same arguments used in the proof of 2.3; therefore, we state it without proof.

PROPOSITION 2.5. *Let T_1 and T_2 be invertible bilateral forward weighted shifts in $\ell^2(\mathbb{Z})$ with a positive weight sequence $\{w_n\}_{n\in\mathbb{Z}}$ and $\{a_n\}_{n\in\mathbb{Z}}$, respectively. Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be closed subspaces of $\ell^2(\mathbb{Z})$, $\mathscr{F}_1 = \{m_j : e_{m_j} \in \mathscr{M}_1 \cap \{e_r\}\}$ and $\mathscr{F}_2 = \{h_j : e_{h_j} \in \mathscr{M}_2 \cap \{e_r\}\}$. Then $T_1 \oplus T_2$ is $\mathscr{M}_1 \oplus \mathscr{M}_2$-transitive if and only if there exist $m_i \in \mathscr{F}_1$, $h_p \in \mathscr{F}_2$ and an increasing sequence of positive integers $\{n_k\}_{k\in\mathscr{N}}$ such that $(T_1 \oplus T_2)^{n_k}(\mathscr{M}_1 \oplus \mathscr{M}_2) \subseteq \mathscr{M}_1 \oplus \mathscr{M}_2$ for all $k \in \mathscr{N}$ and*

$$\lim_{k\to\infty} \max \left\{ \prod_{j=m_i}^{m_i+n_k-1} w_j, \prod_{j=h_p}^{h_p+n_k-1} a_j \right\} = 0 \tag{15}$$

and

$$\lim_{k\to\infty} \max \left\{ \prod_{j=1-m_i}^{n_k-m_i} \frac{1}{w_{-j}}, \prod_{j=1-h_p}^{n_k-h_p} \frac{1}{a_{-j}} \right\} = 0 \tag{16}$$

It can be easily shown that the above theorem does not hold just for two operators but for a finite number of invertible bilateral forward weighted shifts.

In the same way we can characterize the $\mathscr{M}$-hypercyclic backward weighted shifts. The following propositions characterized $\mathscr{M}$-hypercyclic backward weighted shift. We skip their proofs since they can be proved by the same steps as in the proof of the case of $\mathscr{M}$-hypercyclic forward weighted shifts.

PROPOSITION 2.6. *Let T be a bilateral backward weighted shift in $\ell^2(\mathbb{Z})$ with a positive weight sequence $\{w_n\}_{n\in\mathbb{Z}}$, $\mathscr{M}$ be a subspace of $\mathscr{X}$ and $\mathscr{F} = \{m_j : e_{m_j} \in \mathscr{M} \cap \{e_r\}\}$. Then T satisfies $\mathscr{M}$-hypercyclic criterion if and only if, for any $q \in \mathscr{N}$, we have*

(i) $\liminf_{n\to\infty} \max\{\prod_{k=1}^{n} \frac{1}{w_{m_j+k}} : m_j \in \mathscr{F} \text{ and } |m_j| \leqslant q\} = 0,$

(ii) $\liminf_{n\to\infty} \max\{\prod_{k=0}^{n-1} w_{m_j-k} : m_j \in \mathscr{F} \text{ and } |m_j| \leqslant q\} = 0,$

(iii) $T^{n_p}\mathscr{M} \subseteq \mathscr{M}$ *for an increasing sequence of positive integers* $\{n_p\}_{p\in\mathscr{N}}$,

PROPOSITION 2.7. *Let T be an invertible bilateral backward weighted shift in $\ell^2(\mathbb{Z})$ with a positive weight sequence $\{w_n\}_{n\in\mathbb{Z}}$, $\mathscr{M}$ be a subspace of $\ell^2(\mathbb{Z})$ and $\mathscr{F} = \{m_j : e_{m_j} \in \mathscr{M} \cap \{e_r\}\}$. Then T is $\mathscr{M}$-transitive if and only if there exist an increasing sequence of positive integers $\{n_k\}_{k\in\mathscr{N}}$ and $m_i \in \mathscr{F}$ such that $T^{n_k}\mathscr{M} \subseteq \mathscr{M}$ for all $k \in \mathscr{N}$ and*

$$\lim_{k\to\infty} \prod_{j=m_i}^{m_i+n_k-1} w_{-j} = 0 \text{ and } \lim_{k\to\infty} \prod_{j=1+m_i}^{n_k+m_i} \frac{1}{w_j} = 0 \tag{17}$$

PROPOSITION 2.8. *Let $T \in \mathscr{B}(\ell^2(\mathbb{Z}))$ be a bilateral backward weighted shift with a positive weight sequence $\{w_n\}_{n\in\mathbb{Z}}$ such that for all $n<0$, $w_n \geqslant b > 0$. Let $\mathscr{M}$ be a subspace of $\ell^2(\mathbb{Z})$ and $\mathscr{F} = \{m_j : e_{m_j} \in \mathscr{M} \cap \{e_r\}\}$. Then T is $\mathscr{M}$-transitive if and only if there exist an increasing sequence of positive integers $\{n_k\}_{k\in\mathscr{N}}$ and $m_i \in \mathscr{F}$ such that $T^{n_k}\mathscr{M} \subseteq \mathscr{M}$ for all $k \in \mathscr{N}$,*

$$\lim_{k\to\infty} \prod_{j=m_i}^{m_i+n_k-1} w_{-j} = 0 \quad \text{and} \quad \lim_{k\to\infty} \prod_{j=1+m_i}^{n_k+m_i} \frac{1}{w_j} = 0 \tag{18}$$

The following example shows that the Herrero question [6] holds true even on a subspace of a Hilbert space.

EXAMPLE 2.9. There exists an operator T such that both T and T^* are subspace-hypercyclic for some subspaces.

Proof. One can construct a weight sequence $\{w_n\}_{n\in\mathbb{Z}}$ such that it satisfies the conditions of 2.3 for a subspace $\mathscr{M}_1$ and satisfies the conditions of 2.7 for a subspace $\mathscr{M}_2$. If we set T to have the weight sequence $\{w_n\}_{n\in\mathbb{Z}}$, it immediately follows that both T and T^* are subspaces-transitive operators for $\mathscr{M}_1$ and $\mathscr{M}_2$, respectively. □

Since $T^n(\mathscr{M}) \subseteq \mathscr{M}$ if and only if $T^{*n}(\mathscr{M}^\perp) \subseteq \mathscr{M}^\perp$, then one may think that an operator T is $\mathscr{M}$-transitive if and only if T^* is $\mathscr{M}^\perp$-transitive. However, the next example shows that the last statement does not need to be true.

EXAMPLE 2.10. Let B be a unilateral backward shift operator, F be a unilateral forward shift operator and

$$\mathscr{M} = \{\{x_n\}_{n\in\mathscr{N}} : x_{2n} = 0 \text{ for all } n \in \mathscr{N}\} \subset \ell^2(\mathbb{N}).$$

Then, by Example 3.7 of [10], $2B$ is $\mathscr{M}$-hypercyclic where $2B(x_0,x_1,x_2,x_3,\cdots) = (2x_1,2x_2,2x_3\cdots)$. However, $(2B)^* = 2F$, where $2F(x_0,x_1,x_2,x_3,\cdots) = (0,2x_0,2x_1,\cdots)$, cannot be $\mathscr{M}^\perp$-subspace since the unilateral forward shifts are unitary and so cannot be subspace-hypercyclic for any subspace.

QUESTION 1. If T is $\mathscr{M}_1$-transitive and T^* is $\mathscr{M}_2$-transitive, is there any relation between $\mathscr{M}_1$ and $\mathscr{M}_2$?

We now turn to the unilateral weighted shift operators acting on $\ell^2(\mathbb{N})$. Let B be a unilateral backward weighted shift operator with a positive weight sequence $\{w_n\}_{n\in\mathscr{N}}$ then B is defined by $Be_0 = 0$ and $Be_n = w_n e_{n-1}$ for all $n \geqslant 1$. Let F be a unilateral forward weighted shift operator with a positive weight sequence $\{\frac{1}{w_n}\}_{n\in\mathscr{N}}$ then F is defined by $Fe_n = (1/w_{n+1})e_{n+1}$ for all $n \geqslant 0$.

Since unilateral forward weighted shifts cannot be subspace-hypercyclic operators for any subspaces as stated in 2.10; therefore, we characterize only the unilateral backward weighted shifts that are subspace-hypercyclic operators.

THEOREM 2.11. *Let B be a unilateral backward weighted shift operator on $\ell^2(\mathbb{N})$ with positive weight sequence $\{w_n\}_{n\in\mathcal{N}}$, $\mathcal{M}$ be a subspace of $\ell^2(\mathbb{N})$ and $\mathcal{F} = \{m_j : e_{m_j} \in \mathcal{M} \cap \{e_r\}\}$. Then B is $\mathcal{M}$-transitive if and only if there exists an $m_i \in \mathcal{F}$ and an increasing sequence $\{n_k\}_{k\in\mathcal{N}}$ of positive integers such that $B^{n_k}\mathcal{M} \subseteq \mathcal{M}$ for all $k \in \mathcal{N}$ and*

$$\limsup_{n\to\infty}(w_{m_i+1}w_{m_i+2}\cdots w_{m_i+n}) = \infty.$$

Proof. For the "if" part, we verify subspace-hypercyclic criterion. Suppose that $D = D_1 = D_2$ be the dense subsets of $\mathcal{M}$ made up of all finitely supported sequences. Then for all $x \in D$, $B^{n_k}x = 0$ for a large enough k since x has only finite numbers of nonzero elements. Let F be a right inverse to B where $Fe_n = (1/w_{n+1})e_{n+1}$, and let $x_k = F^{n_k}y$. Since $B^{n_k}\mathcal{M} \subseteq \mathcal{M}$ for all $k \in \mathcal{N}$, then $\{n_k : k \in \mathcal{N}\} \subset \mathcal{F}$, and so $\{x_k\}_{k\in\mathcal{N}} \subset \mathcal{M}$. It follows that, $B^{n_k}x_k \to y$ and $\|x_k\| = \|F^{n_k}y\| \to 0$ as $k \to \infty$. Hence T satisfies $\mathcal{M}$-hypercyclic criterion and so T is $\mathcal{M}$-transitive.

For the "only if" part, suppose that T is $\mathcal{M}$-transitive. Let $m_i \in \mathcal{F}$, by 1.4 one may find an $x \in \mathcal{M}$, $n \in \mathcal{N}$ and a small positive number ε_n such that

$$B^n\mathcal{M} \subseteq \mathcal{M} \tag{19}$$

$$\|x - e_{m_i}\| \leqslant \varepsilon_n \tag{20}$$

and

$$\|B^n x - e_{m_i}\| \leqslant \varepsilon_n. \tag{21}$$

By (19), it is easy to find an increasing sequence $\{n_k\}_{k\in\mathcal{N}}$ of positive integers such that

$$B^{n_k}\mathcal{M} \subseteq \mathcal{M} \text{ for all } k \in \mathcal{N} \tag{22}$$

Suppose that $x = (x_0, x_1, \cdots)$. From (20), it follows that

$$|x_j| \leqslant \varepsilon_n \text{ for all } j \in \mathcal{N};\, j \neq m_i \tag{23}$$

and

$$|x_{m_i} - 1| < \varepsilon_n$$

From (21), it follows that

$$|x_{n+m_i}w_{1+m_i}w_{2+m_i}\cdots w_{n+m_i}| - 1 \leqslant \varepsilon_n,$$

that is,

$$(x_{n+m_i}w_{1+m_i}w_{2+m_i}\cdots w_{n+m_i}) \geqslant 1 - \varepsilon_n \tag{24}$$

Since $n + m_i \neq m_i$, then by (23) we have $x_{n+m_i} \leqslant \varepsilon_n$, combining this with (24), we get

$$w_{1+m_i}w_{2+m_i}\cdots w_{n+m_i} \geqslant \frac{1-\varepsilon_n}{\varepsilon_n}. \tag{25}$$

Since $\varepsilon_n \to 0$ when $n \to \infty$, then the proof follows by (22) and (25). $\square$

The next proposition characterizes the direct sum of unilateral backward weighted shifts that are subspace-hypercyclic for some subspaces, in term of their weight sequences.

PROPOSITION 2.12. *Let B_1 and B_2 be unilateral backward weighted shifts in $\ell^2(\mathbb{N})$ with positive weight sequences $\{w_n\}_{n\in\mathscr{N}}$ and $\{a_n\}_{n\in\mathscr{N}}$, respectively. Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be subspaces of $\ell^2(\mathbb{N})$, $\mathscr{G}_1 = \{m_j : e_{m_j} \in \mathscr{M}_1 \cap \{e_r\}\}$ and $\mathscr{G}_2 = \{h_j : e_{h_j} \in \mathscr{M}_2 \cap \{e_r\}\}$. Then $B_1 \oplus B_2$ is $\mathscr{M}_1 \oplus \mathscr{M}_2$-transitive if and only if there exist $m_i \in \mathscr{G}_1$, $h_p \in \mathscr{G}_2$ and an increasing sequence of positive integers $\{n_k\}_{k\in\mathscr{N}}$ such that $(B_1 \oplus B_2)^{n_k}(\mathscr{M}_1 \oplus \mathscr{M}_2) \subseteq \mathscr{M}_1 \oplus \mathscr{M}_2$ for all $k \in \mathscr{N}$ and*

$$\limsup_{n\to\infty}\{\min\{(a_{h_p+1}a_{h_p+2}\cdots a_{h_p+n}),(w_{m_i+1}w_{m_i+2}\cdots w_{m_i+n})\}\} = \infty.$$

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